RECHERCHES sur une question relative au calcul des probabilités*

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One finds at the end of the second volume of the analytic Works of Mr. Euler, printed in Petersburg in 1795, a Memoir entitled: *Solutio quarundam quaestionum difficiorum in calculo probabilitatum.*[1] These questions revolve on the lottery of Genoa, which has been since imitated at Manheim, at Paris, at Berlin & in other cities of Europe, in which out of 90 tickets marked with numbers[2] $1, 2, 3, \ldots 90$, one draws 5 of them at some fixed periods. Mr. Euler supposing any number n tickets, of which one draws at each time a number p, & which one puts back again into the urn which contained them, seeks the probability that in a given number of coups one will have brought forth all n numbers, the probability that one will have brought forth $(n-1)$ numbers at least, the probability that one will have brought forth $(n-2)$ numbers at least &c. *Has igitur quaestiones*, says Mr. Euler, *utpote difficillimas hic ex principiis calculi probabilitatum iam pridem usus receptis resolvere constitui. Qui in huiusmodi investigationibus elaborarunt, facile perspicient resolutionem harum quaestionum calculos maxime intricatos postulare, quos autem mihi beneficio certorum caracterum superare licuit.*[3] The analysis of which Mr. Euler makes use in this Memoir is very ingenious & worthy of this great geometer, but as it is a little indirect & as it would not be easy to apply it to the general problem of which this one is only a particular case, I have undertaken to treat the thing directly according to the theory of combinations, & to give to the question all the extent of which it is susceptible.

§ 1. Let a regular prism of which the number of faces are P, among which there

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[1] *Opuscula Analytica* Vol. II, 1785, p. 331-346.

[2] *Translator's note*: The word used is "numéro." This signifies a number used in the sense of a label.

[3] *Translator's note*: We have here in the first sentence an extract from the first paragraph of Euler's paper (E600). "Therefore here I have decided to resolve these questions, inasmuch as they are the most difficult ones, from the principles of the calculus of probability, which have long since been accepted by practice." It is followed by most, but not all of the first sentence of the second paragraph. "Everyone who has carefully labored in investigations of this sort, easily will observe the solution of these questions to require the most intricate calculations, but which by benefit of certain characters, it is permitted by me to have the upper hand.

are a' marked 1, a'' marked 2, a''' marked 3, $\ldots a^{(n)}$ marked n, one demands the probability that at the end of a number x coups one will have brought forth all the kinds of faces. The probability that 1 will not come at all is $= \left(\frac{P-a'}{P}\right)^x$, therefore the probability that 1 will come with or without 2 is $1 - \left(\frac{P-a'}{P}\right)^x$. The probability that 2 will not come at all is $= \left(\frac{P-a''}{P}\right)^x$, the probability that neither 1 nor 2 will come is $= \left(\frac{P-a'-a''}{P}\right)^x$. The first of these probabilities gives the number of combinations x by x where all the numbers are found except 2, the second gives the number of combinations where neither 1 nor 2 is found. If one subtracts this number of combinations from the preceding, one will have the number of combinations where 1 is found without 2; thus $\left(\frac{P-a''}{P}\right)^x - \left(\frac{P-a'-a''}{P}\right)^x$ expresses the probability that 1 will come without 2. If one subtracts this probability from that which 1 will come with or without 2, there will remain for the probability that one will bring forth 1 & 2,

$$1 - \left(\frac{P-a'}{P}\right)^x - \left(\frac{P-a''}{P}\right)^x + \left(\frac{P-a'-a''}{P}\right)^x.$$

This formula expresses thus the probability that 1 & 2 will come with or without 3, now $\left(\frac{P-a'''}{P}\right)^x$ expresses the probability that 3 will not come at all, that is to say it gives the combinations where all the numbers will come except 3; $\left(\frac{P-a'-a'''}{P}\right)^x$ expresses the probability that neither 1 nor 3 will come, that is to say it gives the combinations where all the numbers will come except 1 & 3; subtracting this number from the preceding one has the combinations where 1 is found & not 3, therefore $\left(\frac{P-a'''}{P}\right)^x - \left(\frac{P-a'-a'''}{P}\right)^x$ expresses the probability that 1 will come without 3, next $\left(\frac{P-a''-a'''}{P}\right)^x$ expresses the probability that neither 2 nor 3 will come, that is to say it gives the combinations where neither 2 nor 3 is found; $\left(\frac{P-a'-a''-a'''}{P}\right)^x$ expresses the probability that neither 1, nor 2, nor 3 will come, that is to say it gives the combinations where neither 1 nor 2 nor 3 is found. Subtracting this number from the preceding one has the combinations where 1 is found without 2 or 3, therefore $\left(\frac{P-a''-a'''}{P}\right)^x - \left(\frac{P-a'-a''-a'''}{P}\right)^x$ expresses the probability that 1 will come without 2 or 3; now subtracting from the number of combinations where 1 is found & not 3, the one where 1 is found without 2 or 3, there will remain the number of combinations where 1 & 2 is found without 3, therefore

$$\left(\frac{P-a'''}{P}\right)^x - \left(\frac{P-a'-a'''}{P}\right)^x - \left(\frac{P-a''-a'''}{P}\right)^x + \left(\frac{P-a'-a''-a'''}{P}\right)^x$$

expresses the probability that 1 & 2 will come without 3; if one subtracts this probability from the one that 1 & 2 will come with or without 3, there will remain the probability that 1 will come with 2 & 3 $=$

$$1 - \left(\frac{P-a'}{P}\right)^x - \left(\frac{P-a''}{P}\right)^x - \left(\frac{P-a'''}{P}\right)^x + \left(\frac{P-a'-a''}{P}\right)^x + \left(\frac{P-a'-a'''}{P}\right)^x + \left(\frac{P-a''-a'''}{P}\right)^x - \left(\frac{P-a'-a''-a'''}{P}\right)^x.$$

This formula expresses the probability that 1, 2 & 3 will come with or without 4; now $\left(\frac{P-a^{\text{iv}}}{P}\right)^x$ expresses the probability that 4 will not come at all, that is to say it gives the combinations where all the numbers can be, except 4; $\left(\frac{P-a'-a^{\text{iv}}}{P}\right)^x$ expresses the probability that neither 1 nor 4 will come, that is to say it gives the combinations where all the letters can be, except 1 & 4; subtracting this number from the preceding, one has the combinations where 1 will be found, but not 4, therefore $\left(\frac{P-a^{\text{iv}}}{P}\right)^x - \left(\frac{P-a'-a^{\text{iv}}}{P}\right)^x$ expresses the probability that 1 will come without 4, next $\left(\frac{P-a''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that neither 2 nor 4 will come, that is to say it gives the combinations where it will be wanting 2 & 4; $\left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that neither 1 nor 2 nor 4 will come, that is to say it gives the combinations where 1, 2 & 4 will be wanting. Subtracting this number from the preceding, one will have the combinations where 1 will be found without 2 or 4, therefore $\left(\frac{P-a''-a^{\text{iv}}}{P}\right)^x - \left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that 1 will come without 2 or 4; subtracting these combinations from those where is found 1 without 4 with or without 2, one will have those where 1 is found with 2 without 4, therefore

$$\left(\frac{P-a^{\text{iv}}}{P}\right)^x - \left(\frac{P-a'-a^{\text{iv}}}{P}\right)^x - \left(\frac{P-a''-a^{\text{iv}}}{P}\right)^x + \left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x$$

expresses the probability that 1 will come with 2 without 4; moreover $\left(\frac{P-a'''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that neither 3 nor 4 will come, that is to say it gives the combinations where 3 & 4 will be wanting; $\left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that neither 1 nor 3 nor 4 will come; subtracting this number from the preceding, one has the combinations where 1 will be found without 3 or 4, therefore $\left(\frac{P-a'''-a^{\text{iv}}}{P}\right)^x - \left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that 1 will come without 3 or 4; finally $\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that neither 2 nor 3 nor 4 will come, that is to say it gives the combinations where 2, 3 & 4 will be wanting; $\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that neither 1 nor 2 nor 3 nor 4 will come. Subtracting this number from the preceding, one will have the combinations where 1 comes without 2, 3 or 4; therefore $\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x - \left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that 1 will come without 2, 3 or 4. Subtracting this probability from that which 1 will come without 3 or 4, one will have the probability that 1 will come with 2 without 3 or

$4 =$

$$\left(\frac{P-a'''-a^{\text{iv}}}{P}\right)^x - \left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x.$$
$$-\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x + \left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x$$

Subtracting this probability from that which 1 will come with 2 without 4, one will have the probability that 1 will come with 2 & 3 without $4 =$

$$\left(\frac{P-a^{\text{iv}}}{P}\right)^x - \left(\frac{P-a'-a^{\text{iv}}}{P}\right)^x - \left(\frac{P-a''-a^{\text{iv}}}{P}\right)^x - \left(\frac{P-a'''-a^{\text{iv}}}{P}\right)^x$$
$$+\left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x + \left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x + \left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x$$
$$-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x$$

Subtracting this probability from that which 1 will come with 2 & 3 with or without 4, one will have the probability that 1, 2, 3 & 4 will come together =

$$\begin{array}{ll}
1-\left(\frac{P-a'}{P}\right)^x & +\left(\frac{P-a'-a''}{P}\right)^x - \left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x + \left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x \\
-\left(\frac{P-a''}{P}\right)^x & +\left(\frac{P-a'-a'''}{P}\right)^x - \left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x \\
-\left(\frac{P-a'''}{P}\right)^x & +\left(\frac{P-a'-a^{\text{iv}}}{P}\right)^x - \left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x \\
-\left(\frac{P-a^{\text{iv}}}{P}\right)^x & +\left(\frac{P-a''-a'''}{P}\right)^x - \left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x \\
& +\left(\frac{P-a''-a^{\text{iv}}}{P}\right)^x \\
& +\left(\frac{P-a'''-a^{\text{iv}}}{P}\right)^x
\end{array}$$

Mr. de Moivre is arrived to the same results in his Doctrine of chances.

