OBSERVATIONS sur la méthode de prendre les milieux entre les observations.*

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The most advantageous way to take the mean among some observations has been detailed by great geometers. Mr. Daniel Bernoulli,[1] Mr. Lambert, Mr. de la Place,[2] Mr. de la Grange[3] have occupied themselves in it. The last has given on this subject a very beautiful memoir in Tome V of the *Mémoires de Turin*. He has employed for this the integral calculus. My design in this memoir is to show how one can arrive to the same results by a simple usage of the theory of combinations. Mr. de la Grange having reduced these questions to those that one can make on the game of dice, I am going to begin by treating this game. Mr. de Moivre has given at the end of his Problem 3 a lemma destined to determine how many cases there are in order to bring forth a certain number of points, with a certain number of dice of which each has the same number of faces. But his demonstration being indirect, it will be simpler to treat the thing analytically.

§ 1. For more clarity, I will suppose first the ordinary case of dice which have six faces, on which are marked the numbers 1, 2, 3, 4, 5, 6. The solution of the more general cases will emanate naturally from this particular case.

§ 2. First, for the case of a single die, it is evident that there is only one case to bring forth each of the points 1, 2, 3, 4, 5, 6.

§ 3. We suppose two dice; to each face of the first die can correspond each of the faces of the second die, this which furnishes the following table, of which here is the

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[1] *Translator's note*: "Diiudicatio maxime probabilis plurium observationem discrepantium atque verisimillima inductio inde formanda." *Acta Acad. Sci. Imp. Petrop.,* 1777 (1778), 1, 3–23. It has been translated into English by C.G. Allen as "The most probable choice between several discrepant observations and the formation therefrom of the most likely induction," *Biometrika*, 1961, 48, 1–18.

[2] *Translator's note*: It is not clear to what memoir Trembley refers. Most probably it is the "Mémoire sur les probabilités," *Mém. Acad. R. Sci. Paris*, 1778 (1781).

[3] *Translator's note*: "Mémoire sur l'utilité de la méthode de prendre milieu entre les résultats de plusiers observations," *Miscellanea Taurinensia,* t. V, 1770–1773.

explication:

2	1	1
3	$1+1$	2
4	$1+1+1$	3
5	$1+1+1+1$	4
6	$1+1+1+1+1$	5
7	$1+1+1+1+1+1$	6
8	$1+1+1+1+1$	5
9	$1+1+1+1$	4
10	$1+1+1$	3
11	$1+1$	2
12	1	1

Joining to the face of the first die each of the faces of the second, I have one case for each of the points 2, 3, 4, 5, 6, 7; joining to the face 2 of the first die each of the faces of the second, I have one case for each of the points 3, 4, 5, 6, 7, 8; joining to the face 3 of the first die each of the faces of the second, I have a case for each of the points 4, 5, 6, 7, 8, 9; & thus in sequence. Finally, joining to the face 6 of the first die each of the faces of the second, I have one case for each of the points 7, 8, 9, 10, 11, 12. Reuniting therefore the case for each point, I see that for the first 6 points 2, 3, 4, 5, 6, 7, I have the sequence of natural numbers 1, 2, 3, 4, 5, 6, & for the 6 last 7, 8, 9, 10, 11, 12, I have the sequence of natural numbers reversed 6, 5, 4, 3, 2, 1. Or if one wishes to continue the sequence of natural numbers, one will form this table:

2	1	1	$6-5$
3	2	2	$6-4$
4	3	3	$6-3$
5	4	4	$6-2$
6	5	5	$6-1$
7	6	6	6
8	$7-2$	$7-2.1$	$6+1-2\,(6-5)$
9	$8-4$	$8-2.2$	$6+2-2\,(6-4)$
10	$9-6$	$9-2.3$	$6+3-2\,(6-3)$
11	$10-8$	$10-2.4$	$6+4-2\,(6-2)$
12	$11-10$	$11-2.5$	$6+5-2\,(6-1)$

By means of this table, one sees that one can continue the progression of the natural numbers, but as soon as one has passed the number of faces, it is necessary to subtract the number two, which is the number of the dice multiplied by the numbers of the same progression of the natural numbers.

§ 4. In the case of three dice, one will have, by joining to each face of the third die the cases that we just found for the first two,

3	1	1
4	$2{+}1$	3
5	$3{+}2+1$	6
6	$4{+}3+2+1$	10
7	$5{+}4+3+2+1$	15
8	$6{+}5+4+3+2+1$	21
9	$7{-}2.1+6+5+4+3+2$	$28{-}3.1$
10	$8{-}2.2+7-2.1+6+5+4+3$	$36{-}3.3$
11	$9{-}2.3+8-2.2+7-2.1+6+5+4$	$45{-}3.6$
12	$10{-}2.4+9-2.3+8-2.2+7-2.1+6+5$	$55{-}3.10$
13	$11{-}2.5+10-2.4+9-2.3+8-2.2+7-2.1+6$	$66{-}3.15$
14	$11{-}2.5+10-2.4+9-2.3+8-2.2+7-2.1$	$78{-}3.21$
15	$11{-}2.5+10-2.4+9-2.3+8-2.2$	$91{-}3.28+\frac{3.2}{1.2}.1$
16	$11{-}2.5+10-2.4+9-2.3$	$105{-}3.36+\frac{3.2}{1.2}.3$
17	$11{-}2.5+10-2.4$	$120{-}3.45+\frac{3.2}{1.2}.6$
18	$11{-}2.5$	$136{-}3.55+\frac{3.2}{1.2}.10$

One sees by this table, that one obtains in this case the triangular numbers; but as soon as one has passed the number of faces, it is necessary to subtract the number 3 which is the number of dice multiplied by the numbers of the same progression; since as soon as one has passed again one time the number of faces, it is necessary to add the number $\frac{3.2}{1.2}$ multiplied by the numbers of the same progression. By taking therefore the form of the general term of the triangular numbers, one can give to the table the following form:

3	$\frac{(6-5)(6-4)}{2}$
4	$\frac{(6-4)(6-3)}{2}$
5	$\frac{(6-3)(6-2)}{2}$
6	$\frac{(6-2)(6-1)}{2}$
7	$\frac{(6-1)\,6}{2}$
8	$\frac{6\,(6+1)}{2}$
9	$\frac{(6+1)(6+2)}{2}-3.\frac{(6-5)(6-4)}{2}$
10	$\frac{(6+2)(6+3)}{2}-3.\frac{(6-4)(6-3)}{2}$
11	$\frac{(6+3)(6+4)}{2}-3.\frac{(6-3)(6-2)}{2}$
12	$\frac{(6+4)(6+5)}{2}-3.\frac{(6-2)(6-1)}{2}$
13	$\frac{(6+5)(6+6)}{2}-3.\frac{(6-1)\,6}{2}$
14	$\frac{(6+6)(6+7)}{2}-3.\frac{6.\,(6+1)}{2}$
15	$\frac{(6+7)(6+8)}{2}-3.\frac{(6+1)(6+2)}{2}+\frac{3.2}{1.2}\frac{(6-5)(6-4)}{2}$
16	$\frac{(6+8)(6+9)}{2}-3.\frac{(6+2)(6+3)}{2}+\frac{3.2}{1.2}\frac{(6-4)(6-3)}{2}$
17	$\frac{(6+9)(6+10)}{2}-3.\frac{(6+3)(6+4)}{2}+\frac{3.2}{1.2}\frac{(6-3)(6-2)}{2}$
18	$\frac{(6+10)(6+11)}{2}-3.\frac{(6+4)(6+5)}{2}+\frac{3.2}{1.2}\frac{(6-2)(6-1)}{2}$

The law of the terms is made known already. One sees that at the end of six terms, it is necessary to add the progression of the triangular numbers. The law of the coefficients is that of the coefficients of the binomial.

§ 5. In the case of 4 dice, one will have by joining to each of the faces of the fourth die upon the case that we just found for the first three dice, one will have, I say, the following table, in which one sees to reveal the law which must take place in the more complicated cases, & the principles which can serve to extend these formulas to the case where the dice would have each f faces.

4	1		
5	$3+1$		
6	$6+3+1$		
7	$10+6+3+1$		
8	$15+10+6+3+1$		
9	$21+15+10+6+3+1$		
10	$28+21+15+10+6+3$	-3.1	
11	$36+28+21+15+10+6$	$-3.3-3.1$	
12	$45+36+28+21+15+10$	$-3.6-3.3-3.1$	
13	$55+45+36+28+21+15$	$-3.10-3.6-3.3-3.1$	
14	$66+55+45+36+28+21$	$-3.15-3.10-3.6-3.3-3.1$	
15	$78+66+55+45+36+28$	$-3.21-3.15-3.10-3.6-3.3-3.1$	
16	$91+78+66+55+45+36$	$-3.28-3.21-3.15-3.10-3.6-3.3$	$+3.1$
17	$105+91+78+66+55+45$	$-3.36-3.28-3.21-3.15-3.10-3.6$	$+3.3+3.1$
18	$120+105+91+78+66+55$	$-3.45-3.36-3.28-3.21-3.15-3.10$	$+3.6+3.3+3.1$
19	$136+120+105+91+78+66$	$-3.55-3.45-3.36-3.28-3.21-3.15$	$+3.10+3.6+3.3+3.1$
20	$136+120+105+91+78$	$-3.55-3.45-3.36-3.28-3.21$	$+3.10+3.6+3.3+3.1$
21	$136+120+105+91$	$-3.55-3.45-3.36-3.28$	$+3.10+3.6+3.3+3.1$
22	$136+120+105$	$-3.55-3.45-3.36$	$+3.10+3.6+3.3$
23	$136+120$	$-3.55-3.45$	$+3.10+3.6$
24	136	-3.55	$+3.10$

This table can be reduced to the following table:

