Summation of a Compound Series, and its Application to a Problem in Probabilities

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The series proposed for solution in the following paper is—

$$\left.\begin{array}{rl} & (\overline{m-q}.\overline{m-q-1}\ldots\ldots m-\overline{q+p}+1)\times(1.2.3\ldots q)\\ + & (\overline{m-q-1}.\overline{m-q-2}\ldots m-\overline{q+p})\times(2.3.4\ldots q+1)\\ & \vdots \\ + & (p.p-1\ldots 1)\times(\overline{m-p}.\overline{m-p+1}\ldots m-\overline{p+q}+1)\end{array}\right\}\quad\text{(A)}$$

The law of this series is manifest. Each term is the product of two factorials, the first consisting of p, and the latter of q factors. And in each successive term, the factors of the first factorial are each diminished by one, and those of the latter increased by one.

Let there be a series $X_nY_1 + X_{n-1}Y_2 + \cdots X_1Y_n$ where $Y_2 = Y_1 + \Delta_1$, $Y_3 = Y_2 + \Delta_2 = Y_1 + \Delta_1 + \Delta_2$, and so on.

Then the series

$$\begin{aligned} &= X_n \times Y_1 \\ &+X_{n-1}\times\overline{Y_1+\Delta_1} \\ &+X_{n-2}\times\overline{Y_1+\Delta_1+\Delta_2}, \\ &\quad\text{\&c.} \\ &= \Sigma X_n\times Y_1 + \Sigma X_{n-1}\times\Delta_1 + \Sigma X_{n-2}\times\Delta_2 + \text{\&c} \end{aligned}$$

where ΣX_n means the sum of all the terms of X from X_1 to X_n inclusive.

Let us then, in the first place, take the differences of the second factorials—

$$\begin{array}{lll} -(1.2.3\ldots\ldots q) & +(2.3.4\ldots q+1) & = (2.3.4\ldots\ldots q).q \\ -(2.3.4\ldots q+1) & +(3.4.5\ldots q+2) & = (3.4.5\ldots q+1).q \\ \quad\text{\&c.} & & \quad\text{\&c.} \end{array}$$

Hence the sum of the whole series =

$$\left.\begin{array}{rl} & \Sigma(m-q.m-q-1\ldots\ldots m-\overline{p+q}+1).1.2.3\ldots\ldots q-1.q \\ + & \Sigma(m-q-1.\overline{m-q-2}\ldots\ldots\ldots\ldots m-\overline{p+q}).2.3.4\ldots\ldots q.q \\ + & \Sigma(m-q-2.m-q-3\ldots\ldots m-\overline{p+q}-1).3.4.5\ldots q+1.q \end{array}\right\}\quad\text{(B)}$$

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Integrating then each line separately, we have the sum

$$\left.\begin{array}{l} = \frac{q}{p+1} \cdot m-q+1.m-q\ldots\ldots m-\overline{p+q}+1 \times 1.2.3\ldots q-1 \\ +\frac{q}{p+1} \cdot m-q.m-q-1 \ldots\ldots\ldots\ldots\ldots m-\overline{p+q} \times 2.3.4\ldots q \\ +\frac{q}{p+1} \cdot m-q-1.m-q-2\ldots m-\overline{p+q}-1 \times .3.4.5\ldots q+1 \\ \qquad\qquad \text{\&c.} \qquad\qquad\qquad \text{\&c.} \end{array}\right\} \quad \text{(C)}$$

If again we treat this form as we have done the original, by taking the differences of the second factorials as they now stand, and again integrating, we reproduce the sum in the form

$$\left.\begin{array}{ll} & \frac{q.q-1}{p+1.p+2} \cdot m-q+2.m-q+1\ldots m-\overline{p+q}+1 \times 1.2.3\ldots q-2 \\ + & \frac{q.q-1}{p+1.p+2} \cdot m-q+1.m-q\ldots\ldots m-p+q \times 2.3.4\ldots q-1 \\ & \qquad\qquad \text{\&c.} \qquad\qquad\qquad \text{\&c.} \end{array}\right\} \quad \text{(D)}$$

It appears, then, that we may continue this differentiation on the one side q times, and integration on the other $q+1$ times; and that at each succeeding operation, an additional next lower factor will be introduced into the numerator of the fractional coefficient, and an additional next highest into the denominator. And after q differentiations, the last factorials will all become unity; and, the middle factorial having acquired an additional higher factor at each of $q+1$ integrations, we have for the sum of the series —

$$\left.\frac{q.q-1.q-2\ldots 1}{p+1.p+2\ldots p+q+1} \times m+1.m\ldots m-\overline{q+p}+1\right\} \quad \text{(E)}$$

II.

The Problem of Probabilities to which the foregoing summation is applicable, is the following:—

Suppose an experiment concerning whose inherent probability of success we know nothing, has been made $\overline{p+q}$ times, and has succeeded p times, and failed q times, what is the probability of success on the $\overline{p+q+1}^{\text{st}}$ trial.