The analogy is now evident, & one sees that if one calls A' the sum of the terms which one obtains by subtracting from P the combinations of a', a'', &c. $a^{(n)}$ one by one, A'' the sum of the terms which one obtains by subtracting from P the sum of the combinations of the same quantities taken two by two, A''' the sum of the terms which one obtains by subtracting from P the sum of the combinations of the same quantities taken three by three, & in general $A^{(n)}$ the sum of the combinations of the same quantities taken n by n, one will have for the probability that 1, 2, 3, $\ldots n$ will have exited at the end of x coups

$$1 - A' + A'' - A''' + A^{\text{iv}} \text{ \&c. } \pm A^{(n)}.$$

§ 2. If one makes now $a' = a'' = a''' \cdots = a^{(n)} = 1$, the preceding formulas will give us the solution of the problem that Mr. Euler treats in first place & that Mr. de la Place has treated in T. 6. of the *Mémoires des Savans étranges* presented to the Academy of Sciences of Paris.[4] One has in this case here, by virtue of the first principles of the doctrine of combinations, the probability that all the numbers of the lottery will have exited at the end of x coups =

$$1 - n\left(\frac{P-1}{P}\right)^x + \frac{n(n-1)}{1.2}\left(\frac{P-2}{P}\right)^x - \frac{n(n-1)(n-2)}{1.2.3}\left(\frac{P-3}{P}\right)^x \text{ \&c.} \pm \left(\frac{P-n}{P}\right)^x.$$

It is necessary now to substitute into this formula for $\left(\frac{P-1}{P}\right)^x$, $\left(\frac{P-2}{P}\right)^x$ &c. their values. Now $\left(\frac{P-1}{P}\right)^x$ expresses the probability that in x coups a number for example o will not exit; as one draws p tickets at each coup, the number of all possible cases results from the combination of n things taken p by p, of which the result must be raised to x. This number is therefore by the doctrine of combinations $= \left(\frac{n(n-1)(n-2)\cdots(n-p+1)}{1.2.3\ldots p}\right)^x$ & the number of cases where 1 is not found, results from the combination of $n-1$ things taken p by p. This number will be by the same rule $= \left(\frac{(n-1)(n-2)\cdots(n-p)}{1.2.3\ldots p}\right)^x$. Therefore by the general rule of probabilities, the probability that in x coups the number 1 will not exit will be $= \left(\frac{(n-1)(n-2)\cdots(n-p)}{n(n-1)(n-2)\cdots(n-p+1)}\right)^x = \left(\frac{n-p}{n}\right)^x$. Likewise $\left(\frac{P-2}{P}\right)^x$ expresses the probability that in x coups, two numbers, for example 1 & 2, will not exit; now the number of possible cases being always the same, the number of cases where 1 & 2 will not be found, results from the combination of $n-2$ things taken p by p, this number is therefore $= \left(\frac{(n-2)(n-3)\cdots(n-p-1)}{1.2.3\ldots p}\right)^x$, therefore the sought probability will be =

$$\left(\frac{(n-2)(n-3)\cdots(n-p-1)}{n(n-1)(n-2)\cdots(n-p+1)}\right)^x = \left(\frac{(n-p)(n-p-1)}{n(n-1)}\right)^x.$$

Next $\left(\frac{P-3}{P}\right)^x$ expresses the probability that the three numbers 1, 2, 3 will not exit; now the number of cases where 1, 2, & 3 are not found, results from the combination of $n-3$ things taken p by p namely $\left(\frac{(n-3)(n-4)\cdots(n-p-2)}{1.2.3\ldots p}\right)^x$, therefore the sought probability will be =

$$\left(\frac{(n-3)(n-4)\cdots(n-p-2)}{n(n-1)(n-2)\cdots(n-p+1)}\right)^x = \left(\frac{(n-p)(n-p-1)(n-p-2)}{n(n-1)(n-2)}\right)^x.$$

One will find likewise

$$\left(\frac{P-4}{P}\right)^x = \left(\frac{(n-4)(n-5)\cdots(n-p-3)}{n(n-1)(n-2)\cdots(n-p+1)}\right)^x = \left(\frac{(n-p)\cdots(n-p-3)}{n(n-1)(n-2)(n-3)}\right)^x.$$

[4] "Mémoire sur les suites récurro-récurrentes et sur leurs usages dans la théorie des hasards," *Mém. Acad. R. Sci. Paris* (Savants étrangers) 6, 1774, pages 353-371.

The law is now rather clear. Substituting therefore these values, our formula will become

$$1-n\left(\frac{(n-1)(n-2)\cdots(n-p)}{n(n-1)(n-2)\cdots(n-p+1)}\right)^x+\frac{n(n-1)}{2}\left(\frac{(n-2)(n-3)\cdots(n-p-1)}{n(n-1)(n-2)\cdots(n-p+1)}\right)^x$$
$$-\frac{n(n-1)(n-2)}{2.3}\left(\frac{(n-3)(n-4)\cdots(n-p-2)}{n(n-1)(n-2)\cdots(n-p+1)}\right)^x \text{ \&c.}$$

This is the formula that Mr. de la Place finds. According to that which we just saw, one can set it under the following form which is much simpler.

$$1-n\left(\frac{n-p}{n}\right)^x+\frac{n(n-1)}{2}\left(\frac{(n-p)(n-p-1)}{n(n-1)}\right)^x$$
$$-\frac{n(n-1)(n-2)}{1.2.3}\left(\frac{(n-p)(n-p-1)(n-p-2)}{n(n-1)(n-2)}\right)^x$$
$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\left(\frac{(n-p)(n-p-1)(n-p-2)(n-p-3)}{n(n-1)(n-2)(n-3)}\right)^x \text{ \&c.}$$

For that which regards the arithmetic calculation of these formulas, one will consider that

$$\frac{A'}{1}=n\left(\frac{n-p}{n}\right)^x,$$
$$\frac{A''}{A'}=\frac{(n-1)}{2}\left(\frac{(n-p-1)}{n-1}\right)^x,$$
$$\frac{A'''}{A''}=\frac{(n-2)}{3}\left(\frac{n-p-2}{n-2}\right)^x,$$
$$\frac{A^{\text{iv}}}{A'''}=\frac{(n-3)}{4}\left(\frac{n-p-3}{n-3}\right)^x\cdots$$

& in general

$$\frac{A^{(\lambda)}}{A^{(\lambda-1)}}=\frac{(n-\lambda)}{\lambda+1}\left(\frac{n-p-\lambda}{n-\lambda}\right)^x.$$

Therefore

$$\begin{aligned}
\ln\ A' &= \ln n + x\ln\left(\tfrac{n-p}{n}\right),\\
\ln\ A'' &= \ln A' + \ln\left(\tfrac{n-1}{2}\right) + x\ln\left(\tfrac{n-p-1}{n-1}\right),\\
\ln\ A''' &= \ln\ A'' + \ln\left(\tfrac{n-2}{3}\right) + x\ln\left(\tfrac{n-p-2}{n-2}\right) \text{ \&c.}\\
\ln\ A^{(\lambda)} &= \ln\ A^{(\lambda-1)} + \ln\ \left(\tfrac{n-\lambda}{\lambda+1}\right) + x\ln\left(\tfrac{n-p-\lambda}{n-\lambda}\right).
\end{aligned}$$

Let as in the lottery of Berlin $n=90$, $p=5$, & make $x=100$, we will have $A'=0.29640$, $A''=0.04066$, $A'''=0.00344$, $A^{\text{iv}}=0.00020$, therefore

$$1-A'+A''-A'''+A^{\text{iv}} \text{ \&c. } =0.74102.$$

Make $x=200$, we will have $A'=0.000976$, $1-A'=0.999024$. There are therefore nearly odds of three against one that all the numbers will have exited at the end of 100 drawings, & nearly odds of one thousand against one that all the numbers will have exited at the end of 200 drawings, as Mr. Euler finds it. If one would wish to know what is the number of coups where one can wager even that all the numbers will have exited, one would find this number between 85 and 86, so that there is advantage to wager that all the numbers will have exited at the end of 86 drawings, & disadvantage to wager that all the numbers will have exited at the end of 85 drawings. The exact number is nearer 85 than 86.

§ 3. Supposing the same things as in § 1. one demands the probability that at the end of x coups one will have brought forth at least $n-1$ kinds of faces. Reasoning always in the same manner, I say: $\left(\frac{P-a'-a''}{P}\right)^x$ is the probability that in x drawings neither 1 nor 2 will come, therefore $1-\left(\frac{P-a'-a''}{P}\right)^x$ expresses the probability that of the two numbers there will exit at least one, $\left(\frac{P-a'-a'''}{P}\right)^x-\left(\frac{P-a'-a''-a'''}{P}\right)^x$ expresses the probability that 2 will come without 1 or 3, $\left(\frac{P-a''-a'''}{P}\right)^x-\left(\frac{P-a'-a''-a'''}{P}\right)^x$ expresses the probability that 1 will come without 2 or 3. Subtracting these two probabilities from the first, one will see that

$$1-\left(\frac{P-a'-a''}{P}\right)^x+2\left(\frac{P-a'-a''-a'''}{P}\right)^x -\left(\frac{P-a'-a'''}{P}\right)^x -\left(\frac{P-a''-a'''}{P}\right)^x$$

expresses the probability that of the three numbers there will exit at least two of them, since 1 & 2 must necessarily exit together or separately, & since 1 must come with 2 & 3 together or separately, & 2 with 1 & 3 together or separately.

One finds likewise that $\left(\frac{P-a''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that 3 will come without 2 or 4; that $\left(\frac{P-a'-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that 2 will come without 1 or 4; that $\left(\frac{P-a'''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that 1 will come without 3 or 4. One finds again that $\left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that 3 will come without 1 or 2 or 4; that $\left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that 2 will come without 1, 3 or 4; that $\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that 1 will come without 2, 3 or 4. Therefore by subtracting these probabilities two by two, one will have the following expressions:

$$\left(\frac{P-a'-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x +\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x=$$

the probability that 2 will come with 3 without 1 or 4;

$$\left(\frac{P-a''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x$$
$$+\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x=$$

the probability that 3 will come with 1 without 2 or 4;

$$\left(\frac{P-a'''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x$$
$$+\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x=$$

the probability that 1 will come with 2 without 3 or 4.

Indeed, if the probability that 2 will come without 1 or 4 with or without 3, one takes off the probability that 3 will come without 1, 2 or 4, there remains the probability that 2 will come with 3 without 1 or 4, & it is the first of our three new formulas. The two others are obtained precisely in the same manner. If one subtracts now these three formulas from the first which expresses the probability that at least two of the three numbers 1, 2, 3 will exit, it is easy to see that which remains. These three numbers furnish three combinations two by two 1, 2; 1, 3; 2, 3.