4	1	17	$560-4.120+\frac{4.3}{2}.4$
5	4	18	$680-4.165+\frac{4.3}{2}.10$
6	10	19	$816-4.220+\frac{4.3}{2}.20$
7	20	20	$969-4.286+\frac{4.3}{2}.35$
8	35	21	$1140-4.364+\frac{4.3}{2}56$
9	56	22	$1330-4.455+\frac{4.3}{2}.84-\frac{4.3.2}{1.2.3}.1$
10	$84-4.1$	23	$1540-4.560+\frac{4.3}{1.2}.120-\frac{4.3.2}{1.2.3}.4$
11	$120-4.4$	24	$1771-4.680+\frac{4.3}{2}.165-\frac{4.3.2}{1.2.3}.10$
12	$165-4.10$		
13	$220-4.20$		
14	$286-4.35$		
15	$364-4.56$		
16	$455-4.84+\frac{4.3}{2}.1$		

This table gives, as one sees, the pyramidal numbers, & by taking the formula of the general term of these numbers, one can give to the table the following form:

4	$\frac{(6-5)(6-4)(6-3)}{1.2.3}$				$6-2$
5	$\frac{(6-4)(6-3)(6-2)}{1.2.3}$				$6-1$
6	$\frac{(6-3)(6-2)(6-1)}{1.2.3}$				6
7	$\frac{(6-2)(6-1)6}{1.2.3}$				$6+1$
8	$\frac{(6-1)(6.)(6+1)}{1.2.3}$				$6+2$
9	$\frac{6.(6+1)(6+2)}{1.2.3}$				$6+3$
10	$\frac{(6+1)(6+2)(6+3)}{1.2.3}$	$-4.\frac{(6-5)(6-4)(6-3)}{1.2.3}$			$6+4$
11	$\frac{(6+2)(6+3)(6+4)}{1.2.3}$	$-4.\frac{(6-4)(6-3)(6-2)}{1.2.3}$			$6+5$
12	$\frac{(6+3)(6+4)(6+5)}{1.2.3}$	$-4.\frac{(6-3)(6-2)(6-1)}{1.2.3}$			$6+6$
13	$\frac{(6+4)(6+5)(6+6)}{1.2.3}$	$-4.\frac{(6-2)(6-1)6}{1.2.3}$			$6+7$
14	$\frac{(6+5)(6+6)(6+7)}{1.2.3}$	$-4.\frac{(6-1)6.(6+1)}{1.2.3}$			$6+8$
15	$\frac{(6+6)(6+7)(6+8)}{1.2.3}$	$-4.\frac{(6.)(6+1)(6+2)}{1.2.3}$			$6+9$
16	$\frac{(6+7)(6+8)(6+9)}{1.2.3}$	$-4.\frac{(6+1)(6+2)(6+3)}{1.2.3}$	$+\frac{4.3}{1.2}\frac{(6-5)(6-4)(6-3)}{1.2.3}$		$6+10$
17	$\frac{(6+8)(6+9)(6+10)}{1.2.3}$	$-4.\frac{(6+2)(6+3)(6+4)}{1.2.3}$	$+\frac{4.3}{1.2}\frac{(6-4)(6-3)(6-2)}{1.2.3}$		$6+11$
18	$\frac{(6+9)(6+10)(6+11)}{1.2.3}$	$-4.\frac{(6+3)(6+4)(6+5)}{1.2.3}$	$+\frac{4.3}{1.2}\frac{(6-3)(6-2)(6-1)}{1.2.3}$		$6+12$
19	$\frac{(6+10)(6+11)(6+12)}{1.2.3}$	$-4.\frac{(6+4)(6+5)(6+6)}{1.2.3}$	$+\frac{4.3}{1.2}\frac{(6-2)(6-1)6}{1.2.3}$		$6+13$
20	$\frac{(6+11)(6+12)(6+13)}{1.2.3}$	$-4.\frac{(6+5)(6+6)(6+7)}{1.2.3}$	$+\frac{4.3}{1.2}\frac{(6-1)6.(6+1)}{1.2.3}$		$6+14$
21	$\frac{(6+12)(6+13)(6+14)}{1.2.3}$	$-4.\frac{(6+6)(6+7)(6+8)}{1.2.3}$	$+\frac{4.3}{1.2}\frac{6.(6+1)(6+2)}{1.2.3}$		$6+15$
22	$\frac{(6+13)(6+14)(6+15)}{1.2.3}$	$-4.\frac{(6+7)(6+8)(6+9)}{1.2.3}$	$+\frac{4.3}{1.2}\frac{(6+1)(6+2)(6+3)}{1.2.3}$	$-\frac{4.3.2}{1.2.3}\frac{(6-5)(6-4)(6-3)}{1.2.3}$	$6+16$
23	$\frac{(6+14)(6+15)(6+16)}{1.2.3}$	$-4.\frac{(6+8)(6+9)(6+10)}{1.2.3}$	$+\frac{4.3}{1.2}\frac{(6+2)(6+3)(6+4)}{1.2.3}$	$-\frac{4.3.2}{1.2.3}\frac{(6-4)(6-3)(6-2)}{1.2.3}$	$6+17$
24	$\frac{(6+15)(6+16)(6+17)}{1.2.3}$	$-4.\frac{(6+9)(6+10)(6+11)}{1.2.3}$	$+\frac{4.3}{1.2}\frac{(6+3)(6+4)(6+5)}{1.2.3}$	$-\frac{4.3.2}{1.2.3}\frac{(6-3)(6-2)(6-1)}{1.2.3}$	$6+18$

One sees by this table that it is necessary to take the pyramidal numbers; after the first six, it is necessary to subtract the sequence of these same numbers multiplied by 4, which is the number of dice; six numbers after, it is necessary to add the sequence of these same numbers multiplied by $\frac{4.3}{1.2}$; six numbers after, it is necessary to subtract the sequence of these same numbers multiplied by $\frac{4.3.2}{1.2.3}$.

§ 6. Now, in order to have any case of this table without forming all the table, for example, to have the case which gives 22, one will note that $22 = 6 + 16$, one will diminish this number by 1, & one will have the first term $\frac{(6+15)(6+14)(6+13)}{1.2.3}$, one will diminish $6 + 15$ by 6, & one will have the second term $-4.\frac{(6+9)(6+8)(6+7)}{1.2.3}$, one will diminish $6 + 9$ by 6, & one will have the third term $\frac{4.3}{1.2}\frac{(6+3)(6+2)(6+1)}{1.2.3}$, one will diminish $6 + 3$ by 6, & one will have the fourth term $-\frac{4.3.2}{1.2.3}\frac{(6-3)(6-4)(6-5)}{1.2.3}$. I will go no further, because the term following would become zero. The coefficients are, as I have said above, those of the binomial, & the signs alternate. The number of cases will be therefore

$$\frac{(6+15)(6+14)(6+13)}{1.2.3} - 4.\frac{(6+9)(6+8)(6+7)}{1.2.3} + \frac{4.3}{1.2}\frac{(6+3)(6+2)(6+1)}{1.2.3} - \frac{4.3.2}{1.2.3}\frac{(6-3)(6-2)(6-1)}{1.2.3},$$

as one finds it by our table.

§ 7. One sees clearly that if there were 5 dice, it would be necessary to take the figurate numbers of the 5$^{\text{th}}$ order, to add a new term of six by six terms, to multiply the second term by 5 number of dice, the third by $\frac{5.4}{1.2}$, the fourth by $\frac{5.4.3.}{1.2.3}$, the fifth by $\frac{5.4.3.2}{1.2.3.4}$.

§ 8. And in general if there were n dice, it would be necessary to take the figurate numbers of the n^{th} order, to add a new term of 6 by 6 terms, & to multiply the second by n number of dice, the third by $\frac{n(n-1)}{1.2}$, the fourth by $\frac{n(n-1)(n-2)}{1.2.3}$, & thus in sequence.

§ 9. If one demands the number of cases that one has in order to bring forth 27 points with 6 dice, I make $27 = 6 + 21$, & I have

$$\frac{(6+20)(6+19)(6+18)(6+17)(6+16)}{1.2.3.4.5} - 6.\frac{(6+14)(6+13)(6+12)(6+11)(6+10)}{1.2.3.4.5}$$
$$+\frac{6.5}{1.2}\frac{(6+8)(6+7)(6+6)(6+5)(6+4)}{1.2.3} - \frac{6.5.4}{1.2.3}\frac{(6+2)(6+1)\,6.\,(6-1)(6-2)}{1.2.3.4.5}$$

I stop myself here because the following term is $= 0$, the number of sought cases is therefore $65780 + 30030 - 93024 - 1120 = 1666$, it is the number that Mr. de Moivre finds. The formulas of the general terms of the different orders of the figurate numbers teaches us thus that for the 2$^{\text{nd}}$ order, there is only one factor in the general term, two for the 3$^{\text{rd}}$, three for the 4$^{\text{th}}$, & in general $n-1$ for the n^{th}.

§ 10. In order to bring together now our formulas to those of Mr. de Moivre, instead of making, for example, $27 = 6 + 21$, we will make $27 = p + 1$, $n = 6$, subtracting 1 from $p + 1$, the 1$^{\text{st}}$ term will be

$$\frac{p(p-1)(p-2)(p-3)(p-4)}{1.2.3.4.5},$$

subtracting 6 from p, the second term will be

$$-n\frac{(p-6)(p-7)(p-8)(p-9)(p-10)}{1.2.3.4.5},$$

subtracting 6 from $p-6$ the third term will be

$$\frac{n(n-1)}{1.2}\frac{(p-12)(p-13)(p-14)(p-15)(p-16)}{1.2.3.4.5},$$

subtracting 6 from $p-12$ the fourth term will be

$$\frac{n(n-1)(n-2)}{1.2.3}\frac{(p-18)(p-19)(p-20)(p-21)(p-22)}{1.2.3.4.5},$$

and thus in sequence, & one obtains thus the formula of Mr. de Moivre, if one makes in this formula $f = 6$.

