The Law of Large Numbers

Sections extracted from

Researches sur la probabilité des jugements en matière criminelle et en matière civile *

Simon Denis Poisson

1837

pp. 138–145

§52 In a very great number μ of consecutive trials, we represent the chance of the event E of any nature, by p_1 in the first trial, by p_2 in the second,... by p_μ in the last. Let also p' be the mean of all these chances, or their sum divided by their number, that is,

$$p' = \frac{1}{\mu}(p_1 + p + 2 + p_3 + \cdots + p_\mu);$$

at the same time, the mean chance of the contrary event F will be the sum of the fractions $1 - p_1$, $- - p_2$, $\ldots 1 - p_\mu$, divided by μ; and by designating it by q', one will have $p' + q' = 1$. This being, the one of the general propositions, which we wish to consider, consists in this that if one calls m and n the numbers of times that E and F will arrive or are arrived during the series of these trials, the ratios of m and n to the total number μ or $m+n$, will be, very nearly and with a very great probability, the values of the mean chances p' and q', and reciprocally, p' and q' will be the approximate values of $\frac{m}{\mu}$ and $\frac{n}{\mu}$.

When these ratios will have been deduced from a long series of trials, they will make known therefore the mean chances p' and q', likewise they determine, by the rule of n° 49, the same chances p and q of E and F, when they are constants. But in order that these approximate values of p' and q' are able to serve, also by approximation, to evaluate the numbers of times that E and F will arrive in a new series of a great number of trials, it is necessary that it be certain, or at least very probable, that the mean chances of E and F will be exactly, or quite nearly the same, for this second series, and for the first. Now, it is this which holds effectively by virtue of another general proposition of which here is the enunciation.

I suppose that by the nature of the events E and F, the one which will arrive at each trial is able to be due to one of the causes C_1, C_2, C_3, $\ldots C_\nu$, of which ν is the number, which is mutually exclusive, and which I will regard first as equally possible.

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 4, 2010

I designate by c_1 the chance that any cause C_i will give to the arrival of the event E; in a manner that at each determined trial, at the first, for example, the chance of E is c_1 when it will be the cause C_1 which will intervene, c_2 when it will be C_2, etc. If there was only a single possible cause, the chance of E would be necessarily the same in all the trials; but under our hypothesis, it will be susceptible, at each trial, of a number ν of equally probable values, and will vary, in consequence, from one trial to another. Now, if one makes

$$\gamma = \frac{1}{\nu}(c_1 + c_2 + c_3 + \cdots + c_\nu),$$

the sum of the chances that E will have had, in a very great number of trials already effected, or that this event will have in a long series of future trials, divided by their number, will be, very nearly and very probably, equal to the fraction γ, of which the magnitude is independent of this number; consequently, the mean chance p' of E will be able to be regarded as being the same in two or many series, of which each will be composed of a very great number of trials.

By combining this second general proposition with the first, one concludes from it that if m is the number of times that the event E will arrive or is arrived in a very great number μ of trials, and m' in another very great number μ', one will have, very nearly and very probably,

$$\frac{m}{\mu} = \frac{m'}{\mu'}.$$

These two ratios would be rigorously equal between them, and to the unknown quantity γ, if the numbers μ and μ' would be able to be infinite. When their values given by observation will differ notably from one another, there will be place to think that in the interval of the two series of trials, some of the causes C_1, C_2, C_3, etc., will have ceased to be possible, and that some others will be become it; this which will have changed the chances c_1, c_2, c_3, etc., and hence the value of γ. However, this change will not be certain, and we will give in the following the expression of its probability, as function of the observed difference $\frac{m'}{\mu'} - \frac{m}{\mu}$, and of the number of trials μ and μ'.

One will make this consequence return from two preceding propositions, in the same theorem of Jakob Bernoulli, by observing that under the hypothesis on which the second is founded, the fraction γ is the chance of E, unknown, but constant during the two series of trials. Indeed, this event is able to happen at each trial, by virtue of each of the causes C_1, C_2, C_3, etc., which have each one same probability $\frac{1}{\nu}$; the chance of its arrival by virtue of any cause C_i will be the product $\frac{1}{\nu}c_i$, according to the rule of n^o 5; and according to that of n^o 10, its complete chance will have for value the sum of the products $\frac{1}{\nu}c_1$, $\frac{1}{\nu}c_2$, $\frac{1}{\nu}c_3$, etc., equal to the quantity γ.