Now 1, 2 can not come without 3 or 4, by virtue of subtracting it from the third formula; 1, 3 can not come without 2 or 4, by subtracting it from the second; 2, 3 can not come without 1 or 4, by subtracting it from the first. Therefore the formula

$$\begin{aligned}1-&\left(\frac{P-a'-a''}{P}\right)^x+2\left(\frac{P-a'-a''-a'''}{P}\right)^x-3\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x\\
-&\left(\frac{P-a'-a'''}{P}\right)^x+2\left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x\\
-&\left(\frac{P-a'-a^{\text{iv}}}{P}\right)^x+2\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x\\
-&\left(\frac{P-a''-a'''}{P}\right)^x+2\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x\\
-&\left(\frac{P-a''-a^{\text{iv}}}{P}\right)^x\\
-&\left(\frac{P-a'''-a^{\text{iv}}}{P}\right)^x\end{aligned}$$

expresses the probability, that of the numbers 1, 2, 3, 4 there will exit at least three of them. These four numbers taken three by three form the following four combinations, 1, 2, 3; 1, 2, 4; 2, 3, 4. It is necessary to seek the probability that each of these combinations will come without the two other numbers, for example that 1, 2, 3 will come without 4 & 5, & subtracting these four probabilities from the probability that we just found, one will have the probability that of the numbers $1,\ 2,\ 3,\ 4,\ 5$ there will exit at least four of them, since there must exit at least three of the first four, & since any of the combinations three by three of these first four can not come without bringing forth at least one of the two other numbers. Now, in order to find the probability that $1,\ 2,\ 3$ will come without 4 & 5, it is necessary to seek the probability that $1,\ 2$ will come without 4 or 5 with or without 3, & by subtracting the probability that $1,\ 2$ will come without $3,\ 4$ or 5. Now the probability that $1,\ 2$ will come without 4 & 5 is by that which precedes,

$$\left(\frac{P-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x+\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x.$$

In order to have the probability that $1,\ 2$ will come without $3,\ 4$ or 5, it is necessary to seek the probability that 1 will come without $3,\ 4$ or 5 with or without 2, & by subtracting the probability that 1 will come without $2,\ 3,\ 4$ or 5. Now the probability that 1 will come without $3,\ 4$ or 5 is by that which precedes $\left(\frac{P-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$, & the probability that 1 will come without $2,\ 3,\ 4$ or 5 is $\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$. Therefore the probability that $1,\ 2$ will come without $3,\ 4$ or 5 is

$$\begin{aligned}&\left(\frac{P-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\\&-\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x+\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\end{aligned}$$

& the probability that $1,\ 2,\ 3$ will come without 4 or 5, is $=$

$$\begin{aligned}&\left(\frac{P-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x+\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\\&-\left(\frac{P-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x+\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\\&-\left(\frac{P-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x+\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\end{aligned}$$

One will have likewise the probability that $1,\ 2,\ 4$ will come without 3 or 5 $=$

$$\begin{aligned}&\left(\frac{P-a'''-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a'''-a^{\text{v}}}{P}\right)^x+\left(\frac{P-a'-a''-a'''-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\\&-\left(\frac{P-a''-a'''-a^{\text{v}}}{P}\right)^x+\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\\&-\left(\frac{P-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x+\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\end{aligned}$$

One will have likewise the probability that $1,\ 3,\ 4$ will come without 2 or $5 =$

$$\left(\frac{P-a''-a^{\text{v}}}{P}\right)^x - \left(\frac{P-a'-a''-a^{\text{v}}}{P}\right)^x + \left(\frac{P-a'-a''-a'''-a^{\text{v}}}{P}\right)^x - \left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$$
$$-\left(\frac{P-a''-a'''-a^{\text{v}}}{P}\right)^x + \left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$$
$$-\left(\frac{P-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x + \left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$$

One will have likewise the probability that $2,\ 3,\ 4$ will come without 1 or $5 =$

$$\left(\frac{P-a'-a^{\text{v}}}{P}\right)^x - \left(\frac{P-a'-a''-a^{\text{v}}}{P}\right)^x + \left(\frac{P-a'-a''-a'''-a^{\text{v}}}{P}\right)^x - \left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$$
$$-\left(\frac{P-a'-a'''-a^{\text{v}}}{P}\right)^x + \left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$$
$$-\left(\frac{P-a'-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x + \left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$$

The sum of these four probabilities will be therefore

$$\left(\frac{P-a'-a^{\text{v}}}{P}\right)^x - 4\left(\frac{P-a'-a''-a'''}{P}\right)^x$$
$$+3\left(\frac{P-a'-a''-a'''-a^{\text{v}}}{P}\right)^x - 4\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$$
$$+\left(\frac{P-a''-a^{\text{v}}}{P}\right)^x - 2\left(\frac{P-a'-a'''-a^{\text{v}}}{P}\right)^x + 3\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$$
$$+\left(\frac{P-a'''-a^{\text{v}}}{P}\right)^x - 3\left(\frac{P-a'-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x + 3\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$$
$$+\left(\frac{P-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x - \left(\frac{P-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x + 3\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$$
$$-2\left(\frac{P-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x$$

& by subtracting it from the preceding formula, one will have the formula,

$$
\begin{aligned}
1- &\left(\frac{P-a'-a''}{P}\right)^x + 2\left(\frac{P-a'-a''-a'''}{P}\right)^x \\
- &\ 3\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x + 4\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x \\
&-\left(\frac{P-a'-a'''}{P}\right)^x + 2\left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x - 3\left(\frac{P-a'-a''-a'''-a^{\text{v}}}{P}\right)^x \\
&-\left(\frac{P-a'-a^{\text{iv}}}{P}\right)^x + 2\left(\frac{P-a'-a''-a^{\text{v}}}{P}\right)^x - 3\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x \\
&-\left(\frac{P-a'-a^{\text{v}}}{P}\right)^x + 2\left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x - 3\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x \\
&-\left(\frac{P-a''-a'''}{P}\right)^x + 2\left(\frac{P-a'-a'''-a^{\text{v}}}{P}\right)^x \\
&-\left(\frac{P-a''-a^{\text{iv}}}{P}\right)^x + 2\left(\frac{P-a'-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x \\
&-\left(\frac{P-a''-a^{\text{v}}}{P}\right)^x + 2\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x \\
&-\left(\frac{P-a'''-a^{\text{iv}}}{P}\right)^x + 2\left(\frac{P-a''-a'''-a^{\text{v}}}{P}\right)^x \\
&-\left(\frac{P-a'''-a^{\text{v}}}{P}\right)^x + 2\left(\frac{P-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x \\
&-\left(\frac{P-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x + 2\left(\frac{P-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x
\end{aligned}
$$

which expresses the probability that of the numbers $1,\ 2,\ 3,\ 4,\ 5$ there will exit at least four of them.

The analogy is now evident, & one sees that by conserving the denominations of § 1 one will have for the probability that in x coups one will have brought forth at least $n-1$ kinds of faces =

$$1 - A'' + 2A''' - 3A^{\text{iv}} + 4A^{\text{v}} - 5A^{\text{vi}} + 6A^{\text{vii}} \text{ \&c.}$$

§ 4. If $a = b = c = d =$&c.$= 1$ one will have by the theory of combinations,[5] the probability that out of n numbers one will have brought forth at least $n-1$ of them in x coups=

$$
\begin{aligned}
1-&\frac{n(n-1)}{1.2}\left(\frac{P-2}{p}\right)^x + \frac{2n(n-1)(n-2)}{1.2.3}\left(\frac{P-3}{P}\right)^x \\
&-\frac{3n(n-1)(n-2)(n-3)}{1.2.3.4}\left(\frac{P-4}{P}\right)^x + \frac{4n(n-1)\cdots(n-4)}{1\ldots5}\left(\frac{P-5}{P}\right)^x \text{ \&c.}
\end{aligned}
$$

& by setting for $\left(\frac{P-2}{p}\right)^x$, $\left(\frac{P-3}{P}\right)^x$, $\left(\frac{P-4}{P}\right)^x$ &c. the values found § 2. one will have this probability =

$$
\begin{aligned}
1-&\frac{n(n-1)}{1.2}\left(\frac{(n-p)(n-p-1)}{n(n-1)}\right)^x \\
+&\frac{2n(n-1)(n-2)}{1.2.3}\left(\frac{(n-p)(n-p-1)(n-p-2)}{n(n-1)(n-2)}\right)^x \\
-&\frac{3n(n-1)(n-2)(n-3)}{1.2.3.4}\left(\frac{(n-p)\cdots(n-p-3)}{n(n-1)(n-2)(n-3)}\right)^x \text{ \&c.}
\end{aligned}
$$

[5] *Translator's note*: $a = a'$, $b = a''$, $c = a'''$, $d = a^{iv}, \ldots$ Trembley repeatedly uses this alternate notation.

This is that which Mr. Euler finds.

§ 5. Supposing always the same things as in § 1. one demands the probability that at the end of x coups one has brought forth at least $n-2$ faces. I reason always in the same manner, & I say: $\left(\frac{P-a'-a''-a'''}{P}\right)^x$ is the probability that there will come neither 1 nor 2 nor 3 in x drawings, therefore $1-\left(\frac{P-a'-a''-a'''}{P}\right)^x$ expresses the probability that of the numbers $1,\ 2,\ 3$ there will exit at least one of them. In order to have the probability that of the numbers $1,\ 2,\ 3,\ 4$ there will exit at least two of them, it is necessary to subtract from the preceding probability the probability that one of the numbers $1,\ 2,\ 3$ comes without one of the three others. Now the probability that 1 will come without $2,\ 3$ or 4 is by that which precedes $\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{v}}}{P}\right)^x$; the probability that 2 will come without $1,\ 3$ or 4 is $\left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x$; the probability that 3 will come without $1,\ 2,$ or 4 is $\left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x$. Therefore

$$\begin{aligned}1-&\left(\frac{P-a'-a''-a'''}{P}\right)^x+3\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x\\ -&\left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x\\ -&\left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x\\ -&\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x\end{aligned}$$

expresses the probability that of the numbers $1,\ 2,\ 3,\ 4$ there will exit at least 2 of them. In order to have the probability that of the numbers $1,\ 2,\ 3,\ 4,\ 5$ there will exit at least three of them, it is necessary to subtract from the preceding probability the probability that each of the six combinations two by two of the numbers $1,\ 2,\ 3,\ 4$ will come without the three other numbers. Now by that which precedes, the probability that $1,\ 2$ will come without $3,\ 4$ or 5 is

$$\left(\frac{P-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\\ -\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x+\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x.$$

The probability that $1,\ 3$ will come without $2,\ 4$ or 5 will be

$$\left(\frac{P-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\\ -\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x+\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x.$$

The probability that $1,\ 4$ will come without $2,\ 3$ or 5 will be

$$\left(\frac{P-a''-a'''-a^{\mathrm{v}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\mathrm{v}}}{P}\right)^x \\ -\left(\frac{P-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x+\left(\frac{P-a'-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x.$$

The probability that $2,\ 3$ will come without $1,\ 4$ or 5 will be

$$\left(\frac{P-a'-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x-\left(\frac{P-a'-a''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x \\ -\left(\frac{P-a'-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x+\left(\frac{P-a'-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x.$$

The probability that $2,\ 4$ will come without $1,\ 3$ or 5 will be

$$\left(\frac{P-a'-a'''-a^{\mathrm{v}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\mathrm{iv}}}{P}\right)^x \\ -\left(\frac{P-a'-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x+\left(\frac{P-a'-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x.$$

The probability that $3,\ 4$ will come without $1,\ 2$ or 5 will be

$$\left(\frac{P-a'-a''-a^{\mathrm{v}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\mathrm{v}}}{P}\right)^x \\ -\left(\frac{P-a'-a''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x+\left(\frac{P-a'-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x.$$

One will have therefore by subtracting these six probabilities from the preceding,

$$\begin{array}{lll}
1-\left(\frac{P-a'-a''-a'''}{P}\right)^x & +3\left(\frac{P-a'-a''-a'''-a^{\mathrm{iv}}}{P}\right)^x & -6\left(\frac{P-a'-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x \\
\quad-\left(\frac{P-a'-a''-a^{\mathrm{iv}}}{P}\right)^x & +3\left(\frac{P-a'-a''-a'''-a^{\mathrm{v}}}{P}\right)^x & \\
\quad-\left(\frac{P-a'-a''-a^{\mathrm{v}}}{P}\right)^x & +3\left(\frac{P-a'-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x & \\
\quad-\left(\frac{P-a'-a'''-a^{\mathrm{iv}}}{P}\right)^x & +3\left(\frac{P-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x & \\
\quad-\left(\frac{P-a'-a'''-a^{\mathrm{v}}}{P}\right)^x & & \\
\quad-\left(\frac{P-a'-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x & & \\
\quad-\left(\frac{P-a''-a'''-a^{\mathrm{iv}}}{P}\right)^x & & \\
\quad-\left(\frac{P-a''-a'''-a^{\mathrm{v}}}{P}\right)^x & & \\
\quad-\left(\frac{P-a''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x & & \\
\quad-\left(\frac{P-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}}{P}\right)^x & &
\end{array}$$

this is the probability that out of the numbers $1,\ 2,\ 3,\ 4,\ 5$ there will exit at least three of them.