§ 11. Now, if the dice had f faces instead of having 6 of them, the same general reasoning would subsist, by applying to the number f that which we have said of the number 6. Repeating therefore exactly the same reasonings & the same operations, one will conclude in general that in the case of n dice of f faces each, it will be necessary to take the figurate numbers of the n^{th} order, to add a new term of f by f terms & to multiply the second by n, the third by $\frac{n(n-1)}{1.2}$, the fourth by $\frac{n(n-1)(n-2)}{1.2.3}$, & thus in sequence. Next being given a number $p+1$, it will be necessary to subtract 1 in order to have the first factor of the first term, to subtract f from this first factor in order to have the first factor of the second term, to subtract f from this factor in order to have the first factor of the third term, & thus in sequence. Making therefore $p-f=1$, $q-f=r$, $r-f=s$ &c., one will have by virtue of these precepts the same formula of Mr. de Moivre,

$$\begin{aligned}\frac{p(p-1)(p-2)(p-3)}{1.2.3.4}\ \&\text{c.} - n\frac{q(q-1)(q-2)q-3)}{1.2.3.4}\ \&\text{c.}\\ + \frac{n(n-1)}{1.2}\frac{r}{1}\frac{(r-1)(r-2)r-3)}{1.2.3.4}\ \&\text{c.}\\ - \frac{n(n-1)(n-2)}{1.2.3}\frac{s(s-1)(s-2)(s-3)}{1.2.3.4}\ \&\text{c.}\end{aligned}$$

for the case of n dice. One will take $(n-1)$ factors, & one will proceed until one finds a term $= 0$.

§ 12. After having treated of dice in general, I myself am going to draw nearer together my subject, & to treat the cases which lead to the method to take the mean among the observations.

§ 13. Let there be n dice with three faces, of which one is marked 0, the second 1, the third -1, one demands the probability to bring forth any number with n dice.

§ 14. If there is only a single die, one has the following table:

1	0
1	1
1	-1

thus the probability to bring forth zero will be $\frac{1}{3}$ & likewise for the other numbers.

§ 15. If there are two dice, to each of the preceding cases correspond the three cases of the first die, this which gives the following table:

1	2
2	1
3	0

The cases which give -1 & -2 being evidently equal in number to those which give 1 & 2, I believed to be able to omit them. Thus the probability to bring forth zero will be $\frac{3}{9}$.

§ 16. If there are three dice, to each of the cases of the preceding table correspond the three cases of the third die, this which gives the following table:

$$\begin{array}{r|l} 1 & 3 \\ 3 & 2 \\ 6 & 1 \\ 7 & 0 \end{array}$$

Thus the probability to bring forth zero will be $\frac{7}{27}$, that to bring forth 2, will be $\frac{3}{27}$, & thus of the rest.

§ 17. One will have, proceeding likewise for four dice,

$$\begin{array}{rrrrr} 4, & 3, & 2, & 1, & 0, \\ 1, & 4, & 10, & 16, & 19, \end{array}$$

for five dice,

$$\begin{array}{rrrrrr} 5, & 4, & 3, & 2, & 1, & 0, \\ 1, & 5, & 15, & 30, & 45, & 51, \end{array}$$

for six dice,

$$\begin{array}{rrrrrrr} 6, & 5, & 4, & 3, & 2, & 1, & 0, \\ 1, & 6, & 21, & 50, & 90, & 126, & 141. \end{array}$$

One will have therefore, by following the same analogy as in the problem of ordinary dice, for n dice,

$$\begin{array}{lllll} n, & n-1, & n-2, & n-3, & n-4 \\ 1, & n, & \frac{n(n+1)}{1.2}, & \frac{n(n+1)(n+2)}{1.2.3} - n.1, & \frac{n(n+1)\cdots(n+3)}{1.2.3.4} - n.n \end{array}$$

$$\begin{array}{ll} n-5 & n-6 \\ \frac{n(n+1)\cdots(n+4)}{1.2...5} - n.\frac{n(n+1)}{1.2} & \frac{n(n+1)\cdots(n+5)}{1.2...6} - n\frac{n(n+1)(n+2)}{1.2.3} + \frac{n(n-1)}{1.2}1 \end{array}$$

...

$$n-\nu$$

$$\frac{n(n+1)\cdots(n+\nu-1)}{1.2...\nu} - n.\frac{n(n+1)\cdots(n+\nu-4)}{1.2\cdots(\nu-3)} + \frac{n(n-1)}{1.2}.\frac{n(n+1)\cdots(n+\nu-3)}{1.2\cdots(\nu-6)}\&c.$$

$$0$$

$$\frac{n(n+1)\cdots(2n-1)}{1.2...n} - n.\frac{n(n+1)\cdots(2n-4)}{1.2\cdots(n-3)} + \frac{n(n-1)}{1.2}\cdot\frac{n(n+1)\cdots(2n-7)}{1.2\cdots(n-6)}\&c$$

It is necessary to pay attention in this formula, when the last factor of the denominator of the last term will become zero, to put 1 in the place of this term. Thus for the case of $n=6$ one has for the case where one will obtain zero

$$\frac{6.7.8.9.10.11}{1.2.3.4.5.6} - 6.\frac{6.7.8}{1.2.3} + \frac{6.5}{1.2}1 = 141.$$

§ 18. Let there be now n dice with four faces, of which two are marked zero, one 1 & the other -1, one demands the probability to bring forth any number with n dice.

If there is only one die, one has the following table:

$$\begin{array}{rr} 1, & 0, \\ 1, & 2. \end{array}$$

If there are two dice, one proceeds as above, & one has

$$\begin{matrix} 2, & 1, & 0, \\ 1, & 4, & 6. \end{matrix}$$

If there are three dice, one has

$$\begin{matrix} 3, & 2, & 1, & 0, \\ 1, & 6, & 15, & 20. \end{matrix}$$

If there are four dice, one has

$$\begin{matrix} 4, & 3, & 2, & 1, & 0, \\ 1, & 8, & 28, & 56, & 70. \end{matrix}$$

If there are five dice, one has

$$\begin{matrix} 5, & 4, & 3, & 2, & 1, & 0, \\ 1, & 10, & 45, & 120, & 210, & 252. \end{matrix}$$

If there are six dice, one has

$$\begin{matrix} 6, & 5, & 4, & 3, & 2, & 1, & 0, \\ 1, & 12, & 66, & 220, & 495, & 792, & 924. \end{matrix}$$

If there are n dice, one has

$$\begin{matrix} n & n-1 & n-2 & n-3 & n-4 \\ 1, & 2n, & \frac{2n(2n-1)}{1.2}, & \frac{2n(2n-1)(2n-2)}{1.2.3}, & \frac{2n(2n-1)(2n-2)(2n-3)}{1.2.3.4} \end{matrix}$$

$$\begin{matrix} n-5 & n-6 & n-\nu \\ \frac{2n(2n-1)\cdots(2n-4)}{1.2\ldots 5}, & \frac{2n(2n-1)\cdots(2n-5)}{1.2\ldots 6}, & \frac{2n(2n-1)\cdots(2n-\nu+1)}{1.2\ldots \nu}, \end{matrix}$$

$$\begin{matrix} 0 \\ \frac{2n(2n-1)\cdots(n+1)}{1.2\ldots n}. \end{matrix}$$

§ 19. Let there be n dice of which the number of faces are $a^{(1)} + a^{(0)} + a^{(-1)}$; let the $a^{(1)}$ faces be marked 1, the $a^{(0)}$ faces be marked 0, & the $a^{(-1)}$ faces be marked -1, one will have

for the case of 1 die,

$$\begin{array}{r|l} 1 & a^{(1)} \\ 0 & a^{(0)} \\ -1 & a^{(-1)} \end{array}$$

for the case of 2 dice,

$$\begin{array}{r|l} 2 & a^{(1)^2} \\ 1 & 2a^{(0)}a^{(1)} \\ 0 & 2a^{(-1)}a^{(1)} + a^{(0)^2} \\ -1 & 2a^{(-1)}a^{(0)} \\ -2 & a^{(-1)^2} \end{array}$$

for the case of 3 dice,

$$\begin{array}{r|l}
3 & a^{(1)^3} \\
2 & 3a^{(0)}a^{(1)^2} \\
1 & 3a^{(-1)}a^{(1)^2} + 3a^{(0)^2}a^{(1)} \\
0 & 6a^{(-1)}a^{(0)}a^{(1)} + 3a^{(0)^3} \\
-1 & 3a^{(-1)^2}a^{(1)} + 3a^{(-1)}a^{0)^2} \\
-2 & 3a^{(-1)^2}a^{(0)} \\
-3 & a^{(-1)^3}
\end{array}$$

for the case of 4 dice,

$$\begin{array}{r|l}
4 & a^{(1)^4} \\
3 & 4a^{(0)}a^{(1)^3} \\
2 & 4a^{(-1)}a^{(1)^3} + 6a^{(0)^2}a^{(1)^2} \\
1 & 12a^{(-1)}a^{(0)}a^{(1)^2} + 4a^{(0)^3}a^{(1)} \\
0 & 6a^{(-1)^2}a^{(1)^2} + 12a^{(-1)}a^{(0)^2}a^{(1)} + a^{(0)^4} \\
-1 & 12a^{(-1)^2}a^{(0)}a^{(1)} + 4a^{(-1)}a^{0)^3} \\
-2 & 4a^{(-1)^3}a^{(1)} + 6a^{(-1)^2}a^{(0)^2} \\
-3 & 4a^{(-1)^3}a^{(0)} \\
-4 & 4a^{(-1)^4}
\end{array}$$