For more simplicity, we have regarded all the causes C_1, C_2, C_3, etc., as equally possible; but one is able to suppose that each of them enter one or many times in their total number ν; this which will render them unequally probable. One will designate then by $\nu\gamma_i$ the number of times that any cause C_i will be repeated in this number ν; the fraction γ_i will express the probability of this cause; and the expression of γ will become

$$\gamma = \gamma_1 c_1 + \gamma_2 c_2 + \gamma_3 c_3 + \cdots + \gamma_\nu c_\nu.$$

One will have, at the same time,

$$\gamma_1 + \gamma_2 + \gamma_3 + \cdots + \gamma_\nu = 1,$$

since one of the causes to which these probabilities are referred, must have place certainly at each trial. Since the number of possible causes will be infinite, the probability of each of them will become infinitely small; by representing, in this case, by x one of the chances c_1, c_2, c_3, ..., c_ν, of which the value will be able to be extended from $x = 0$ to $x = 1$, and by Ydx, the probability of the cause which gives this chance any x to the event E, one will have, as in n° 45,

$$\gamma = \int_0^1 Yxdx, \qquad \int_0^1 Ydx = 1.$$

§53. We suppose actually that instead of two possible events E and F, there are a given number λ, of which one alone must arrive at each trial. This case is the one where one considers a thing A of any nature, susceptible of a number λ of values, known or unknown, which I will represent by a_1, a_2, a_3, ..., a_λ, and among which one alone must take place at each trial, so that that which is arrived or which will arrive will be, in this question, the observed event or the future event. Let also $c_{i,'nu}$ be the chance that the cause C_i, if it was certain, would give to the value a_ν of A. The values of $c_{i,i'}$, relative to the diverse indices i and i', from $i = 1$ to $i = \nu$ and from $i' = 1$ to $i' = \lambda$, will be known or unknown; but for each index i', one must have

$$c_{i,1} + c_{i,2} + c_{i,3} + \cdots + c_{i,\lambda} = 1;$$

for if the cause C_i was certain, one of the values $a_1, a_2, \ldots a_\lambda$, would certainly arrive by virtue of its cause. We designate, beyond, by α_ν, the sum of the chances of a_ν, which will have or which have taken place in a very great number μ of consecutive trials, divided by this number, that is, the mean chance of this value a_μ of A, in this series of experiences. By considering a_ν as an event E, and the set of the $\lambda - 1$ other values of A as the contrary event F, one will be able to take, according to the second general proposition of the preceding number,

$$\alpha_\nu = \gamma_1 c_{1,\nu} + \gamma_2 c_{2,\nu} + \gamma_3 c_{3,\nu} + \cdots + \gamma_\nu c_{\nu,\nu};$$

γ_1, γ_2, γ_3, $\ldots \gamma_\nu$, being always the probabilities of the diverse causes which are able to bring forth the events during the series of trials, or otherwise said, which are able to produce the values of A that one has observed or that one will observe. This put, the third general proposition which remains for us to make known, consists in this that the sum of these μ values of A, divided by their number, or the mean value of this thing, will differ very probably quite little from the sum of all the possible values, multiplied respectively by their mean chances. Thus, by calling s the sum of the effective values of A, one will have, very nearly and with a great probability,

$$\frac{s}{\mu} = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \cdots + a_\nu\alpha_\nu;$$

in such a way that if one designates by δ a fraction as small as one will wish, one will be able to suppose always the number μ great enough in order to render as little different as one will wish from unity, the probability that the difference of the two numbers of this equation will be less than δ. We observe moreover that according to the preceding expression of α_ν and of the values of α_1, α_2, α_3, etc., which are deduced from them, the second member is independent of μ; when this number is very great, the sum s is therefore sensibly proportional to it; consequently, if one represents by s' the sum of the values of A in another series of a great number μ' of trials, the difference of the ratios $\frac{s}{\mu}$ and $\frac{s'}{\mu'}$ will be very probably quite small; and by neglecting it, one will have