In order to have the probability that of the numbers 1, 2, 3, 4, 5, 6 there will exit at least four of them, it is necessary to subtract from the preceding probability the probability that each of the ten combinations three-by-three of the numbers of the first five numbers will come without the three other numbers. Now these probabilities result easily from the preceding calculations; here is an example of it for the probability that $1,\ 2,\ 3$ will come without $4, 5,\ 6$. The probability that $1,\ 2$ will come without $4,\ 5,\ 6$ is found by that which precedes $=$

$$\left(\frac{P-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x-\left(\frac{P-a'-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x+\left(\frac{P-a'-a''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x.$$

The probability that $1,\ 2$ will come without $3,\ 4,\ 5,\ 6$ is $=$

$$\left(\frac{P-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x-\left(\frac{P-a'-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x+\left(\frac{P-a'-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x.$$

Therefore subtracting the second formula from the first, the probability that $1,\ 2, 3$ will come without $4,\ 5,\ 6$ is $=$

$$\text{I.}\quad\left(\frac{P-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x-\left(\frac{P-a'-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x$$
$$+\left(\frac{P-a'-a''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x+\left(\frac{P-a'-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x+\left(\frac{P-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x$$

One will have likewise the probability that $1,\ 2,\ 4$ will come without $3,\ 5$ & 6 by exchanging in the preceding c into d & d into c,

$$\text{II.}\quad\left(\frac{P-a'''-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x-\left(\frac{P-a'-a'''-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x$$
$$+\left(\frac{P-a'-a''-a'''-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a''-a'''-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x+\left(\frac{P-a'-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x+\left(\frac{P-a''-a'''-a^{\mathrm{iv}}-a^{\mathrm{v}}-a^{\mathrm{vi}}}{P}\right)^x$$

One will have the probability that $1,\ 2,\ 5$ will come without $3,\ 4,$ & 6 by exchanging in the preceding d into e & e into d,

$$\begin{aligned}\text{III.}\quad &\left(\frac{P-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x-\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x\\ &+\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x\\ &-\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x+\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x\\ &-\left(\frac{P-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x+\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x\end{aligned}$$

One will have the probability that $1,\ 3,\ 4$ will come without $2,\ 5,\ 6$ by exchanging in the second b into c & c into b,

$$\begin{aligned}\text{IV.}\quad &\left(\frac{P-a''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x-\left(\frac{P-a'-a''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x\\ &+\left(\frac{P-a'-a''-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x\\ &-\left(\frac{P-a''-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x+\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x\\ &-\left(\frac{P-a''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x+\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x\end{aligned}$$

One will have the probability that $1,\ 3,\ 5$ will come without $2,\ 4,\ 6$ by exchanging in the preceding d into e & e into d,

$$\begin{aligned}\text{V.}\quad &\left(\frac{P-a''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x-\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x\\ &+\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x\\ &-\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x+\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x\\ &-\left(\frac{P-a''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x+\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x\end{aligned}$$

One will have the probability that $1,\ 4,\ 5$ will come without $2,\ 3,\ 6$ by exchanging in the preceding c into d & d into c,

$$\text{VI.}\quad \left(\frac{P-a''-a'''-a^{\text{vi}}}{P}\right)^x - \left(\frac{P-a'-a''-a'''-a^{\text{vi}}}{P}\right)^x$$
$$+\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x - \left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x + \left(\frac{P-a'-a''-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a''-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x + \left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$

One will have the probability that $2,\ 3,\ 4$ will come without $1,\ 5,\ 6$ by exchanging in formula II a into c & c into a,

$$\text{VII.}\quad \left(\frac{P-a'-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x - \left(\frac{P-a'-a''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$
$$+\left(\frac{P-a'-a''-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x - \left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a'-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x + \left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a'-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x + \left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$

One will have the probability that $2,\ 3,\ 5$ will come without $1,\ 4,\ 6$ by exchanging in the preceding formula d into e & e into d,

$$\text{VIII.}\quad \left(\frac{P-a'-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x - \left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x$$
$$+\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x - \left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x + \left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a'-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x + \left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$

One will have the probability that $2,\ 4,\ 5$ will come without $1,\ 3,\ 6$ by exchanging in the preceding formula c into d & d into c,

$$\text{IX.}\quad \left(\frac{P-a'\ -a'''-a^{\text{vi}}}{P}\right)^x - \left(\frac{P-a'-a''-a'''-a^{\text{vi}}}{P}\right)^x$$
$$+\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x - \left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x + \left(\frac{P-a'-a''-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a'-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x + \left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$

One will have the probability that $3,\ 4,\ 5$ will come without $1,\ 2,\ 6$ by exchanging in the preceding formula b into c & c into b,

$$\text{X.}\quad \left(\frac{P-a'\ -a''-a^{\text{vi}}}{P}\right)^x - \left(\frac{P-a'-a''-a'''-a^{\text{vi}}}{P}\right)^x$$
$$+\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x - \left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x + \left(\frac{P-a'-a''-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$
$$-\left(\frac{P-a'-a''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x + \left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x$$

Subtracting these ten formulas from the preceding, one will have the probability that out of the numbers 1, 2, 3, 4, 5, 6 there will exit at least four of them, =

$$
\begin{array}{llll}
1 & -\left(\frac{P-a'-a''-a'''}{P}\right)^x & +3\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x & -6\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x \\
 & +10\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x & & \\
 & -\left(\frac{P-a'-a''-a^{\text{iv}}}{P}\right)^x & +3\left(\frac{P-a'-a''-a'''-a^{\text{v}}}{P}\right)^x & -6\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x \\
 & -\left(\frac{P-a'-a''-a^{\text{v}}}{P}\right)^x & +3\left(\frac{P-a'-a''-a'''-a^{\text{vi}}}{P}\right)^x & -6\left(\frac{P-a'-a''-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x \\
 & -\left(\frac{P-a'-a''-a^{\text{vi}}}{P}\right)^x & +3\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x & -6\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x \\
 & -\left(\frac{P-a'-a'''-a^{\text{iv}}}{P}\right)^x & +3\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x & -6\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x \\
 & -\left(\frac{P-a'-a'''-a^{\text{v}}}{P}\right)^x & +3\left(\frac{P-a'-a''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x & -6\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x \\
 & -\left(\frac{P-a'-a'''-a^{\text{vi}}}{P}\right)^x & +3\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x & \\
 & -\left(\frac{P-a'-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x & +3\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x & \\
 & -\left(\frac{P-a'-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x & +3\left(\frac{P-a'-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x & \\
 & -\left(\frac{P-a'-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x & +3\left(\frac{P-a'-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x & \\
 & -\left(\frac{P-a''-a'''-a^{\text{iv}}}{P}\right)^x & +3\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x & \\
 & -\left(\frac{P-a''-a'''-a^{\text{v}}}{P}\right)^x & +3\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x & \\
 & -\left(\frac{P-a''-a'''-a^{\text{vi}}}{P}\right)^x & +3\left(\frac{P-a''-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x & \\
 & -\left(\frac{P-a''-a'''-a^{\text{vi}}}{P}\right)^x & +3\left(\frac{P-a''-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x & \\
 & -\left(\frac{P-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x & +3\left(\frac{P-a''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x & \\
 & -\left(\frac{P-a''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x & +3\left(\frac{P-a'''-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x & \\
 & -\left(\frac{P-a''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x & & \\
 & -\left(\frac{P-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x & & \\
 & -\left(\frac{P-a'''-a^{\text{iv}}-a^{\text{vi}}}{P}\right)^x & & \\
 & -\left(\frac{P-a'''-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x & & \\
 & -\left(\frac{P-a^{\text{iv}}-a^{\text{v}}-a^{\text{vi}}}{P}\right)^x. & &
\end{array}
$$

The analogy is now evident, & one sees that by conserving the same denominations as above, one will have for the probability that out of n faces one will bring forth at least $n-3$ of them,

$$1 - A''' + 3A^{\text{iv}} - 6A^{\text{v}} + 10^{\text{vi}} \text{ \&c.}$$

§ 6. If $a' = a'' = a''' = a^{\text{iv}}$ &c. $= a^{(n)} = 1$, one will have by the theory of combinations,

$$1 - \frac{n(n-1)(n-2)}{1.2.3}\left(\frac{P-3}{P}\right)^x + \frac{3n\cdots(n-3)}{1\ldots4}\left(\frac{P-4}{P}\right)^x + \frac{6n\cdots(n-4)}{1\ldots5}\left(\frac{P-5}{P}\right)^x + \frac{10n\cdots(n-5)}{1\ldots6}\left(\frac{P-6}{P}\right)^x \text{ \&c.}$$

for the probability sought, & by putting for $\left(\frac{P-3}{P}\right)^x$, $\left(\frac{P-4}{P}\right)^x$, $\left(\frac{P-5}{P}\right)^x$ &c. the values found above, one will have this probability, =

$$1-\frac{n(n-1)(n-2)}{1.2.3}\left(\frac{(n-p)\cdots(n-p-2)}{n\cdots(n-2)}\right)^x+\frac{3n\cdots(n-3)}{1\ldots 4}\left(\frac{(n-p)\cdots(n-p-3)}{n\cdots(n-3)}\right)^x$$
$$-\frac{6n\cdots(n-4)}{1\ldots 5}\left(\frac{(n-p)\cdots(n-p-4)}{n\cdots(n-4)}\right)^x \text{ \&c.}$$

This is the result that Mr. Euler finds.