The analogy is now evident, the numerical coefficients being only the number of permutations of the same quantities; thus the coefficient of $a^{(-1)}a^{(0)^2}a^{(1)}$ is equal to the number of permutations of 4 things, of which 2 are the same, namely $\frac{1.2.3.4}{1.2} = 12$, & as for the same terms, they are in all for n dice $(a^{(1)} + a^{(0)} + a^{(-1)})^n$ & for each number it is necessary to take the terms of which the sum of the exponents is equal to this number, thus for four dice it is necessary to take $a^{(-1)^2}a^{(1)^2}$, $a^{(-1)}a^{(0)^2}a^{(1)}$, $a^{(0)^4}$, in order to have the terms corresponding to zero, it is necessary to take $a^{(-1)}a^{(1)^3}$, $a^{(0)^2}a^{(1)^2}$ in order to have the terms corresponding to two, having need to multiply always the exponent of $a^{(1)}$ by 1, the exponent of $a^{(0)}$ by 0, the exponent of $a^{(-1)}$ by -1. One will have therefore for n dice

$$\begin{array}{l|l}
0 & a^{(0)^n} + n(n-1)a^{(0)^{n-2}}a^{(-1)}a^{(1)} + \frac{n(n-1)\cdots(n-3)}{1.2\ 1.2}a^{(0)^{n-4}}a^{(-1)^2}a^{(1)^2} \\
 & \quad + \frac{n(n-1)\cdots(n-5)}{1.2.3\ 1.2.3}a^{(0)^{n-6}} - 6a^{(-1)^3}a^{(1)^3} \text{ \&c.} \\
1 & \frac{n}{1}a^{(0)^{n-1}}a^{(1)} + \frac{n(n-1)(n-2)}{1.2}a^{(0)^{n-3}}a^{(1)^2}a^{(-1)} \\
 & \quad + \frac{n(n-1)\cdots(n-4)}{1.2\ 1.2.3}a^{(0)^{n-5}}a^{(1)^3}a^{(-1)^2} \\
 & \quad + \frac{n(n-1)\cdots(n-6)}{1.2.3\ 1\ldots4}a^{(0)^{n-7}}a^{(1)^4}a^{(-1)^3} \text{ \&c} \\
2 & \frac{n(n-1)}{1.2}a^{(0)^{n-2}}a^{(1)^2} + \frac{n(n-1)\cdots(n-3)}{1.2.3}a^{(0)^{n-4}}a^{(1)^3}a^{(-1)} \\
 & \quad + \frac{n(n-1)\cdots(n-5)}{1.2\ 1.2.3.4}a^{(1)^4}a^{(-1)^2} \\
 & \quad + \frac{n(n-1)\cdots(n-7)}{1\ldots3\ 1\ldots5}a^{(0)^{n-8}}a^{(1)^5}a^{(-1)^3} \text{ \&c} \\
3 & \frac{n(n-1)(n-2)}{1.2.3}a^{(0)^{n-3}}a^{(1)^3} + \frac{n(n-1)\cdots(n-4)}{1.2\ldots4}a^{(0)^{n-5}}a^{(1)^4}a^{(-1)} \\
 & \quad + \frac{n(n-1)\cdots(n-6)}{1.2\ 1.2\ldots5}a^{(0)^{n-7}}a^{(-1)^2}a^{(1)^5} \\
 & \quad + \frac{n(n-1)\cdots(n-8)}{1\ldots3\ 1\ldots6}a^{(0)^{n-9}}a^{(1)^6}a^{(-1)^3} \text{ \&c} \\
 & \ldots \\
\nu & \frac{n(n-1)\cdots(n-\nu+1)}{1.2\ldots\nu}a^{(0)^{n-\nu}}a^{(1)^\nu} + \frac{n(n-1)\cdots(n-\nu-1)}{1.2\cdots(\nu+1)}a^{(0)^{n-\nu-2}}a^{(-1)}a^{(1)^{\nu+1}} \\
 & \quad + \frac{n(n-1)\cdots(n-\nu-3)}{1.2\ 1.2\ldots(\nu+2)}a^{(0)^{n-\nu-4}}a^{(1)^{\nu+2}}a^{(-1)^2} \\
 & \quad + \frac{n(n-1)\cdots(n-\nu-5)}{1.2.3\ 1.2\ldots(\nu+3)}a^{(0)^{n-\nu-6}}a^{(1)^{\nu+3}}a^{(-1)^3} \\
 & \quad + \frac{n(n-1)\cdots(n-\nu-7)}{1\ldots4\ 1\ldots(\nu+4)}a^{(0)^{n-\nu-8}}a^{(1)^{\nu+4}}a^{(-1)^4} \ldots \\
 & \quad + \frac{n(n-1)\cdots(n-\nu-2\varrho+1)}{1\ldots\varrho 1\ldots(\nu+\varrho)}a^{(0)^{n-\nu-2\varrho}}a^{(1)^{\nu+\varrho}}a^{(-1)^\varrho}
\end{array}$$

One will have the values $-1,\ -2,\ -3,\ \ldots -\nu$ by exchanging $a^{(1)}$ & $a^{(-1)}$ with the values $1,\ 2,\ 3, \ldots \nu$.

§ 20. Let there be n dice of which the number of faces is $a^{(r)} + a^{(0)} + a^{(-1)}$, of which the faces $a^{(r)}$ are marked r, the faces $a^{(0)}$ are marked 0, & the faces $a^{(-1)}$ are marked 1, one demands the probability to bring forth of any number with n dice.

One has for the case of 1 die

$$\begin{array}{ccc} r, & 0, & -1 \\ a^{(r)}, & a^{(0)}, & a^{(-1)} \end{array}$$

for the case of two dice

$$\begin{array}{cccccc} 2r, & r, & r-1, & 0, & -1, & -2 \\ a^{(r)^2}, & 2a^{(0)}a^{(r)}, & 2a^{(-1)}a^{(r)}, & a^{(0)^2}, & 2a^{(0)}a^{(-1)}, & a^{(-1)^2} \end{array}$$

for the case of three dice

$$\begin{array}{ccccc} 3r, & 2r, & 2r-1, & r, & r-1, \\ a^{(r)^3}, & 3a^{(0)}a^{(r)^2}, & 3a^{(-1)}a^{(r)^2}, & 3a^{(0)^2}a^{(r)}, & 6a^{(-1)}a^{(0)}a^{(r)}, \end{array}$$

$$\begin{array}{cccc} r-2, & -1, & -2, & -3, \\ 3a^{(-1)^2}a^{(r)}a^{(0)^3}, & 3a^{(-1)}a^{(0)^2}, & 3a^{(-1)^2}a^{(0)}, & a^{(-1)^3} \end{array}$$

for the case of four dice

$$\begin{array}{ccccc} 4r, & 3r, & 3r-1, & 2r, & 2r-1, \\ {a^{(r)}}^4, & 4a^{(0)}{a^{(r)}}^3, & 4a^{(-1)}{a^{(r)}}^3, & 6{a^{(0)}}^2{a^{(r)}}^2, & 12a^{(-1)}a^{(0)}{a^{(r)}}^2, \\ 2r-2, & r, & r-1, & r-2, & \\ 6{a^{(-1)}}^2{a^{(r)}}^2, & 4{a^{(0)}}^3a^{(r)}, & 12a^{(-1)}{a^{(0)}}^2a^{(r)}, & 12{a^{(-1)}}^2 2a^{(0)}a^{(r)} & \&c. \end{array}$$

The analogy is now evident, the numerical coefficients are those of the same quantities, & for each number it is necessary to take the one, of which the sum of the exponents multiplied each by its sign (namely $a^{(r)}$ by r, $a^{(0)}$ by 0, $a^{(-1)}$ by -1), is equal to this number; thus the term which corresponds to $2r-1$ is $12a^{(-1)}a^{(0)}{a^{(r)}}^2$, because $2r-1$ is the sum of the exponents, namely $2.r+1.0+1.(-1)$, the coefficient is equal to the number of permutations of 4 things, of which 2 are the same, namely $\frac{1.2.3.4}{1.2}=12$. As for the same terms, they are in all for n dice $(a^{(r)}+a^{(0)}+a^{(-1)})^n$. One will have therefore for n dice,

$$\begin{array}{ccc} nr & (n-1)r & (n-1)r-1 \\ {a^{(r)}}^n, & n{a^{(r)}}^{n-1}a^{(0)}, & n{a^{(r)}}^{n-1}a^{(-1)} \\ (n-2)r & (n-2)r-1 & (n-2)r-2 \\ \frac{n(n-1)}{2}{a^{(r)}}^{n-2}{a^{(0)}}^2, & n(n-1){a^{(r)}}^{n-2}a^{(-1)}a^{(0)}, & \frac{n(n-1)}{2}{a^{(r)}}^{n-2}{a^{(-1)}}^2, \end{array}$$

$$\begin{array}{cc} (n-3)r & (n-3)r-1 \\ \frac{n(n-1)(n-2)}{1.2.3}{a^{(r)}}^{n-3}{a^{(0)}}^3, & \frac{n(n-1)(n-2)}{1.2}{a^{(r)}}^{n-3}{a^{(0)}}^2a^{(-1)}, \\ (n-3)r-2 & (n-3)r-3 \\ \frac{n(n-1)(n-2)}{1.2}{a^{(r)}}^{n-3}a^{(0)}{a^{(-1)}}^2, & \frac{n(n-1)(n-2)}{1.2.3}{a^{(r)}}^{n-3}{a^{(-1)}}^3, \\ (n-4)r & (n-4)r-1 \\ \frac{n(n-1)\cdots(n-3)}{1\ldots4}{a^{(r)}}^{n-4}{a^{(0)}}^4, & \frac{4n(n-1)\cdots(n-3)}{1\ldots4}{a^{(r)}}^{n-4}{a^{(0)}}^3a^{(-1)}, \\ (n-4)r-2 & (n-4)r-3 \\ \frac{4.3}{1.2}\frac{n(n-1)\cdots(n-3)}{1.2\ldots4}{a^{(r)}}^{n-4}{a^{(0)}}^2{a^{(-1)}}^2, & \frac{4.3.2}{1.2.3}\frac{n(n-1)\cdots(n-3)}{1.2\ldots4}{a^{(r)}}^{n-4}a^{(0)}{a^{(-1)}}^3, \\ (n-4)r-4 & \\ \frac{4.3.2.1}{1.2.3.4}\frac{n(n-1)\cdots(n-3)}{1.2\ldots4}{a^{(r)}}^{n-4}{a^{(-1)}}^4, & \\ (n-5)r & (n-5)r-1 \\ \frac{n(n-1)\cdots(n-4)}{1.2\ldots5}{a^{(r)}}^{n-5}{a^{(0)}}^5, & \frac{5}{1}\frac{n(n-1)\cdots(n-4)}{1.2\ldots5}{a^{(r)}}^{n-5}{a^{(0)}}^4a^{(-1)}, \end{array}$$

$$(n-5)r-2 \qquad\qquad (n-5)r-1$$
$$\tfrac{5.4}{1.2}\tfrac{n(n-1)\cdots(n-4)}{1.2\ldots5}a^{(r)^{n-5}}a^{(0)^3}a^{(-1)^2}, \qquad \tfrac{5.4.3}{1.2.3}\tfrac{n(n-1)\cdots(n-4)}{1\ldots5}a^{(r)^{n-5}}a^{(0)^2}a^{(-1)^3},$$