$$\frac{s}{\mu} = \frac{s'}{mu'}.$$

In the greater part of the questions, the number λ of the possible values of A is infinite; they increase by infinitely small degree, and are comprehended between some given limits; and the probability that any cause C_i gives to each of these values becomes, consequently, infinitely small. By representing these limits by l and l', and by $Z_i dz$ the chance that C_i will give to any value z, which will be able to be extended from $z = l$ to $z = l'$, one will have

$$\int_l^{l'} Z_i dz = 1;$$

the total chance of this value z, or very nearly its mean chance during the series of trials, will be Zdz, by making, for brevity,

$$\gamma_1 Z_1 + \gamma_2 Z_2 + \gamma_3 Z_3 + \cdots + \gamma_\nu Z_\nu = Z;$$

and there will result from it

$$\frac{s}{\mu} = \int_l^{l'} Zz dz.$$

The quantity Z will be a function known or unknown of z; but the sum of the fractions γ_1, γ_2, γ_3, etc., being unity, in the same way each of the integrals $\int_l^{l'} Z_1 dz$, $\int_l^{l'} Z_2 dz$, $\int_l^{l'} Z_3 dz$, etc., one will have always

$$\int_l^{l'} Z dz = 1,$$

either if there is only a limited number ν of possible causes, or if there is an unlimited number, or if one has $\nu = \infty$.

§54. Now, the law of large numbers resides in these two equations

$$\frac{m}{\mu} = \frac{m'}{\mu'}, \qquad \frac{s}{\mu} = \frac{s'}{\mu'},$$

applicable to all the cases of eventuality of physical things and of moral things. It has two different significations of which each corresponds to one of these equations, and which both are verified constantly, as one is able to see it by the varied examples that I have cited in the preamble of this work. These examples of each specie were able to leave no doubt on its generality and its exactitude; but it was good, because of the importance of this law, that it was demonstrated *a priori*; because it is the necessary base of the applications of the calculus of probabilities, which interest us most; and besides its demonstration, founded on the propositions of the preceding two sections, has the advantage of making known to us the reason itself of its existence.

By virtue of the first equation, the number m of times that an event E, of any nature, takes place in a very great number μ of trials, is able to be regarded as proportional to μ. For each nature of a thing, the ratio $\frac{m}{\mu}$ has a special value γ, which it would attain rigorously, if μ was to become infinite; and the theory shows us that this value is the sum of the possible chances of E at each trial, multiplied respectively by the probabilities of the causes which correspond to them. That which characterizes the set of these causes, is the relation which exists for each of them between its probability and the chance tat it would give, if it were certain, at the arrival of E. As much as this law of probability does not change, we observe the permanence of the ratio $\frac{m}{\mu}$, in the diverse series composed of a great number of trials; if on the contrary, between two series of trials, this law has changed, and if there is resulted from it in mean chance γ, a notable change, we will be cautioned by a similar change in the value of $\frac{m}{\mu}$: when, in the interval of two series of observations, some circumstances will have rendered more probable the causes, physical or moral, which give the greatest chances to the arrival of E, there will result an increase in the value of γ in this interval, and the ratio $\frac{m}{\mu}$ will be found greater in the second series than it was in the first; the contrary will happen, when the circumstances will have increased the probabilities of the causes which give the least chances to the arrival of E. By the nature of this event, if all its possible causes are equally probable, one will have $Y = 1$ and $\gamma = \frac{1}{2}$; and very probably, the number of times that E will arrive in a long series of trials will deviate very little from the mean of their number. Likewise, if the causes of E have some probabilities proportional to the chances that these causes give to its arrival, and if their number is again infinite, one will have $Y = ax$; in order that the integral $\int_0^1 Y dx$ be unity, it is necessary that one have $a = 2$; there will result therefore $\gamma = \frac{2}{3}$; consequently in a long series of trials, there will be a probability very close to certitude that the number of arrivals of E will be very nearly double of that of the arrivals of the contrary event. But in the greater part of the questions, the law of probability of causes is unknown to us, the mean chance γ is not able to be calculated *a priori*, and it is experience which gives the approximate and very probable value of it, by prolonging the series of trials far enough in order that the ratio $\frac{m}{\mu}$ becomes sensibly invariable, and taking then this ratio for that value.