§ 7. Supposing always the same things, we demand the probability that at the end of x coups one will have brought forth at least $n-3$ faces. I continue to reason likewise, & I say: $\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x$ is the probability that there will come neither 1 nor 2 nor 3 nor 4 in x coups, therefore $1-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x$ expresses the probability that of the numbers, 1, 2, 3, 4 there will exit at least one of them. In order to have the probability that of the numbers 1, 2, 3, 4, 5 there will exit at least two of them, it is necessary to subtract from the preceding probability the probability that one of the numbers 1, 2, 3, 4 will come without the four others. Now

$$\left(\frac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x=$$

the probability that 1 will come without 2, 3, 4, 5.

$$\left(\frac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x=$$

the probability that 2 will come without 1, 3, 4, 5.

$$\left(\frac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x=$$

the probability that 3 will come without 1, 2, 4, 5.

$$\left(\frac{P-a'-a''-a'''-a^{\text{v}}}{P}\right)^x-\left(\frac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x=$$

the probability that 4 will come without 1, 2, 3, 5. Therefore

$$\begin{aligned}1&-\left(\tfrac{P-a'-a''-a'''-a^{\text{iv}}}{P}\right)^x \qquad +4\left(\tfrac{P-a'-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\\ &-\left(\tfrac{P-a'-a''-a'''-a^{\text{v}}}{P}\right)^x\\ &-\left(\tfrac{P-a'-a''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\\ &-\left(\tfrac{P-a'-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x\\ &-\left(\tfrac{P-a''-a'''-a^{\text{iv}}-a^{\text{v}}}{P}\right)^x=\end{aligned}$$

the probability that of the numbers $1,\ 2,\ 3,\ 4,\ 5$ there will exit at least two of them. One will find by continuing the operation, & conserving always the same denominations, that

$$1 - A^{\text{iv}} + 4A^{\text{v}} - 10A^{\text{vi}} + 20A^{\text{vii}} - 35A^{\text{viii}} \text{ \&c.}$$

expresses the probability that out of n numbers there will exit at least $n-3$ of them.

§ 8. The analogy is now evident, the numeric coefficients being the figurate numbers, & one will have the following formula, which is general:

$$\begin{aligned}1 - &A^{(\lambda)} + (\lambda+1)A^{(\lambda+1)} - \frac{(\lambda+1)(\lambda+2)}{1.2}A^{(\lambda+2)}\\ &+ \frac{(\lambda+1)(\lambda+2)(\lambda+3)}{1.2.3}A^{(\lambda+3)} - \frac{(\lambda+1)\cdots(\lambda+4)}{1\ldots 4}A^{(\lambda+4)}\\ &+ \frac{(\lambda+1)\cdots(\lambda+5)}{1.2\ldots 5}A^{(\lambda+5)} \text{ \&c. } =\end{aligned}$$

the probability that of n numbers there will exit at least $n-\lambda$ of them.

§ 9. If now we make $a' = a'' = a''' = a^{\text{iv}}$ &c. $= 1$, we will have

$$\begin{aligned}A' &= \frac{n}{1}\left(\frac{P-1}{P}\right)^x,\\ A'' &= \frac{n(n-1)}{1.2}\left(\frac{P-2}{P}\right)^x,\\ A''' &= \frac{n(n-1)(n-2)}{1.2.3}\left(\frac{P-3}{P}\right)^x,\\ A^{\text{iv}} &= \frac{n\cdots(n-3)}{1\ldots 4}\left(\frac{P-4}{P}\right)^x \cdots\\ A^{(\lambda)} &= \frac{n\cdots(n-\lambda+1)}{1.2\ldots\lambda}\left(\frac{P-\lambda}{P}\right)^x\end{aligned}$$

One will have therefore by substituting these values,

$$\begin{aligned}1 - &\frac{n\cdots(n-\lambda+1)}{1.2\ldots\lambda}\left(\frac{P-\lambda}{P}\right)^x + (\lambda+1)\frac{n\cdots(n-\lambda)}{1.2\ldots\lambda+1}\left(\frac{P-\lambda-1}{P}\right)^x\\ &- \frac{(\lambda+1)(\lambda+2)}{1.2}\left(\frac{n\cdots(n-\lambda-1)}{1\ldots\lambda+2}\right)\left(\frac{P-\lambda-2}{P}\right)^x\\ &+ \frac{(\lambda+1)\cdots(\lambda+3)}{1.2.3}\left(\frac{n\cdots(n-\lambda-2)}{1\ldots\lambda+3}\right)\left(\frac{P-\lambda-3}{P}\right)^x\\ &- \frac{(\lambda+1)\cdots(\lambda+4)}{1\ldots 4}\left(\frac{n\cdots(n-\lambda-3)}{1\ldots\lambda+4}\right)\left(\frac{P-\lambda-4}{P}\right)^x \text{ \&c. } =\end{aligned}$$

$$
\begin{aligned}
1- & \frac{n\cdots(n-\lambda+1)}{1.2\ldots\lambda}\left(\frac{(n-p)\cdots(n-p-\lambda+1)}{n\ldots(n-\lambda+1)}\right)^x \\
& +\frac{(\lambda+1)}{1}\frac{n\cdots(n-\lambda)}{1\ldots(\lambda+1)}\left(\frac{(n-p)\cdots(n-p-\lambda)}{n\ldots n-\lambda}\right)^x \\
& -\frac{(\lambda+1)(\lambda+2)}{1.2}\left(\frac{n\cdots(n-\lambda-1)}{1\ldots(\lambda+2)}\right)\left(\frac{(n-p)\cdots(n-p-\lambda-1)}{n\ldots(n-\lambda-1)}\right)^x \\
& +\frac{(\lambda+1)(\lambda+2)(\lambda+3)}{1.2.3}\left(\frac{n\cdots(n-\lambda-2)}{1\ldots\lambda+3}\right)\left(\frac{(n-p)\cdots(n-p-\lambda-2)}{n\ldots(n-\lambda-2)}\right)^x \\
& -\frac{(\lambda+1)\cdots(\lambda+4)}{1.2.3.4}\left(\frac{n\cdots(n-\lambda-3)}{1\ldots\lambda+4}\right)\left(\frac{(n-p)\cdots(n-p-\lambda-3)}{n\ldots n-\lambda-3}\right)^x \text{ \&c.}
\end{aligned}
$$

Mr. Euler finds an analogous formula. One sees that the direct use of the doctrine of combinations has led us quite simply to the solution of the general problem of which the one of Mr. Euler is only a particular case, since a', a'', $a''' \ldots a^{(n)}$ which in the problem of Mr. Euler are $= 1$ can be anything in ours. The general problem would have place in a lottery where there would be a tickets marked 1, b marked 2, c marked 3 &c. & where one would demand the probability that there would have exited a certain number of kinds of them. One will find always the value of the expressions $\frac{P-a'}{P}$, $\frac{P-a'-a''}{P}$ &c. by way of the following formula which expresses the probability that by taking t things out of n, of which u are of a certain kind, there will be found m of this kind of them. This formula is, as one knows,

$$
\frac{(n-t)(n-t-1)\cdots(n-t-u+m+1)t(t-1)\cdots(t-m+1)}{n(n-1)\cdots n-u+1}\,\frac{u(u-1)\cdots(u-m+1)}{1.2.3\ldots m}
$$

§ 10. It is evident by the same nature of things, that in problem 1, if the number of drawings x is smaller than $\frac{n}{p}$, it will be impossible that all the numbers have exited at the end of a number x of drawings; thus in all the cases where $x < np$, the found probability must be $= 0$, & the *minimum* of x where the probability is not null, has place when $x = \frac{n}{p}$ or $n = px$. Mr. Euler remarks that in this case the sum of the series which gives the probability, can be expressed by some products. This can be demonstrated by the integral calculus, by the following method which is quite simple.

§ 11. The series that the question is to sum is in this case here

$$
\begin{aligned}
1-px & \left(\frac{px-p}{px}\right)^x+\frac{px(px-1)}{1.2}\left(\frac{(px-p)(px-p-1)}{px(px-1)}\right)^x \\
& -\frac{px(px-1)(px-2)}{1.2.3}\left(\frac{(px-p)\cdots(px-p-2)}{px\ldots(px-2)}\right)^x \\
& -\frac{px\cdots(px-3)}{1\ldots4}\left(\frac{(px-p)\cdots(px-p-3)}{px\ldots(px-3)}\right)^x \text{ \&c.}
\end{aligned}
$$

I see first that if $x = 1$, all the series $= 1$, because all the terms vanish with the exception of the first.

If $x = 2$, one has the series

$$\begin{aligned}
&1-2p\left(\frac{p}{2p}\right)^x + \frac{2p(2p-1)}{1.2}\left(\frac{p(p-1)}{2p(2p-1)}\right)^2\\
&\quad -\frac{2p(2p-1)(2p-2)}{1.2.3}\left(\frac{p(p-1)(p-2)}{2p(2p-1)(2p-2)}\right)^2\\
&\quad +\frac{2p(2p-1)(2p-2)(2p-3)}{1.2.3.4}\left(\frac{p(p-1)(p-2)(p-3)}{2p(2p-1)(2p-2)(2p-3)}\right)^2 \text{ \&c.}\\
&= 1-\frac{p^2}{2p} + \frac{1}{1.2}\frac{p^2(p-1)^2}{2p(2p-1)} - \frac{1}{1.2.3}\frac{p^2(p-1)^2(p-2)^2}{2p(2p-1)(2p-2)}\\
&\quad +\frac{1}{1.2.3.4}\frac{p^2(p-1)^2(p-2)^2(p-3)^2}{2p(2p-1)(2p-2)(2p-3)} \text{ \&c.}
\end{aligned}$$

In order to find the sum of this series, I form the summatory sequence of that there, & I have by making successively $p = 1, 2, 3, 4$ &c. the sequence

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \frac{1}{2} & \frac{1.2}{3.4} & \frac{1.2.3}{4.5.6} & \frac{1.2.3.4}{5.6.7.8} & \text{\&c.} \end{array}$$

The general term of this sequence, that is to say the sum of the sought series when $x = 2$, is therefore $\frac{1.2.3...p}{(p+1)(p+2)\cdots 2p}$.

If $x = 3$, one has the series,

$$\begin{aligned}
&1-\frac{(2p)^3}{(3p)^2} + \frac{1}{1.2}\frac{(2p)^3(2p-1)^3}{(3p)^2(3p-1)^2} - \frac{1}{1.2.3}\frac{(2p)^3(2p-1)^3(2p-2)^3}{(3p)^2(3p-1)^2(3p-2)^2}\\
&\quad +\frac{1}{1.2.3.4}\frac{(2p)^3(2p-1)^3(2p-2)^3(2p-3)^3}{(3p)^2(3p-1)^2(3p-2)^2(3p-3)^2} \text{ \&c.}
\end{aligned}$$

Make successively $p = 1, 2, 3, 4$ & we will have the summatory series,

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \frac{1.2}{3^2} & \frac{1.2.3.4}{5^2.6^2} & \frac{1.2.3.4.5.6}{7^2.8^2.9^2} & \frac{1.2.3.4.5.6.7.8}{9^2.10^2.11^2.12^2} & \text{\&c.} \end{array}$$

of which the general term is evidently $\frac{1.2.3...2p}{((2p+1)(2p+2)\cdots(3p))^2}$.