$$(n-5)r-4 \qquad\qquad (n-5)r-5$$
$$\tfrac{5.4.3.2}{1.2.3.4}\cdot\tfrac{n(n-1)\cdots(n-4)}{1.2\ldots5}a^{(r)^{n-5}}a^{(0)}a^{(-1)^4}, \qquad \tfrac{5.4.3.2.1}{1.2.3.4.5}\tfrac{n(n-1)\cdots(n-4)}{1\ldots5}a^{(r)^{n-5}}a^{(-1)^5},$$
$$\ldots \qquad\qquad \ldots$$

$$(n-\nu)r \qquad\qquad (n-\nu)r-1$$
$$\tfrac{n(n-1)\cdots(n-\nu+1)}{1.2\ldots\nu}a^{(r)^{n-\nu}}a^{(0)^\nu}, \qquad \tfrac{\nu n(n-1)\cdots(n-\nu+1)}{1.2\ldots\nu}a^{(r)^{n-\nu}}a^{(0)^{\nu-1}}a^{(-1)},$$

$$(n-\nu)r-2$$
$$\tfrac{\nu(\nu-1)}{1.2}\ \tfrac{n(n-1)\cdots(n-\nu+1)}{1.2\ldots\nu}a^{(r)^{n-\nu}}a^{(0)^{\nu-2}}a^{(-1)^2},$$

$$(n-\nu)r-3$$
$$\tfrac{\nu(\nu-1)(\nu-2)}{1.2.3}\cdot\tfrac{n(n-1)\cdots(n-\nu+1)}{1.2\ldots\nu}a^{(r)^{n-\nu}}a^{(0)^{\nu-3}}a^{(-1)^3}$$
$$\ldots \qquad\qquad \ldots$$

$$(n-\nu)r-\varrho$$
$$\tfrac{\nu(\nu-1)\cdots(\nu-\varrho+1)}{1.2\ldots\varrho}\cdot\tfrac{n(n-1)\cdots(n-\nu+1)}{1.2\ldots\nu}a^{(r)^{n-\nu}}a^{(0)^{\nu-\varrho}}a^{(-1)^\varrho}$$

This formula is general. Besides, it will have some terms which will be found the same, & this will depend on the values of r; thus, if $r = 2$, the terms $2r-2$ & r will be the same, & thus in sequence. In order to have a general expression of this reduction, we seek according to the preceding table the terms which correspond to the index $(n-\nu)r-\varrho$, & we will obtain the following,

$$\begin{aligned}
&\tfrac{\nu(\nu-1)\ldots(\nu-\varrho+1)}{1.2\ldots\varrho}\cdot\tfrac{n(n-1)\cdots(n-\nu+1)}{1.2\ldots\nu}\,a^{(r)^{n-\nu}}a^{(0)^{\nu-\varrho}}a^{(-1)^\varrho}\\
&+\tfrac{(\nu-1)(\nu-2)\ldots(\nu-\varrho-r)}{1.2\ldots(\varrho+r)}\cdot\tfrac{n(n-1)\cdots(n-\nu+2)}{1.2\ldots(\nu-1)}\,a^{(r)^{n-\nu+1}}a^{(0)^{\nu-\varrho-r-1}}a^{(-1)^{\varrho+r}}\\
&+\tfrac{(\nu-2)(\nu-3)\ldots(\nu-\varrho-2r-1)}{1.2\ldots(\varrho+2r)}\cdot\tfrac{n(n-1)\cdots(n-\nu+3)}{1.2\ldots(\nu-2)}\,a^{(r)^{n-\nu+2}}a^{(0)^{\nu-\varrho-2r-2}}a^{(-1)^{\varrho+2r}}\\
&+\tfrac{(\nu-3)(\nu-4)\ldots(\nu-\varrho-3r-2)}{1.2\ldots(\varrho+3r)}\cdot\tfrac{n(n-1)\cdots(n-\nu+4)}{1.2\ldots(\nu-3)}\,a^{(r)^{n-\nu+3}}a^{(0)^{\nu-\varrho-3r-3}}a^{(-1)^{\varrho+3r}}\\
&\ldots\\
&+\tfrac{(\nu-Q)(\nu-Q-1)\ldots(\nu-\varrho-Qr-(Q-1))}{1.2\ldots(\varrho+Qr)}\cdot\tfrac{n(n-1)\cdots(n-\nu+Q+1)}{1.2\ldots(\nu+Q)}\,a^{(r)^{n-\nu+Q}}a^{(0)^{\nu-\varrho-Qr-Q}}a^{(-1)^{\varrho+Qr}}.
\end{aligned}$$

§ 21. Let there be n dice with five faces, which are marked respectively with the characters 2, 1, 0, -1, -2, one demands the probability to bring forth any number with n dice.

For the case of 1 die, one has

$$\begin{array}{ccccc} 2, & 1, & 0, & -1, & -2\\ 1, & 1, & 1, & 1, & 1. \end{array}$$

If there are 2 dice, one has

$$\begin{array}{ccccccccc} 4, & 3, & 2, & 1, & 0, & -1, & -2, & -3, & -4\\ 1, & 2, & 3, & 4, & 5. & 4, & 3, & 2, & 1. \end{array}$$

For the case of 3 dice, one has

6,	5,	4,	3,	2,	1,	0,	−1,	−2,	−3,	−4,	−5,	−6
1,	3,	6,	10,	15.	18,	19,	18,	15.	10,	6,	3,	1.

For the case of 4 dice, one has

8,	7,	6,	5,	4,	3,	2,	1,	0,	−1,	−2,	−3,	−4,	−5,	−6,	−7,	−8
1,	4,	10,	20,	35.	52,	68,	80,	85.	80,	68,	52,	35.	20,	10,	4,	1.

For the case of 5 dice, one has

10,	9,	8,	7,	6,	5,	4,	3,	2,	1,	0,
1,	5,	15,	35,	70.	121,	185,	255,	320.	365,	381,

−1,	−2,	−3,	−4,	−5,	−6,	−7,	−8,	−9,	−10
365,	320.	255,	185,	121,	70.	35,	15,	5,	1.

For the case of 6 dice, one has

12,	11,	10,	9,	8,	7,	6,	5,	4,	3,	2,	1,	0
1,	6,	21	56,	126,	246,	426,	666,	951,	1246,	1506,	1686,	1751,

−1,	−2,	−3,	−4,	−5,	−6,	−7,	−8,	−9,	−10,	−11,	−12
1686,	1506,	1246,	951,	666,	426,	246,	126,	56,	21,	6,	1.

One will have in following the same analogy that in the problem of the ordinary dice, for n dice

$$\begin{array}{ccccc} 2n & 2n-1 & 2n-2 & 2n-3 & 2n-4 \\ 1, & n, & \frac{n(n+1)}{1.2}, & \frac{n(n+1)(n+2)}{1.2.3}, & \frac{n(n+1)\cdots(n+3)}{1.2...4}, \end{array}$$

$$\begin{array}{cc} 2n-5 & 2n-6 \\ \frac{n(n+1)\cdots(n+4)}{1.2...5} - n.1, & \frac{n(n+1)\cdots(n+5)}{1.2...6} - n.n, \\ 2n-7 & 2n-8 \\ \frac{n(n+1)\cdots(n+6)}{1.2...7} - n\frac{n(n+1)}{1.2}, & \frac{n(n+1)\cdots(n+7)}{1.2...8} - n\cdot\frac{n(n+1)(n+2)}{1.2.3}, \\ 2n-9 & 2n-10 \\ \frac{n(n+1)\cdots(n+8)}{1.2...9} - \frac{n.n(n+1)\cdots(n+3)}{1.2...4}, & \frac{n(n+1)\cdots(n+9)}{1.2...10} - \frac{n.n(n+1)\cdots(n+4)}{1.2...5} + \frac{n(n-3)}{1.2}.1, \end{array}$$

$$\cdots$$

$$2n-\nu$$

$$\frac{n(n+1)\cdots(n+\nu-1)}{1.2...\nu} - n\frac{n(n+1)\cdots(n+\nu-6)}{1.2...(\nu-5)} + \frac{n(n-1)}{1.2}\frac{n(n+1)\cdots(n+\nu-11)}{1.2...(\nu-10)}$$
$$-\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{n(n+1)\cdots(n+\nu-16)}{1.2...(\nu-15)} \text{ \&c. } \cdots$$

$$0$$

$$\frac{n(n+1)\cdots(2n-1)}{1.2...n} - n\cdot\frac{n(n+1)\cdots(2n-6)}{1.2...(n-5)} + \frac{n(n-1)}{1.2}\frac{n(n+1)\cdots(2n-11)}{1.2...(n-10)} \text{ \&c.}$$

One sees that the solution of this problem differs in nothing from that of the preceding, except that in the first, the die having only three faces, one took only three factors, instead that in this case here the die having five faces, one takes five factors; if the die had seven faces, one would take seven factors, & thus in sequence.