This Problem is interesting, because it tends to the discovery of a rational measure for those expectations of success which constitute the motive for a large portion of human actions. The force of such expectations commonly depends, not upon reason, but upon temperament; and, according, as a man is naturally sanguine or the reverse, so in all the contingencies of life, does he over-estimate or under-estimate the chances in his favour.

It would be going much too far to think that we can give an algebraic formula, by the application of which a man may, in every practical case, correct his natural tendency to error, and arrive at a strictly rational amount of expectation. All that we can say is, that experience has led dispassionate men to come to nearly the same conclusion as the mathematician: for while he asserts the probability of success to be $\frac{p+1}{p+q+2}$, they act upon the supposition that the probabilities of success and failure are proportioned to the number of experienced cases of success and failure: and when either p or q is a large number, that is, when the experience is great, the conclusion and the supposition coincide.

In order to realise the Problem, we shall use the ordinary illustration, and suppose the bag contains m balls in unknown proportions of black and white, but all either black or white; that p white and q black balls have been drawn, and that it is required to find the probability of drawing a white at the $\overline{p+q+1}^{\text{th}}$ drawing.

The problem as thus stated, admits of four varieties.

1. m may be given, and the balls drawn may have been replaced in the bag.

2. m may be given, and the balls drawn not replaced.

3. m may be infinite or indefinite, and the balls replaced.

4. m may be infinite or indefinite, and the balls not replaced.

Of these, the 3*d* is the only case which I find solved in the treatises which I have consulted. I propose to solve the 2*d* case, and therein the 4*th*; and, in conclusion, to make an attempt at the solution of the 1*st* case.

To render the observed event, that is, the drawing of p white and q black balls (or E), possible, the original number of whites may have been any number from $m-q$ to p inclusive, and the number of blacks any number from q to $m-p$.

Let us call the hypothesis of $m-q$ white and q black, H_1
and $m-q-1$ white and $q+1$ black, H_2, &c.

Then H_1 gives for probability of E $\frac{m-q.m-q-1...m-q-p+1\times 1.2.3...q}{m.m-1...m-q-p+1}$ [1] or calling the denominator A,

$$\left.\begin{array}{ll} H_1 \text{ gives } \frac{1}{A}\cdot m-q.m-q-1\ldots m-q-p+1\times 1.2.3\ldots q & (\alpha) \\ H_2 \text{ gives } \frac{1}{A}\cdot m-q-1.m-q-2\ldots m-q-p\times 2.3.4\ldots q+1 & (\beta) \\ H_3 \text{ gives } \frac{1}{A}\cdot m-q-2.m-q-3\ldots m-q-p-1\times 3.4\ldots q+2 & (\gamma) \end{array}\right\} \text{(F)}$$

Now, $\alpha+\beta+\gamma$, &c. by the former proposition (E)

$$=\frac{1}{A}\cdot\frac{q.q-1\ldots 1}{p+1.p+2\ldots p+q+1}\times m+1.m\ldots m-p-q+1$$

$\therefore$ probability of $H_1=\frac{\alpha}{\alpha+\beta+\gamma,\text{ \&c.}}$

$$=\frac{p+1.p+2\ldots p+q+1}{m+1.m\ldots m-p-q+1\times 1.2.3\ldots q}\times(m-q.m-q-1\ldots m-q-p)\times(1\times 1.2.3\ldots q)$$

But the probability of a white at $\overline{p+q+1}^{\text{th}}$ drawing on H_1 is $\frac{m-p-q}{m-p-q}$. $\therefore$ probability of white derived from H_1 is

$$\frac{p+1.p+2\ldots p+q+1}{m+1.m\ldots m-p-q+1\times 1.2.3\ldots q}\times(m-q.m-q-1\ldots m-q-p)\times(1.2.3\ldots q)\quad\text{(G)}$$

[1] The coefficient (U of GALLOWAY's Treatise), expressing the number of different ways in which p white and q black balls can be combined in $p+q$ trials, is here omitted. This is immaterial, as it disappears in the expression $\frac{\alpha}{\alpha+\beta+\gamma,\text{ \&c.}}$.

So probability from H_2

$$\frac{p+1\ldots p+q+1}{m+1.m\ldots m-p-q\times 1.2.3\ldots q}\times(m-q-1.m-q-2\ldots m-q-p-1)\times(2.3\ldots\overline{q+1})$$

And so for all the other hypotheses in succession.

Now this series, omitting for the present the consideration of the fraction which is a factor common to them all, is a series of the same form as that summed in the last proposition, only that now $p+1$ must be substituted for p.

We have therefore the whole probability of a white at $\overline{p+q+1}^{\text{th}}$ drawing

$$= \left.\begin{array}{l}\frac{p+1.p+2\ldots p+q+1}{m+1.m\ldots m-p-q\times 1.2\ldots q}\times\frac{1.2\ldots q}{p+2\ldots p+q+2}\\ \times m+1.m\ldots m-p-q=\frac{p+1}{p+q+2}\end{array}\right\} \qquad \text{(H)}$$

Note.—It may be worth observing, that, had we summed the original series in Prop. 1 upwards instead of downwards, we should have got for a first factor $\frac{1.2.3\ldots p}{q+1.q+2\ldots p+q+1}$, which must therefore $=\frac{1.2.3\ldots q}{p+1.p+2\ldots p+q+1}$. And that these fractions are equal may be proved independently, for if we divide each by $1.2.3\ldots p\times 1.2.3\ldots q$, we have on both sides the same quotient $\frac{1}{1.2.3\ldots p+q+1}$.