If $x = 4$, one has the series,

$$\begin{aligned}
&1-\frac{(3p)^4}{(4p)^3} + \frac{1}{1.2}\frac{(3p)^4(3p-1)^4}{(4p)^3(4p-1)^3} - \frac{1}{1.2.3}\frac{(3p)^4(3p-1)^4(3p-2)^4}{(4p)^3(4p-1)^3(4p-2)^3}\\
&\quad +\frac{1}{1.2.3.4}\frac{(3p)^4(3p-1)^4(3p-2)^4(3p-3)^4}{(4p)^3(4p-1)^3(4p-2)^3(4p-3)^3} \text{ \&c.}
\end{aligned}$$

Make successively $p = 1, 2, 3, 4$ & we will have the summatory sequence,

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \frac{1.2.3}{4^3} & \frac{1.2.3.4.5.6}{7^3.8^3} & \frac{1.2.3.4.5.6.7.8.9}{10^3.11^3.12^3} & \frac{1.2.3...12}{13^3.14^3.15^3.16^3} & \text{\&c.} \end{array}$$

of which the general term is evidently $\frac{1.2.3...3p}{((3p+1)(3p+2)\cdots(4p))^3}$. The analogy is now evident, & one sees that the proposed series will be =

$$\frac{1.2.3\ldots(x-1)p}{(((x-1)p+1)((x-1)p+2)\cdots(xp))^{x-1}} =$$

(by making $px = n$)

$$\frac{1.2.3\ldots(x-1)p}{((n(n-1)\cdots n-p+1)^{x-1}} = \frac{1.2.3\ldots xp}{((n(n-1)\cdots(n-p+1)^{x}} = \frac{1.2.3\ldots n}{((n(n-1)\cdots(n-p+1)^{x}}.$$

§ 12. One can demonstrate also that the sum of our series is $= 0$ in the case where $n > px$. Let $n = p(x+1)$, the series will become

$$1-p(x+1)\left(\frac{px}{p(x+1)}\right)^x + \frac{p(x+1)((x+1)p-1)}{1.2}\left(\frac{(px)(px-1)}{(x+1)p((x+1)p-1)}\right)^x - \frac{(x+1)p\cdots((x+1)p-2)}{1.2.3}\left(\frac{px\cdots(px-2)}{(x+1)p\cdots((x+1)p-2)}\right)^x \ \&\text{c.}$$

If $x = 1$, this series becomes

$$1-p+\frac{p(p-1)}{1.2}-\frac{p(p-1)(p-2)}{1.2.3} \ \&\text{c.} \ = (1-1)^p = 0.$$

If $x = 2$, this series becomes

$$\begin{aligned} &1-\frac{(2p)^2}{(3p)}+\frac{1}{1.2}\frac{(2p)^2(2p-1)^2}{3p(3p-1)}-\frac{1}{1.2.3}\frac{(2p)^2(2p-1)^2(2p-2)^2}{(3p)(3p-1)(3p-2)} \\ &\quad+\frac{1}{1.2.3.4}\frac{(2p)^2(2p-1)^2(2p-2)^2(2p-3)^2}{(3p)(3p-1)(3p-2)(3p-3)} \ \&\text{c.} \\ &= 1-\frac{2p}{1}\left(\frac{2p}{3p}\right)+\frac{2p(2p-1)}{1.2}\left(\frac{2p}{3p}\,\frac{(2p-1)}{(3p-1)}\right) \\ &\quad-\frac{2p(2p-1)(2p-2)}{1.2.3}\left(\frac{2p}{3p}\,\frac{(2p-1)}{3p-1}\,\frac{(2p-2)}{3p-2}\right) \ \&\text{c.} \end{aligned}$$

If one forms the summatory sequence of this series by making $p = 1, 2, 3, 4$, &c. one will have each term $= 0$, the general term of the series is therefore $= 0$, & the sum sought $= 0$.

If $x = 3$, this series becomes

$$\begin{aligned} &1-\frac{3p}{1}\left(\frac{3p}{4p}\right)^2+\frac{3p(3p-1)}{1.2}\left(\frac{3p}{4p}\,\frac{(3p-1)}{(4p-1)}\right)^2 \\ &\quad-\frac{3p(3p-1)(3p-2)}{1.2.3}\left(\frac{3p}{4p}\,\frac{(3p-1)}{(4p-1)}\,\frac{(3p-2)}{(4p-2)}\right)^2 \ \&\text{c..} \end{aligned}$$

which one will find in the same manner $= 0$.

If $x = 4$, this series becomes

$$1 - \frac{4p}{1}\left(\frac{4p}{5p}\right)^3 + \frac{4p(4p-1)}{1.2}\left(\frac{4p}{5p}\,\frac{(4p-1)}{(5p-1)}\right)^3 - \frac{4p(4p-1)(4p-2)}{1.2.3}\left(\frac{4p}{5p}\,\frac{(4p-1)}{(5p-1)}\,\frac{(4p-2)}{(5p-2)}\right)^3 \text{ \&c..}$$

which one will find likewise $= 0$.

Therefore the series itself which one can put under this form

$$1 - \frac{xp}{1}\left(\frac{xp}{(x+1)p}\right)^{x-1} + \frac{xp(xp-1)}{1.2}\left(\frac{xp}{(x+1)p}\,\frac{(xp-1)}{((x+1)p-1)}\right)^{x-1} - \frac{xp(xp-1)(xp-2)}{1.2.3}\left(\frac{xp}{(x+1)p}\,\frac{(xp-1)}{((x+1)p-1)}\,\frac{(xp-2)}{((x+1)p-2)}\right)^{x-1} \text{ \&c..}$$

is $= 0$.

Let now $n = p(x+2)$, & one will prove likewise that the series

$$1 - p(x+2)\left(\frac{p(x+1)}{p(x+2)}\right)^x + \frac{p(x+2)((x+2)p-1)}{1.2}\left(\frac{px(x+1)}{p(x+2)}\,\frac{(p(x+1)-1)}{(p(x+2)p-1)}\right)^x \text{ \&c.}$$

$$= 1 - (x+1)p\left(\frac{(x+1)p}{(x+2)p}\right)^{x-1} + \frac{(x+1)p((x+1)p-1)}{1.2}\left(\frac{(x+1)p}{(x+2)p}\,\frac{(p(x+1)-1)}{(p(x+2)p-1)}\right)^x \text{ \&c.} - \frac{(x+1)p((x+1)p-1)((x+2)p-2)}{1.2.3}\left(\frac{(x+1)p}{(x+2)p}\,\frac{((x+1)p-1)}{((x+2)p-1)}\,\frac{((x+1)p-2)}{((x+2)p-2)}\right)^x \text{ \&c..}$$

$= 0$.

Therefore finally if $n = p(x+m)$, m being any whole number, one will find in the same manner that the series

$$1 - \frac{(x+m-1)p}{1}\left(\frac{(x+m-1)p}{(x+m)p}\right)^{x-1} + \frac{(x+m-1)p((x+m-1)p-1)}{1.2}\left(\frac{(x+m-1)p}{(x+m)p}\,\frac{((x+m-1)p-1)}{((x+m)p-1)}\right)^{x-1} \text{ \&c.} - \frac{(x+m-1)p((x+m-1)p-1)((x+m-1)p-2)}{1.2.3} \times \left(\frac{(x+m-1)p}{(x+m)p}\left(\frac{(x+m-1)p-1}{(x+m)p-1}\right)\frac{(x+m-1)p-2}{x+mp-2}\right)^x \text{ \&c.} = 0.$$

§ 13. If n & x are very great numbers, & p a small number, the calculation would become impractical by its length, but the analysis furnishes in this case the way to abbreviation. One will have under this assumption,

$$\left(\frac{n-p}{n}\right)^x = \left(1-\frac{p}{n}\right)^x = 1 - \frac{xp}{n} + \frac{x^2}{2}\cdot\frac{p^2}{n^2} \text{ \&c.} = e^{-\frac{px}{n}},$$

& likewise

$$\left(\frac{n-p-1}{n-1}\right)^x = e^{-\frac{px}{n-1}} \cdots \left(\frac{n-p-\lambda}{n-\lambda}\right)^x = e^{-\frac{px}{n-\lambda}}.$$

We will have therefore for the probability that all the numbers have exited,

$$\begin{aligned}
&1-\frac{n}{1}e^{-\frac{px}{n}} + \frac{n(n-1)}{1.2}e^{-\frac{px}{n-1}-\frac{px}{n}}\\
&\quad-\frac{n(n-1)(n-2)}{1.2.3}e^{-\frac{px}{n-2}-\frac{px}{n-1}-\frac{px}{n}}\\
&\quad+\frac{n\cdots(n-3)}{1\ldots4}e^{-\frac{px}{n-3}-\frac{px}{n-2}-\frac{px}{n-1}-\frac{px}{n}} \text{ \&c.}\\
=\ &1-ne^{-\frac{px}{n}} + \frac{n^2}{2}e^{-\frac{2px}{n}} - \frac{n^3}{2.3}e^{-\frac{3px}{n}} + \frac{n^4}{1.2\ldots4}e^{-\frac{4px}{n}} \text{ \&c.}\\
&\quad= e^{-ne^{-\frac{px}{n}}} = e^{-nq}
\end{aligned}$$

by making for brevity $q = e^{-\frac{px}{n}}$.

If one wishes to push the approximation further, one will have

$$\begin{aligned}
\left(1-\frac{p}{n}\right)^x &= 1 - \frac{px}{n} + \frac{x^2}{2}\cdot\frac{p^2}{n^2} - \frac{x^3}{2.3}\cdot\frac{p^3}{n^3} + \frac{x^4}{2.3.4}\cdot\frac{p^4}{n^4} \text{ \&c.}\\
-\frac{p^2x}{2n^2}&\left(1-\frac{px}{n}+\frac{x^2}{2}\cdot\frac{p^2}{n^2}-\frac{x^3}{2.3}\cdot\frac{p^3}{n^3}\text{ \&c.}\right) = e^{-\frac{px}{n}}\left(1-\frac{p^2x}{2n^2}\right).
\end{aligned}$$

One will have likewise

$$\begin{aligned}
\left(1-\frac{p}{n-1}\right)^x &= e^{-\frac{px}{n-1}}\left(1-\frac{p^2x}{2(n-1)^2}\right),\\
\left(1-\frac{p}{n-2}\right)^x &= e^{-\frac{px}{n-1}}\left(1-\frac{p^2x}{2(n-1)^2}\right),\\
\cdots\left(1-\frac{p}{n-\lambda}\right)^x &= e^{-\frac{px}{n-\lambda}}\left(1-\frac{p^2x}{2(n-\lambda)^2}\right).
\end{aligned}$$