§ 22. Let therefore there be a die which has $2\alpha+1$ faces, marked respectively with characters α, $\alpha-1$, $\alpha-2$, $\alpha-3, \ldots 3$, 2, 1, 0, -1, -2, $-3, \ldots -(\alpha-3)$, $-(\alpha-2)$, $-(\alpha-1)$, $-\alpha$, one will have by proceeding as in the preceding problems, for the case of n dice

$$\begin{array}{cccccc} n\alpha & n\alpha-1 & n\alpha-2 & n\alpha-3 & n\alpha-4 & n\alpha-2\alpha \\ 1, & n, & \frac{n(n+1)}{1.2}, & \frac{n(n+1)(n+2)}{1.2.3}, & \frac{n(n+1)\cdots(n+3)}{1\ldots 4}, & \frac{n(n+1)\cdots(n+2\alpha-1)}{1.2\ldots 2\alpha}, \end{array}$$

$$\begin{array}{cc} n\alpha-(2\alpha+1) & n\alpha-(2\alpha+2) \\ \frac{n(n+1)\cdots(n+2\alpha)}{1\ldots(2\alpha+1)}-n.1, & \frac{n(n+1)\cdots(n+2\alpha+1)}{1\ldots(2\alpha+2)}-n.n, \end{array}$$

$$\begin{array}{cc} n\alpha-(2\alpha+3) & n\alpha-(4\alpha+1) \\ \frac{n(n+1)\cdots(n-2\alpha+2)}{1.2\ldots(2\alpha+3)}-n.\frac{n(n+1)}{1.2}, & \frac{n(n+1)\cdots(n+4\alpha)}{1\ldots(4\alpha+1)} \end{array}$$

$$\begin{array}{cc} n\alpha-(4\alpha+1) & n\alpha-(4\alpha+2) \\ -\frac{n(n+1)\cdots(n+2\alpha-1)}{1.2\ldots 2\alpha}, & \frac{n(n+1)\cdots(n+4\alpha+1)}{1.2\ldots(4\alpha+2)} \end{array}$$

$$\begin{array}{c} n\alpha-(4\alpha+2) \\ -n.\frac{n(n+1)\cdots(n+2\alpha)}{1.2\ldots(2\alpha+1)}+\frac{n(n+1)}{1.2}.1\ldots \end{array}$$

$$\begin{array}{c} n\alpha-(6\alpha+3) \\ \frac{n(n+1)\cdots(n+6\alpha+2)}{1.2\ldots(6\alpha+3)}-n.\frac{n(n+1)\cdots(n+4\alpha+1)}{1.2\ldots(4\alpha+1)} \\ +\frac{n(n-1)}{1.2}\frac{n(n+1)\cdots(n+2\alpha)}{1.2\ldots(2\alpha+1)}-\frac{n(n-1)(n-2)}{1.2.3}.1 \\ \cdots \end{array}$$

$$\begin{array}{c} n\alpha-(8\alpha+4) \\ \frac{n(n+1)\cdots(n+8\alpha+3)}{1.2\ldots(8\alpha+4)}-n.\frac{n(n+1)\cdots(n+6\alpha+2)}{1.2\ldots(6\alpha+3)} \\ +\frac{n(n-1)}{1.2}\frac{n(n+1)\cdots(n+4\alpha+1)}{1.2\ldots(4\alpha+2)}-\frac{n(n-1)(n-2)}{1.2.3}\frac{n(n+1)\cdots(n+2\alpha)}{1.2\ldots(2\alpha+1)} \\ +\frac{n(n+1)\cdots(n-3)}{1.2\ldots 4}\cdots \end{array}$$

$$\begin{array}{c} n\alpha-(2\mu\alpha+\mu) \\ \frac{n(n+1)\cdots(n+2\mu\alpha+\mu-1)}{1.2\ldots 2\mu\alpha+\mu}-n.\frac{n(n+1)\cdots(n+2(\mu-1)\alpha+\mu-2)}{1.2\ldots(2(\mu-1)\alpha+\mu-1)} \\ +\frac{n(n-1)}{1.2}\frac{n(n+1)\cdots(n+2(\mu-2)\alpha+\mu-3)}{1.2\ldots(2(\mu-2)\alpha+\mu-2)} \\ -\frac{n(n-1)(n-2)}{1.2.3}\frac{n(n+1)\cdots(n+2(\mu-3)\alpha+\mu-4)}{1.2\ldots 2(\mu-3)\alpha+\mu-3} \\ \cdots \\ \pm\frac{n(n-1)\cdots(n-\mu+2)}{1.2\ldots(\mu-1)}\cdot\frac{n(n+1)\cdots(n+2\alpha)}{1.2\ldots(2\alpha+1)}\mp\frac{n(n-1)\cdots(n-\mu+1)}{1.2\ldots\mu}.1 \end{array}$$

$$n\alpha-(2\mu\alpha+\mu+1)$$

$$\tfrac{n(n+1)\cdots(n+2\mu\alpha+\mu)}{1.2...(2\mu\alpha+\mu+1)}-n.\tfrac{n(n+1)\cdots(n+2(\mu-1)\alpha+\mu-1)}{1.2...(2(\mu-1)\alpha+\mu)}$$
$$+\tfrac{n(n-1)}{1.2}\,\tfrac{n(n+1)\cdots(n+2(\mu-2)\alpha+\mu-2)}{1.2...(2(\mu-2)\alpha+\mu-1)}$$
$$-\tfrac{n(n-1)(n-2)}{1.2.3}\,\tfrac{n(n+1)\cdots(n+2(\mu-3)\alpha+\mu-3)}{1.2...2(\mu-3)\alpha+\mu-2}$$
$$\dots$$
$$\pm\tfrac{n(n-1)\cdots(n-\mu+2)}{1.2...(\mu-1)}\cdot\tfrac{n(n+1)\cdots(n+2\alpha+1)}{1.2...(2\alpha+2)}\mp\tfrac{n(n-1)\cdots(n-\mu+1)}{1.2...\mu}.\tfrac{n}{1}$$

$$n\alpha-(2\mu\alpha+\mu+2)$$

$$\tfrac{n(n+1)\cdots(n+2\mu\alpha+\mu+1)}{1.2...(2\mu\alpha+\mu+2)}-n.\tfrac{n(n+1)\cdots(n+2(\mu-1)\alpha+\mu)}{1.2...2(\mu-1)\alpha+\mu+1}$$
$$+\tfrac{n(n-1)}{1.2}\,\tfrac{n(n+1)\cdots(n+2(\mu-2)\alpha+\mu-1)}{1.2...(2(\mu-2)\alpha+\mu)}$$
$$-\tfrac{n(n-1)(n-2)}{1.2.3}\,\tfrac{n(n+1)\cdots(n+2(\mu-3)\alpha+\mu-2)}{1.2...(2(\mu-3)\alpha+\mu-1)}\dots$$
$$\pm\tfrac{n(n-1)\cdots(n-\mu+2)}{1.2...(\mu-1)}\cdot\tfrac{n(n+1)\cdots(n+2\alpha+2)}{1.2...(2\alpha+3)}$$
$$\mp\tfrac{n(n-1)\cdots(n-\mu+1)}{1.2...\mu}.\tfrac{n(n+1)}{1.2}\dots$$
$$\dots$$

$$n\alpha-(2\mu\alpha+\mu+3)$$

$$\tfrac{n(n+1)\cdots(n+2\mu\alpha+\mu+2)}{1.2...(2\mu\alpha+\mu+3)}-n.\tfrac{n(n+1)\cdots(n+2(\mu-1)\alpha+\mu+1)}{1.2...(2(\mu-1)\alpha+\mu+2)}$$
$$+\tfrac{n(n-1)}{1.2}\,\tfrac{n(n+1)\cdots(n+2(\mu-2)\alpha+\mu)}{1.2...(2(\mu-2)\alpha+\mu+1)}$$
$$-\tfrac{n(n-1)(n-2)}{1.2.3}\,\tfrac{n(n+1)\cdots(n+2(\mu-3)\alpha+\mu-1)}{1.2...2(\mu-3)\alpha+\mu}$$
$$\dots$$
$$\pm\tfrac{n(n-1)\cdots(n-\mu+2)}{1.2...(\mu-1)}\cdot\tfrac{n(n+1)\cdots(n+2\alpha+3)}{1.2...(2\alpha+4)}$$
$$\mp\tfrac{n(n-1)\cdots(n-\mu+1)}{1.2...\mu}.\tfrac{n(n+1)(n+2)}{1.2.3}$$
$$\dots$$

$$n\alpha-(2\mu\alpha+\mu+\varrho)$$

$$\tfrac{n(n+1)\cdots(n+2\mu\alpha+\mu+\varrho-1)}{1.2...(2\mu\alpha+\mu+\varrho)}-n.\tfrac{n(n+1)\cdots(n+2(\mu-1)\alpha+\mu+\varrho-2)}{1.2...(2(\mu-1)\alpha+\mu-\varrho-1)}$$
$$+\tfrac{n(n-1)}{1.2}\,\tfrac{n(n+1)\cdots(n+2(\mu-2)\alpha+\mu+\varrho-3)}{1.2...(2(\mu-2)\alpha+\mu+\varrho-2)}$$
$$-\tfrac{n(n-1)(n-2)}{1.2.3}\,\tfrac{n(n+1)\cdots(n+2(\mu-3)\alpha+\mu+\varrho-4)}{1.2...(2(\mu-3)\alpha+\mu+\varrho-3)}$$
$$\dots$$
$$\pm\tfrac{n(n-1)\cdots(n-\mu+2)}{1.2...(\mu-1)}\cdot\tfrac{n(n+1)\cdots(n+2\alpha+\varrho)}{1.2...(2\alpha+\varrho+1)}$$
$$\mp\tfrac{n(n-1)\cdots(n-\mu+1)}{1.2...\mu}.\tfrac{n(n+1)\cdots(n+\varrho-1)}{1.2...\varrho}$$

$$0$$

$$\tfrac{n(n+1)\cdots(n\alpha+n-1)}{1.2...n}-n.\tfrac{n(n+1)\cdots(n\alpha+n-(2\alpha+2))}{1.2...(n-(4\alpha+1))}$$
$$+\tfrac{n(n-1)}{1.2}\,\tfrac{n(n+1)\cdots(n\alpha+n-(4\alpha+3))}{1.2...n-(2\alpha+2)}$$
$$-\tfrac{n(n-1)(n-2)}{1.2.3}\,\tfrac{n(n+1)\cdots(n\alpha+n-(6\alpha+4))}{1.2...(n-(6\alpha+3))}\ \text{\&c.}$$
$$\dots$$

§ 23. One can, by making $2\mu\alpha+\mu+\varrho=\nu$, set the general formula under this form:

$$n\alpha-\nu$$

$$\frac{n(n+1)\cdots(n+\nu-1)}{1.2\ldots\nu}-n.\frac{n(n+1)\cdots(n+\nu-2\alpha-2)}{1.2\ldots(\nu-2\alpha-1)}$$
$$+\frac{n(n-1)}{1.2}\cdot\frac{n(n+1)\cdots(n+\nu-4\alpha-3)}{1.2\ldots(\nu-4\alpha-2)}$$
$$-\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{n(n+1)\cdots(n+\nu-6\alpha-4)}{1.2\ldots(\nu-6\alpha-3)}\cdots\text{ \&c.}$$