There now remains for solution only the first case of the problem in chances, that is, to find the probability of drawing a white ball, when m the number of balls is given, and p white and q black have already been drawn and returned.

The main object in this case is to sum the series

$$\overline{m-1}^p\times 1^q+\overline{m-2}^p\times 2^q\ldots 1^p.\overline{m-1}^q \qquad \text{(I)}$$

This may be done much as in the preceding case, by taking the successive differences of the right-hand factors till the differences vanish, and multiplying the successive terms of the last or $\overline{q+1}^{\text{st}}$ row of differences into the $\overline{q+1}^{\text{st}}$ summation of the successive terms of the series $(1+2^p\cdots+\overline{m-1}^p)+(1+2^p\cdots+\overline{m-2}^p)$, &c.

This may be sufficiently explained by going through the operation in a low particular case. Let $p=2$, $q=3$.

Then the series written perpendicularly is

$$\begin{array}{lllll}
\overline{m-1}^2\times 1 & \Sigma_1\overline{m-1}^2\times 1 & \Sigma_2\overline{m-1}^2\times 1 & \Sigma_3\overline{m-1}^2\times 1 & \Sigma_4\overline{m-1}^2\times 1\\
\overline{m-2}^2\times 8 & \Sigma_1\overline{m-2}^2\times 7 & \Sigma_2\overline{m-2}^2\times 6 & \Sigma_3\overline{m-2}^2\times 5 & \Sigma_4\overline{m-2}^2\times 4\\
\overline{m-3}^2\times 27 = & \Sigma_1\overline{m-3}^2\times 19 = & \Sigma_2\overline{m-3}^2\times 12 = & \Sigma_3\overline{m-3}^2\times 6 = & \Sigma_4\overline{m-3}^2\times 1\\
\overline{m-4}^2\times 64 & \Sigma_1\overline{m-4}^2\times 37 & \Sigma_2\overline{m-4}^2\times 18 & \Sigma_3\overline{m-4}^2\times 6 & \\
\overline{m-5}^2\times 125 & \Sigma_1\overline{m-5}^2\times 61 & \Sigma_2\overline{m-5}^2\times 20 & \Sigma_3\overline{m-5}^2\times 6 & \\
\quad\&\text{c.} & \quad\&\text{c.} & \quad\&\text{c.} & \quad\&\text{c.} &
\end{array}$$

The value of the different sigmas is easily found by the method of finite differences.

Generally, since the differences of 1^q, 2^q, 3^q, &c., always vanish in the $\overline{q+1}^{\text{th}}$ line and after th q^{th} term of it, the general expression is

$$\Sigma_{q+1}\overline{m-1}^p+d_2\Sigma_{q+1}\overline{m-2}^p\cdots d_q\Sigma_{q+1}\overline{m-q}^p;$$

d_1, d_2, d_3, &c., signifying the 1st, 2d, 3d, &c., terms of the $\overline{q+1}^{\text{th}}$ row of differences.

This summation may be applied to find the probability in the case now under consideration, for it expresses the $\alpha+\beta+\gamma$, &c., of the preceding case. Applying it as we did the value of $\alpha+\beta+\gamma$, &c., there found, we shall find the probability of a white ball at the $\overline{p+q+1}^{\text{th}}$ trial to be

$$\frac{\Sigma_{q+1}\overline{m-1}^{p+1} + d_2\Sigma_{q+1}\overline{m-2}^{p+1}\cdots d_q\Sigma_{q+1}\overline{m-q}^{p+1}}{\Sigma_{q+1}\overline{m-1}^{p} + d_2\Sigma_{q+1}\overline{m-2}^{p}\cdots d_q\Sigma_{q+1}\overline{m-q}^{p}} \qquad \text{(K)}$$

If m be infinite, the expression becomes

$$\frac{(1+d_2+\cdots d_q).\Sigma_{q+1}m^{p+1}}{m(1+d_2\cdots d_q).\Sigma_{q+1}m^p} = \frac{\Sigma_{q+1}m^{p+1}}{m\Sigma_{q+1}m^p}$$

But if x be a quantity varying between the limits 0, x,

$$\frac{\Sigma_1 m^{p+1}}{m\Sigma_1 m^p} = \frac{\int_x^0 x^{p+1}dx}{x\int_x^0 x^p dx} = \frac{p+1}{p+2}\cdot\frac{x^{p+2}}{x.x^{p+1}}$$

And by continuation

$$\frac{\Sigma_{q+1}m^{p+1}}{m\Sigma_{q+1}m^p} = \frac{p+1.p+2\ldots p+q+1}{p+2.p+3\ldots p+q+2} = \frac{p+1}{p+q+2} \qquad \text{(L)}$$

We have thus found the probability in every case of the problem; the 2d and 4th at H, for the result being independent of m, must be true for an infinite as well as a finite number. The 1st case is solved at K, and the 3rd at L.