We will have therefore for the probability that all the numbers have exited,

$$1{-}ne^{-\frac{px}{n}}\left(1-\frac{p^2x}{2n^2}\right)+\frac{n(n-1)}{1.2}\left(1-\frac{p^2x}{2n^2}\right)$$
$$\times\left(1-\frac{p^2x}{2(n-1)^2}\right)e^{-\frac{px}{n}}e^{-\frac{px}{n-1}}$$
$$-\frac{n(n-1)(n-2)}{1.2.3}e^{-\frac{px}{n}}e^{-\frac{px}{n-1}}e^{-\frac{px}{n-2}}$$
$$\times\left(1-\frac{p^2x}{2n^2}\right)\left(1-\frac{p^2x}{2(n-1)^2}\right)\left(1-\frac{p^2x}{2(n-2)^2}\right)$$
$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}e^{-\frac{px}{n}}e^{-\frac{px}{n-1}}e^{-\frac{px}{n-2}}e^{-\frac{px}{n-3}}$$
$$\times\left(1-\frac{p^2x}{2n^2}\right)\left(1-\frac{p^2x}{2(n-1)^2}\right)\left(1-\frac{p^2x}{2(n-2)^2}\right)\left(1-\frac{p^2x}{2(n-3)^2}\right)\ \text{\&c.}$$

$$=1{-}ne^{-\frac{px}{n}}\left(1-\frac{p^2x}{2n^2}\right)+\frac{n(n-1)}{1.2}e^{-\frac{2px}{n}}\left(1-\frac{p^2x}{2n^2}\right)^2$$
$$-\frac{n(n-1)(n-2)}{1.2.3}e^{-\frac{3px}{n}}\left(1-\frac{p^2x}{2n^2}\right)^3$$
$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}e^{-\frac{4px}{n}}\left(1-\frac{p^2x}{2n^2}\right)^4\ \text{\&c.}$$

$$=1{-}ne^{-\frac{px}{n}}\left(1-\frac{p^2x}{2n^2}\right)+\frac{n^2}{2}e^{-\frac{2px}{n}}\left(1-\frac{p^2x}{2n^2}\right)^2$$
$$-\frac{n^3}{1.2.3}e^{-\frac{3px}{n}}\left(1-\frac{p^2x}{2n^2}\right)^3+\frac{n^4}{2.3.4}e^{-\frac{4px}{n}}\left(1-\frac{p^2x}{2n^2}\right)^4\ \text{\&c.}$$
$$-\frac{n}{2}e^{-\frac{2px}{n}}\left(1-\frac{p^2x}{2n^2}\right)^2\left(1-ne^{-\frac{px}{n}}\left(1-\frac{p^2x}{2n^2}\right)\right.$$
$$\left.+\frac{n^2}{2}e^{-\frac{2px}{n}}\left(1-\frac{p^2x}{2n^2}\right)^2-\frac{n^3}{2.3}e^{-\frac{px}{n}}\left(1-\frac{p^2x}{2n^2}\right)^3\ \text{\&c.}\right)$$

=(by making $e^{-\frac{px}{n}}=q$)

$$e^{-nq}\left(1-\frac{p^2x}{2n^2}\right)+\frac{n}{2}q^2\left(1-\frac{p^2x}{2n^2}\right)^2e^{-nq}\left(1-\frac{p^2x}{2n^2}\right)=$$
$$e^{-nq}e^{\frac{p^2x}{2n^2}nq}-\frac{n}{2}q^2\left(1-\frac{p^2x}{2n^2}\right)^2e^{-nq}e^{\frac{p^2x}{2n^2}nq}=$$
$$e^{-nq}e^{\frac{p^2x}{2n}q}-\frac{n}{2}q^2\left(1-\frac{p^2x}{2n^2}\right)^2e^{-nq}e^{\frac{p^2x}{2n}q}.$$

Now $e^{\frac{p^2x}{2n}q} = 1 + \frac{p^2x}{2n}q$ &c. Therefore the sought probability will be

$$e^{-nq}\left(1+\frac{p^2x}{2n}q\right) - \frac{n}{2}q^2e^{-nq} = e^{-nq}\left(1+\frac{p^2x}{2n}q - \frac{n}{2}q^2\right).$$

§ 14. If one considers this process with attention, one will see that one can put $\frac{1}{n^2}$ in place of $\frac{1}{(n-1)^2}$, $\frac{1}{(n-2)^2}$ &c. without committing an error greater than that of the terms where q^3 enters. It is necessary to except the terms $e^{-\frac{px}{n-1}}$, $e^{-\frac{px}{n-2}}$ &c. One has $\frac{1}{n-1} = \frac{1}{n} + \frac{1}{n^2}$, $\frac{1}{n-2} = \frac{1}{n} + \frac{2}{n^2}$, $\ldots$ $\frac{1}{n-\lambda} = \frac{1}{n} + \frac{\lambda}{n^2}$. We will have therefore

$$\begin{aligned}
e^{-\frac{px}{n-1}} &= e^{-\frac{px}{n}}e^{-\frac{px}{n^2}},\\
e^{-\frac{px}{n-2}} &= e^{-\frac{px}{n}}e^{-\frac{2px}{n^2}},\\
e^{-\frac{px}{n-3}} &= e^{-\frac{px}{n}}e^{-\frac{3px}{n^2}},\ldots\\
e^{-\frac{px}{n-\lambda}} &= e^{-\frac{px}{n}}e^{-\frac{\lambda px}{n^2}}.
\end{aligned}$$

Therefore taking the preceding formulas, & applying these corrections to the first part of the formula alone, it will become

$$\begin{aligned}
&1 - ne^{-\frac{px}{n}}\left(1-\frac{p^2x}{2n}\right) + \frac{n^2}{2}e^{-\frac{2px}{n}-\frac{px}{n^2}}\left(1-\frac{p^2x}{2n}\right)^2\\
&-\frac{n^3}{2.3}e^{-\frac{3px}{n}-\frac{3px}{n^2}}\left(1-\frac{p^2x}{2n}\right)^3\\
&-\frac{n^4}{2.3.4}e^{-\frac{4px}{n}-\frac{6px}{n^2}}\left(1-\frac{p^2x}{2n}\right)^4 \text{ \&c.}
\end{aligned}$$

Now

$$\begin{aligned}
e^{-\frac{px}{n^2}} &= 1 - \frac{px}{n^2},\\
e^{-\frac{3px}{n^2}} &= 1 - \frac{3px}{n^2},\\
e^{-\frac{6px}{n^2}} &= 1 - \frac{6px}{n^2} \text{ \&c.}
\end{aligned}$$

whence it follows that it is necessary to add to the found formula,

$$\begin{aligned}
&-\frac{n^2}{2}e^{-\frac{2px}{n}}\frac{px}{n^2} + \frac{n^3}{2.3}e^{-\frac{3px}{n}}\frac{3px}{n^2} - \frac{n^4}{2.3.4}e^{-\frac{4px}{n}}\frac{6px}{n^2} \text{ \&c.} =\\
&-\frac{px}{2}e^{-\frac{2px}{n}}\left(1 - ne^{-\frac{px}{n}} + \frac{n^2}{2}e^{-\frac{2px}{n}} - \frac{n^3}{2.3}e^{-\frac{3px}{n}} \text{ \&c.}\right) =\\
&-\frac{px}{2}q^2e^{-nq}.
\end{aligned}$$

Thus the probability sought will be $e^{-nq}\left(1+\frac{p^2x}{2n}q - \left(\frac{n+px}{2}\right)q^2\right)$.

If one makes $p = 1$, one will have $e^{-nq}\left(1 + \frac{x}{2n}q - \frac{(n+x)}{2}q^2\right)$. It is that which Mr. de la Place finds in the *Mémoires de l'Académie des Sciences de Paris* for 1783[6] by a very profound analysis, drawn from the integral calculus in finite differences. It is remarkable that one can arrive to the same conclusions without integral calculus & by some quite simple considerations.

§ 15. Let for example, as Mr. de la Place supposes it $n = 10000$, $p = 1$, & let one demand the number of coups at the end of which one can wager even that all the numbers will have exited. We will have in this case to resolve the equation

$$e^{-nq}\left(1 + \frac{x}{2n}q - \frac{(n+x)}{2}q^2\right) = \frac{1}{2}.$$

Now[7] $q = e^{-\frac{x}{n}}$, therefore $-\frac{x}{n} = \ln q$, $x = -n\ln q$. For first approximation I make $e^{-nq} = \frac{1}{2}$, therefore $-nq = -\ln 2$, $q = \frac{\ln 2}{n} = \frac{0.6931472}{10000} = 0.00006931472, \ln q = -4.1591745$, $-\ln q = 4.1591745$. Multiplying this quantity by 2.302585, one will have $-\ln q = 9.576729$. Therefore $x = -\ln q = 95767.29$ first value. I substitute now these values of x & q into the formula $1 + \frac{x}{2n}q - \frac{(n+x)}{2}q^2$ & I have

$$\begin{aligned}
\ln q &= 5.8408255 & \ln q^2 &= 1.6816510\\
\ln x &= 4.9812173 & \ln \frac{x+n}{2} &= 4.7233213\\
\ln xq &= \overline{0.8220428} & \ln\left(\frac{x+n}{2}\right)q^2 &= \overline{6.4049723}\\
\ln 2n &= 4.3010300 & \left(\frac{x+n}{2}\right)q^2 &= 0.00025408\\
\ln \frac{x}{2n}q &= \overline{6.5210128} & &\\
\frac{x}{2n}q &= 0.00033190 & &\\
\left(\frac{x+n}{2}\right)q^2 &= 0.00025408 & &\\
\frac{x}{2n}q - \left(\frac{x+n}{2}\right)q^2 &= 0.00007782 & &
\end{aligned}$$

Therefore $1 + \frac{x}{2n}q - \frac{(n+x)}{2}q^2 = 1.00007782$. Therefore $e^{-nq} = \frac{1}{2(1.00007782)}$, $q = \frac{\ln(2.00015564)}{n} = 0.000069322548$, $-\ln q = 4.1591254$. Multiplying this quantity by 2.3025851, one will have $-\ln q = 9.5767402$. Therefore

$$x = -n\ln q = 95767.401.$$

The sought number is therefore a little nearer to 95767 than to 95768. Mr. de la Place arrives to the same conclusion at the end of his Memoir. One sees next that in this case

[6] "Suite du mémoire sur les approximations des Formules qui sont fonctions de très-grands nombres," *Mém. Acad. R. Sci. Paris* 1783 (1786), p. 423-467.

[7] *Translator's note*: Where Trembley writes log. hyp., I have written ln.

the first operation sufficed, so that the correction was superfluous, this which confirms the goodness of the method.

§ 16. One can find likewise an approximation for the second of the formulas found above, when x & n are very great numbers.