Making now $n\alpha-\nu=\mu'$, $2\alpha+1=\varrho'$, one will have the following formula,

$$\mu'$$

$$\frac{n(n+1)\cdots(n\alpha+n-\mu'-1)}{1.2\ldots(n\alpha-\mu)}-n.\frac{n(n+1)\cdots(n\alpha+n-\mu'-\varrho'-1)}{1.2\ldots(n\alpha-\mu'-\varrho')}$$
$$+\frac{n(n-1)}{1.2}\cdot\frac{n(n+1)\cdots(n\alpha+n-\mu'-2\varrho'-1)}{1.2\ldots(n\alpha-\mu'-2\varrho')}$$
$$-\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{n(n+1)\cdots(n\alpha+n-\mu'-3\varrho'-1)}{1.2\ldots(n\alpha-\mu'-3\varrho')}\cdots\text{ \&c.}$$

Now

$$\frac{n(n+1)\cdots(n\alpha+n-\mu'-1)}{1.2\ldots(n\alpha-\mu')}=n.\frac{n(n+1)\cdots(n\alpha-\mu')(n\alpha-\mu'+1)\cdots(n\alpha-\mu'+n-1)}{1.2\ldots(n-1)n(n+1)\cdots(n\alpha-\mu')}$$
$$=\frac{(n\alpha-\mu'+1)(n\alpha-\mu'+2)\cdots(n\alpha-\mu'+n-1)}{1.2\ldots(n-1)}.$$

One will have likewise,

$$\frac{n(n+1)\cdots(n\alpha+n-\mu'-\varrho'-1)}{1.2\cdots(n\alpha-\mu'-\varrho')}=$$
$$\frac{n(n+1)\cdots(n\alpha-\mu'-\varrho')(n\alpha+n-\mu'-\varrho'+1)\cdots(n\alpha+n-\mu'-\varrho'-1)}{1.2\cdots(n-1)n(n+1)\cdots(n\alpha-\mu'-\varrho')}=$$
$$\frac{(n\alpha-\mu'-\varrho'+1)(n\alpha-\mu'-\varrho'+2)\cdots(n\alpha-\mu'-\varrho'+n-1)}{1.2\cdots(n-1)},$$

& thus in sequence.

§ 24. The formula will become therefore

$$\mu'$$

$$\tfrac{1}{1.2\ldots(n-1)}((n\alpha-\mu'+1)(n\alpha-\mu'+2)\cdots(n\alpha-\mu'+n-1)$$
$$-(n\alpha-\mu'-\varrho'+1)(n\alpha-\mu'-\varrho'+2)\cdots(n\alpha-\mu'-\varrho'+n-1)n$$
$$+(n\alpha-\mu'-2\varrho'+1)(n\alpha-\mu'-2\varrho'+2)\cdots(n\alpha-\mu'-2\varrho'+n-1)\tfrac{n(n-1)}{1.2}$$
$$-(n\alpha-\mu'-3\varrho'+1)(n\alpha-\mu'-3\varrho'+2)\cdots(n\alpha-\mu'-3\varrho'+n-1)\tfrac{n(n-1)(n-2)}{1.2.3}\text{ \&c.}$$

§ 25. This formula can also serve to resolve the following problem: let there be a die which has $\alpha+\beta+1$ faces marked respectively with the characters α, $\alpha-1$, $\alpha-2, \ldots 3, 2, 1, 0, -1, -2, -3, -4, \ldots -\beta$, to find the same values as above. It will suffice to put into the preceding formulas $\alpha+\beta$ instead of 2α. Thus the general formula for μ' will subsist by making $\varrho' = \alpha+\beta+1$. In the first case, one will be able to give to μ' all the values in whole numbers from $n\alpha$ to $-n\alpha$, & in the second from $n\alpha$ to $-n\beta$. This formula returns to that which Mr. de la Grange finds (*Mém. de Turin*, Volume V, p. 210),[4] & that he has deduced from the differential calculus. One sees that a simple use of the theory of combinations suffices in order to arrive to it.

§ 26. If μ' is negative & $> n\alpha$, the formula will become $=0$, making therefore $\mu' = -n\alpha-\beta$, ($\boldsymbol{\beta}$ being any positive whole number), one will have the following general formula

$$\begin{aligned}&(2n\alpha+\beta+1)(2n\alpha+\beta+2)\cdots(2n\alpha+\beta+n-1)\\&-n(2(n-1)\alpha+\beta)(2(n-1)\alpha+\beta+1)\cdots(2(n-1)\alpha+\beta+n-2)\\&+\tfrac{n(n-1)}{1.2}(2(n-2)\alpha+\beta-1)(2(n-2)\alpha+\beta)\cdots(2(n-2)\alpha+\beta+n-3)\\&-\tfrac{n(n-1)(n-2)}{1.2.3}(2(n-3)\alpha+\beta-2)(2(n-3)\alpha+\beta-1)\cdots(2(n-3)\alpha+\beta+n-4)\cdots\ \text{\&c.}=0\end{aligned}$$

where n, α, β represent all the positive whole numbers.

§ 27. Let there be a sequence of which the general term is,

$$(s+1)(s+2)\cdots(s+n-1),$$

I suppose that its sum is $=$

$$(s+1)(s+2)\cdots(s+n-1)(\alpha+\gamma s),$$

α & γ being some indeterminate constants. I make $s=0$, & I have

$$1.2.3\ldots(n-1) = 1.2.3\ldots(n-1)\alpha,$$

this which gives $\alpha=1$. I make $s=1$, & I have

$$1.2.3\ldots(n-1)+2.3.4\ldots n = (2.3\ldots n)(1+\gamma),$$

therefore

$$\gamma = \frac{1.2.3\ldots(n-1)}{1.2\ldots n}+1-1 = \frac{1}{n},$$

the sought sum will be therefore $=$

$$(s+1)(s+2)\cdots(s+n-1)\left(1+\frac{s}{n}\right) = \frac{(s+1)(s+2)\cdots(s+n)}{n}.$$

This method can serve in a multitude of cases, as I will show elsewhere. If one takes the sum of one such series from $s=\mu$ to $s=\nu$, one will have the formula,

$$\frac{(\nu+1)(\nu+2)\cdots(\nu+n)-\mu(\mu+1)\cdots(\mu+n-1)}{n}.$$

[4] *Translator's note*: See § 25 of Lagrange.

§28. One will have by applying this formula to the preceding formulas, the sum of the cases contained between two given numbers. For this I will make first for abridgment, $n\alpha - \mu' = s$, & the formula in question will become,

$$\frac{1}{1.2\ldots(n-1)}\big((s+1)(s+2)\cdots(s+n-1)$$
$$-n(s-\varrho+1)(s-\varrho+2)\cdots(s-\varrho+n-1)$$
$$+\frac{n(n-1)}{1.2}(s-2\varrho+1)(s-2\varrho+2)\cdots(s-2\varrho+n-1)$$
$$-\frac{n(n-1)(n-2)}{1.2.3}(s-3\varrho+1)(s-3\varrho+2)\cdots(s-3\varrho+n-1)\ \&\text{c.}\big)$$

§ 29. Making successively $x - \nu$ & $x = \mu$, one will have

$$\frac{1}{1.2\ldots n}\big((\nu+1)(\nu+2)\cdots(\nu+n) - \mu(\mu+1)\cdots(\mu+n-1)\big)$$
$$-n\big((\nu-\varrho+1)(\nu-\varrho+2)\cdots(\nu-\varrho+n)$$
$$-n(\mu-\varrho)(\mu-\varrho+1)\cdots(\mu-\varrho+n-1)\big)$$
$$+\frac{n(n-1)}{1.2}\big((\nu-2\varrho+1)(\nu-2\varrho+2)\cdots(\nu-2\varrho+n)\big)$$
$$-\big((\mu-2\varrho)(\mu-2\varrho+1)\cdots(\mu-2\varrho+n-1)\big)$$
$$-\frac{n(n-1)(n-2)}{1.2.3}(\nu-3\varrho+1)(\nu-3\varrho+2)\cdots(\nu-3\varrho+n)\,\big)$$
$$-\big((\mu-3\varrho)(\mu-3\varrho+1)\cdots(\mu-3\varrho+n-1)\big)\ \&\text{c.}$$

§ 30. I suppose, in order to give an example, the case treated above § 21, where $\alpha = \beta = 2$, $n = 6$, & let one demand the sum of the cases contained between 8 & 2, one has here $n\alpha = 12$, this which gives

$$12 - 2 = 10 = \nu,\ 12 - 8 = 4 = \mu,$$

& the formula will become,

$$\frac{1}{1.2\ldots6}\left\{\begin{matrix}(11.12.13.14.15.16 - 4.5.6.7.8.9)\\ -6.(6.7.8.9.10.11) + \dfrac{6.5}{1.2}(1.2.3.4.5.6)\end{matrix}\right\} = 5167$$

as one finds it by summing the numbers found § 21 from 8 to 2., because

$$1506 + 1246 + 951 + 666 + 426 + 246 + 126 = 5167.$$

This formula reverts to that which Mr. de la Grange gives in the Memoir cited p. 212.

§ 31. We have treated § 18 of the case of n dice with four faces, of which two were marked zero, the one 1, the other -1; suppose now a die with nine faces, of which three are marked 0, two 1, two -1, one 2, one -2, one demands the probability to bring forth any number with n dice.

If there is only 1 die, one has

2,	1,	0,	−1,	−2,
1,	2,	3,	2,	1.

If there are 2 dice, one has by proceeding as above,

4,	3,	2,	1,	0,	−1,	−2,	−3,	−4,
1,	4,	10,	16,	19,	16,	10,	4,	1.

If there are 3 dice, one has

6,	5,	4,	3,	2,	1,	0,	−1,	−2,	−3,	−4,	−5,	−6,
1,	6,	21,	50,	90	126,	141,	126,	90,	50,	21,	6,	1.

If there are 4 dice, one has

8,	7,	6,	5,	4,	3,	2,	1,	0,
1,	8,	36,	112,	266,	504,	784,	1016,	1107,

−1,	−2,	−3,	−4,	−5,	−6,	−7,	−8,
1016,	784,	504,	266,	112,	36,	8,	1.