The probability that at the end of x drawings there will have exited at least $(n-1)$ numbers is

$$\begin{aligned}1 &- \frac{n(n-1)}{1.2}\left(\frac{(n-p)}{n}\frac{(n-p-1)}{n-1}\right)^x \\ &+ \frac{2n(n-1)(n-2)}{1.2.3}\left(\frac{(n-p)}{n}\frac{(n-p-1)}{n-1}\frac{(n-p-2)}{n-2}\right)^x \\ &- 3\frac{n\cdots(n-3)}{1\ldots4}\left(\frac{(n-p)}{n}\frac{(n-p-1)}{(n-1)}\frac{(n-p-2)}{(n-2)}\frac{(n-p-3)}{(n-3)}\right)^x \text{ \&c. } =\end{aligned}$$

(by being content with a first approximation & making $q = e^{-\frac{px}{n}}$)

$$1 - \frac{n^2}{2}q^2 + \frac{2n^3}{1.2.3}q^3 - \frac{3n^4}{2.3.4}q^4 \text{ \&c.}$$

by making $q = e^{-\frac{px}{n}}$. Let $nq = x$ one will have the sought probability

$$S = 1 - \frac{x^2}{2} + \frac{2x^3}{2.3} - \frac{3x^4}{2.3.4} \text{ \&c.}$$

Therefore

$$\frac{dS}{dx} = -\frac{x}{1} + \frac{x^2}{1} - \frac{x^3}{2} + \frac{x^4}{2.3} \text{ \&c.} = -x\left(1 - \frac{x}{1} + \frac{x^2}{2} - \frac{x^3}{3}\right) \text{ \&c.} = -xe^{-x}.$$

Therefore

$$\begin{aligned}dS &= -\,xdx\,e^{-x},\\ S &= xe^{-x} + e^{-x} + C.\end{aligned}$$

When $x = 0$, one has $S = 1$, therefore $C = 0$. Therefore

$$S = e^{-x}(1+x) = e^{-nq}(1+nq).$$

Thus instead as in the preceding case it was necessary for first approximation to resolve the equation $e^{-nq} = \frac{1}{2}$, it will be necessary to resolve the equation $e^{-nq}(1+nq) = \frac{1}{2}$. In order to come to end, I suppose first $e^{-nq} = \frac{1}{2}$, whence I draw as above $q = .00006931472$. Therefore

$$\begin{aligned}\ln q &= 5.8408255\\ \ln n &= 4.0000000\\ \ln nq &= \overline{9.8408255}\\ nq &= 0.693147\\ 1+nq &= 1.693147\end{aligned}$$

One has now $e^{-nq} = \frac{1}{2(1.693147)}$,

$$q = \ln\frac{3.386294}{n} = \frac{(0.5297247)(2.3025851)}{10000} = (0.00005297247)(2.3025851)$$

$$\begin{aligned} \ln 0.00005297247 =& 5.7240503 \\ \ln 2.3025851 =& 0.3622156 \\ \ln q =& \overline{6.0862659} \\ \ln n =& 4.0000000 \\ \ln nq =& 0.0862659 \end{aligned}$$

$q = 0.000121974,\ nq = 1.21974,\ 1 + nq = 7.21974,\ e^{-nq} = \frac{1}{2(2.21974)}$,

$$q = \frac{\ln(4.43948)}{n} = (0.00006473321)(2.3025851)$$

$$\begin{aligned} \ln 0.00006473321 =& 5.8111271 \\ \ln 2.3025851 =& 0.3622156 \\ \ln q =& \overline{6.1733427} \\ q =& 0.000149054. \end{aligned}$$

Therefore $nq = 1.49054,\ 1 + nq = 2.49054,\ e^{-nq} = \frac{1}{2(2.49054)}$,

$$q = \frac{\ln(4.98108)}{n} = (0.00006973236)(2.3025851).$$

$$\begin{aligned} \ln 0.00006973236 =& 5.8434344 \\ \ln 2.3025851 =& 0.3622156 \\ \ln q =& \overline{6.2056500} \\ q =& 0.000160565. \end{aligned}$$

Therefore $nq = 1.60565,\ 1 + nq = 2.60565,\ e^{-nq} = \frac{1}{2(2.60565)}$,

$$q = \frac{\ln(5.121130)}{n} = (0.00007093659)(2.3025851).$$

$$\begin{aligned} \ln 0.00007093659 =& 5.8508703 \\ \ln 2.3025851 =& 0.3622156 \\ \ln q =& \overline{6.2130859} \\ q =& 0.0001633375. \end{aligned}$$

Therefore $nq = 1.633375,\ 1 + nq = 2.633375,\ e^{-nq} = \frac{1}{2(2.63338)}$,

$$q = \frac{\ln(5.26675)}{n} = (0.00007215427)(2.3025851).$$

$$\begin{aligned} \ln 0.00007215427 =& 5.8582620 \\ \ln 2.3025851 =& 0.3622156 \\ \ln q =& \overline{6.2204776} \\ q =& 0.0001661415. \end{aligned}$$

Therefore $nq = 1.661415,\ 1 + nq = 2.661415,\ e^{-nq} = \frac{1}{2(2.661415)}$,

$$q = \frac{\ln(5.322830)}{n} = (0.00007261426)(2.3025851).$$

$$\begin{aligned}
\ln 0.00007261426 =& 5.8610219 \\
\ln 2.3025851 =& 0.3622156 \\
\ln q =& \overline{6.2232375} \\
q =& 0.0001672.
\end{aligned}$$

Therefore $nq = 1.672,\ 1 + nq = 2.672,\ e^{-nq} = \frac{1}{2(2.672)}$,

$$q = \frac{\ln(5.344)}{n} = (0.00007278664)(2.3025851).$$

$$\begin{aligned}
\ln 0.00007278664 =& 5.8620518 \\
\ln 2.3025851 =& 0.3622156 \\
\ln q =& \overline{6.2242674} \\
q =& 0.0001676.
\end{aligned}$$

Therefore $nq = 1.676,\ 1 + nq = 2.676,\ e^{-nq} = \frac{1}{2(2.676)}$,

$$q = \frac{\ln(5.352)}{n} = (0.00007285161)(2.3025851).$$

$$\begin{aligned}
\ln 0.00007285161 =& 5.8624391 \\
\ln 2.3025851 =& 0.3622156 \\
\ln q =& \overline{6.2246547} \\
q =& 0.000167747.
\end{aligned}$$

Therefore $nq = 1.67747,\ 1 + nq = 2.67747,\ e^{-nq} = \frac{1}{2(2.67747)}$,

$$q = \frac{\ln(5.35494)}{n} = (0.00007287546)(2.3025851).$$

$$\begin{aligned}
\ln 0.00007287546 =& 5.8625813 \\
\ln 2.3025851 =& 0.3622156 \\
\ln q =& \overline{6.2247969} \\
q =& 0.0001678,\ -\ln q = 3.7752031 \\
\ln 3.7752031 =& 0.5769404 \\
\ln 2.3025851 =& 0.3622156 \\
& \overline{0.9391560} \\
-\ln q =& 8.69273
\end{aligned}$$

$$x = -n \ln q = 86927.3.$$

One would have been able to find the limit much more rapidly by supposing q a little too great, but I have preferred to give the direct approximation without making any error.

§ 17. One will find likewise for the probability that at least $n-2$ numbers will have exited,

$$1 - \frac{n^3}{1.2.3}q^3 + \frac{3n^4}{1.2.3.4}q^4 - \frac{6n^5}{1.2.3.4.5}q^5 \text{ \&c. } =$$

(by making $nq = x$)

$$1 - \frac{x^3}{1.2.3} + \frac{3x^4}{1.2.3.4} - \frac{6x^5}{1.2.3.4.5} + \frac{10x^6}{1.2.3.4.5.6} \text{ \&c.} = S.$$

Therefore

$$\begin{aligned}
\left(\frac{dS}{dx}\right) &= -\frac{x^2}{1.2} + \frac{3x^3}{1.2.3} - \frac{6x^4}{1.2.3.4} + \frac{10x^5}{1.2.3.4.5} \text{ \&c.} \\
&= -\frac{x^2}{2}\left(1 - \frac{x}{1} + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} \text{ \&c.}\right) \\
&= -\frac{x^2}{2}e^{-x} \\
dS &= -\frac{x^2dx}{2}e^{-x}, \\
S &= \frac{x^2}{2}e^{-x} - \int x\,dx\,e^{-x} \\
&= \left(\frac{x^2}{2} + x + 1\right)e^{-x} \\
&= e^{-nq}\left(1 + nq + \frac{1}{2}n^2q^2\right) = \frac{1}{2}.
\end{aligned}$$

Finally to shorten the gropings, as q must be greater than in the preceding case, I make $q = 0.001$, & I have $nq = 10$, $\frac{n^2q^2}{2} = 50$, $1 + nq + n^2q^2 = 61$, $e^{-nq} = \frac{1}{122}$, $q = \frac{\ln 122}{n} = (0.000208636)(2.3025851)$,

$$\begin{aligned}
\ln 0.000208636 &= 6.3193893 \\
\ln 2.3025851 &= 0.3622156 \\
\ln q &= \overline{6.6816049} \\
q &= 0.000480402
\end{aligned}$$

I make $q = 0.0004$ & I have $nq = 4$, $\frac{n^2q^2}{2} = 8$, $1 + nq + \frac{n^2q^2}{2} = 13$, $e^{-nq} = \frac{1}{26}$, $q = \frac{\ln 26}{n} = (0.00014149733)(2.302851)$.

$$\begin{aligned}
\ln 0.00014149733 &= 6.1507453 \\
\ln 2.3025851 &= 0.3622156 \\
\ln q &= \overline{6.5129609} \\
q &= 0.00032581
\end{aligned}$$

I make $q = 0.0002$ & I have $nq = 2$, $\frac{n^2q^2}{2} = 2$, $1 + nq + \frac{n^2q^2}{2} = 5$, $e^{-nq} = \frac{1}{10}$, $q = \frac{\ln 10}{n} = 0.00023025851.$ I make $q = 0.00026$ & I have $nq = 2.6$, $\frac{n^2q^2}{2} = 3.38$, $1+nq+\frac{n^2q^2}{2} = 6.98$, $e^{-nq} = \frac{1}{13.96}$, $q = \frac{\ln 13.96}{n} = (0.00011448854)(2.3025851)$.

$$\begin{aligned}
\ln 0.0001144885 &= 6.0587619 \\
\ln 2.3025851 &= 0.3622156 \\
\ln q &= \overline{6.4209775} \\
q &= 0.00026362
\end{aligned}$$

§ 18. One approaches thus quite near to the value of q, but the approximation is less exact, because it is already raised to above the value of $\frac{1}{n}$.

§ 19. One would find likewise for the probability that there will have exited at least $n - 3$ numbers

$$S = \left(\frac{x^3}{2.3} + \frac{x^2}{2} + x + 1\right) e^{-x}$$

by making $x = nq$, & in general for the probability that there will have exited at least $n - \lambda$ numbers

$$S = \left(\frac{x^\lambda}{1 \dots \lambda} + \frac{x^{\lambda-1}}{1 \dots (\lambda - 1)} + x + 1\right) e^{-x}$$

but can not be exact as long as q is of order $\frac{1}{n}$.

There would be many considerations to make on the later approximations, but I myself could not deliver the detail which they require without lengthening this memoir too much.