If there are 5 dice, one has

10,	9,	8,	7,	6,	5,	4,	3,	2,	1,	0,
1,	10,	55,	210,	615,	1452,	2850,	4740,	6765,	8350,	8953,

−1,	−2,	−3,	−4,	−5,	−6,	−7,	−8,	−9,	−10,
8350,	6765,	4740,	2850,	1452,	615,	210,	55,	10,	1.

If there are 6 dice, one has

12,	11,	10,	9,	8,	7,	6,	5,	4,
1,	12,	78,	352,	1221,	3432,	8074,	16236,	28314,

3,	2,	1,	0,	−1,	−2,	−3,	−4,
43252,	58278,	69576,	73789,	69576,	58278,	43252,	28314,

−5,	−6,	−7,	−8,	−9,	−10,	−11,	−12,
16236,	8074,	3432,	1221,	352,	78,	12,	1.

One will have, by following the same analogy as in the problem of the ordinary dice, the following table for n dice,

$$\begin{array}{cccc} 2n & 2n-1 & 2n-2 & 2n-3 \\ 1, & \frac{2n}{1}, & \frac{2n(2n+1)}{1.2}, & \frac{2n(2n+1)(2n+2)}{1.2.3} - 2n.1, \end{array}$$

$$\begin{array}{cc} 2n-4 & 2n-5 \\ \frac{2n(2n+1)\cdots(2n+3)}{1.2...4} - 2n.\frac{2n}{1}, & \frac{2n(2n+1)\cdots(2n+4)}{1.2...5} - 2n.\frac{2n(2n+1)}{1.2}, \end{array}$$

$$\begin{array}{c} 2n-6 \\ \frac{2n(2n+1)\cdots(2n+5)}{1.2...6} - 2n.\frac{2n(2n+1)\cdots(2n+2)}{1.2.3} + \frac{2n(2n-1)}{1.2}.1 \\ \dots \end{array}$$

$$2n - \nu$$

$$\frac{2n(2n+1)\cdots(2n+\nu-1)}{1.2\ldots\nu} - 2n.\frac{2n(2n+1)\cdots(2n+\nu-4)}{1.2\ldots(\nu-3)} + \frac{2n(2n-1)}{1.2}\cdot\frac{2n(2n+1)\cdots(2n+\nu-7)}{1.2\ldots(\nu-6)} - \frac{2n(2n-1)(2n-2)}{1.2.3}\cdot\frac{2n(2n+1)\cdots(2n+\nu-10)}{1.2\ldots(\nu-9)} \text{ \&c.}$$

$$0$$

$$\frac{2n(2n+1)\cdots(4n-1)}{1.2\ldots 2n} - 2n.\frac{2n(2n+1)\cdots(4n-4)}{1.2\ldots(2n-3)} + \frac{2n(2n-1)}{1.2}\cdot\frac{2n(2n+1)\cdots(4n-7)}{1.2\ldots(2n-6)} - \frac{2n(2n-1)(2n-2)}{1.2.3}\cdot\frac{2n(2n+1)\cdots(4n-10)}{1.2\ldots(2n-9)} \text{ \&c.}$$

§ 32. One sees that the solution of this problem coincides with that of the preceding except with two differences, 1°. instead of taking n number of dice, it is necessary to take $2n$; 2°. instead of taking the figurate numbers in number equal to the number of faces, one takes it equal to the number of diverse marks on the faces, which is here 3. By means of these two observations, the solution of the general problem of the preceding case will be able to be applied to this one, so that the formula of § 24 will give the solution of the following problem.

§ 33. Let there be n dice which have each 1 face marked ω, 2 faces marked $\omega - 1$, 3 faces marked $\omega - 2$, $\ldots \omega - 1$ faces marked 2, ω faces marked 1, $\omega + 1$ faces marked 0, ω faces marked -1, $\omega - 1$ faces marked -2, $\ldots$ 3 faces marked $-\omega + 2$, 2 faces marked $-\omega + 1$, 1 face marked $-\omega$, one demands the number of cases which will bring forth the number μ'. Here is the formula in which one has $\varrho' = \omega + 1$.

$$\begin{aligned}
&\frac{1}{1.2\ldots(2n-1)}((n\alpha-\mu'+1)(n\alpha-\mu'+2)\cdots(n\alpha-\mu'+2n-1)) \\
&\quad - 2n((n\alpha-\mu'-\varrho'+1)(n\alpha-\mu'-\varrho'+2)\cdots(n\alpha-\mu'-\varrho'+2n-1)) \\
&\quad + \frac{2n(2n-1)}{1.2}((n\alpha-\mu'-2\varrho'+1)(n\alpha-\mu'-2\varrho'+2)\cdots(n\alpha-\mu'-2\varrho'+2n-1)) \\
&\quad - \frac{2n(2n-1)(2n-2)}{1.2.3}((n\alpha-\mu'-3\varrho'+1)(n\alpha-\mu'-3\varrho'+2)\cdots(n\alpha-\mu'-3\varrho'+2n-1)) \text{ \&c.}
\end{aligned}$$

§ 34. In order to have now the sum of the cases which they give from μ to ν, one will have by the formula of § 29, by making the changes aforesaid,

$$\begin{aligned}&\frac{1}{1.2\dots 2n}((\nu+1)(\nu+2)\cdots(\nu+2n)-\mu(\mu+1)\cdots(\mu+2n-1))\\&\quad-\frac{2n}{1}((\nu-\varrho+1)(\nu-\varrho+2)\cdots(\nu-\varrho+2n)\\&\qquad-(\mu-\varrho)(\mu-\varrho+1)\cdots(\mu-\varrho+2n-1))\\&\quad+\frac{2n(2n-1)}{1.2}((\mu-2\varrho+1)(\mu-2\varrho+2)\cdots(\mu-2\varrho+2n)\\&\qquad-(\mu-2\varrho)(\mu-2\varrho+1)\cdots(\mu-2\varrho+2n-1))\\&\quad-\frac{2n(2n-1)(2n-2)}{1.2.3}((\mu-3\varrho+1)(\mu-3\varrho+2)\cdots(\mu-3\varrho+2n)\\&\qquad-(\mu-3\varrho)(\mu-3\varrho+1)\cdots(\mu-3\varrho+2n-1))\ \text{\&c.}\end{aligned}$$

One has in this formula $\nu = n\omega - \mu',\ \varrho = \omega + 1$.

§ 35. I suppose, for example, the case treated § 31, where $\omega = 2$, $n = 6$, & let one demand the sum of the cases contained between 8 & 2, one has here $n\alpha = 12$, $\varrho = 3$, this which gives $12 - 2 = 10$, $12 - 8 = 4$, one will make therefore $\nu = 10$, $\mu = 4$, & one will have

$$\frac{1}{1.2\dots 12}\left\{\begin{array}{l}11.12\dots 22 - 4.5\dots 15 + \dfrac{12.11}{1.2}(5.6\dots 16)\\ -12(8.9\dots 19 - 1\dots 12 - \dfrac{12.11.10}{1.2.3}(2.3\dots 13)\end{array}\right\} = 158807$$

as one finds it by summing the numbers found § 31 from 8 to 2, because

$$58278 + 43252 + 28314 + 16236 + 8074 + 3432 + 1221 = 158807.$$

This formula reverts to that which Mr. de la Grange finds page 214.[5]

§ 36. The solution of the preceding problems can serve, as Mr. de la Grange said it, to find the mean among the observations, because after having found the case as one wishes, by the method exposed above, one will find the probability to bring forth these cases by dividing the numbers found by the total number of cases. Thus in the general problem of § 29 the denominator should be ϱ^n, & in the one of § 34, it must be ϱ^{2n}, & the two problems will be able to represent the probability that the error is $\frac{\mu'}{n}$, or contained between the limits $\frac{\nu}{n}$ & $\frac{\mu}{n}$.

§ 37. If in the problem of § 29 α & β become infinite, in a fashion that they have between them a finite ratio, & if ν & μ become also infinite under the same condition, one will have $\frac{\beta}{\alpha} = l$, $\frac{\nu}{\alpha} = s$, $\frac{\mu}{\alpha} = r$, $\varrho = \alpha + \beta = (1 + l)\alpha = f\alpha$; the general formula of the § cited will become therefore, by introducing the denominator ϱ^n in it

[5] *Translator's note*: See § 28 of Lagrange.

& dividing above & below by α^n,

$$\frac{1}{1.2\ldots nf^n}\big((s+n)^n - n(s+n-f)^n + \frac{n(n-1)}{1.2}(s+n-2f)^n$$
$$- \frac{n(n-1)(n-2)}{1.2.3}(s+n-3f)^n \text{ \&c.}\big)$$
$$- \big((r+n)^n - n(r+n-f)^n + \frac{n(n-1)}{1.2}(r+n-2f)^n$$
$$- \frac{n(n-1)(n-2)}{1.2.3}(r+n-3f)^n \text{ \&c.}$$

If in the problem of § 34, one makes $\varrho = 2\omega$, by conserving the same assumptions, one will have in the formula of the § cited, by dividing above & below by ω^{2n},

$$\frac{1}{1.2\ldots n2^{2n}}\big((s+2n)^{2n} - 2n(s+2n-2)^{2n} + \frac{2n(2n-1)}{1.2}(s+2n-4)^{2n}$$
$$- \frac{2n.(2n-2)}{1.2.3}(s+2n-6)^{2n} \text{ \&c.}\big)$$
$$- \big((r+2n)^{2n} - 2n(r+2n-2)^{2n} + \frac{2n(2n-1)}{1.2}(r+2n-4)^{2n}$$
$$- \frac{2n\cdots(2n-2)}{1.2.3}(r+2n-6)^{2n} \text{ \&c.}$$

The first formula agrees with that which Mr. de la Grange, gives p. 213,[6] & the second with that which Mr. de la Grange gives, p. 215.[7]

§ 38. I will not push further these researches. I suffices to me to have shown in detail, how the principles of the theory of combinations gives the same results as the calculus of infinity; this last calculus furnishes often some shorter & simpler methods, which do not merit, as one sees, less confidence, since one can confirm them by the simple enumeration of cases.

[6] *Translator's note*: See § 27 of Lagrange.
[7] *Translator's note*: See § 29 of Lagrange.