The near perfect invariability of this ratio $\frac{m}{\mu}$ for each nature of events, is a fact well worthy of note, if one considers all the variations of the chances during a long series of trials. One would know to try to attribute to the intervention of a hidden power, distinct from physical or moral causes of events, and acting in some view of order or conservation; but the theory shows us that this permanence holds necessarily as much as the law of probability of the causes, relative to each specie of events, come not to change at all; so that one must regard it, in each case, as being the natural state of things, which subsist of themselves without the help of any outside cause, and would have, on the contrary, need of a similar cause in order to experience a notable change. One is able to compare it to the state of repose of bodies, which subsist by virtue of only the inertia of matter as long as no outside cause comes to disturb it.

CHAPTER IV

Continuation of the calculus of probabilities which depend on very large numbers.
pp. 246–254

§94. We are now going to occupy ourselves with formulas relative to chance variables, this which will lead us to demonstrate the three general propositions enunciated in n^{os} 52 and 53, and of which we have concluded the *law of large numbers*.

We consider a series of μ or $m+n$ successive trials, during which the chances of the two contrary events E and F vary in any manner whatsoever. We designate these chances by p_1 and q_1 at the first trial, by p_2 and q_2 at the second, by p_μ and q_μ at the last; so that one has

$$p_1+q_1=1, \quad p_2+q_2=1, \ldots p_\mu+q_m u=1.$$

We call U the probability that E and F will arrive according to a any order, m times and n times. According to the rule of n^o 20, U will be the coefficient of $u^m v^n$ in the development of the product

$$(up_1+vq_1)(up_2+vq_2)\cdots(up_\mu+vq_\mu).$$

Now, if one makes

$$u=e^{x\sqrt{-1}}, \qquad v=e^{-x\sqrt{-1}},$$

the term Uu^mv^n of this product will become $Ue^{(m-n)x\sqrt{-1}}$, and all the other terms will contain some exponentials different from $e^{(m-n)x\sqrt{-1}}$, whence one concludes that by designating this product by X, by multiplying it, in the same way its development, by $e^{(m-n)x\sqrt{-1}}dx$, and integrating next from $x=-\pi$ to $x=\pi$, all these other terms will vanish, and one will have simply

$$\int_{-\pi}^{\pi} Xe^{(m-n)x\sqrt{-1}} = 2\pi U;$$

this which results from this that if i and i' express two whole numbers, positive, negative or zero, of which the first will be $i=m-n$, one will have

$$\int_{-\pi}^{\pi} e^{i'x\sqrt{-1}}e^{-ix\sqrt{-1}} = \int_{-\pi}^{\pi}[\cos(i'-i)x+\sin(i'-i)x\sqrt{-1}]dx=0,$$

when i and i' will differ from one another, and, in particular,

$$\int_{-\pi}^{\pi} e^{ix\sqrt{-1}}e^{-ix\sqrt{-1}} = 2\pi,$$

in the case of $ii=i$.

We will have, at the same time,

$$up_1+vq_1=\cos x+(p_1-q_1)\sin x\sqrt{-1};$$

and if we make

$$\cos^2 x+(p_i-q_i)^2\sin^2 x=\rho_i^2,$$

there will be a real angle r_i, such that one has

$$\frac{1}{\rho_i}\cos x = \cos r_i, \qquad \frac{1}{\rho_i}(p_i - q_i)\sin x = \sin r_i;$$

whence there will result

$$up_i + vq_i = \rho_i e^{r_i\sqrt{-1}}.$$

The sign ρ_i will be ambiguous; in order to fix the ideas, we will regard this quantity as positive. By making, for brevity,

$$\rho_1\rho_2\rho_3\ldots\rho_\mu = Y,$$
$$r_1 + r_2 + r_3 + \cdots + r_m u = y,$$

the product designated by X will become

$$X = Ye^{y\sqrt{-1}};$$

and we will have, consequently,

$$U = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y\cos[y-(m-n)x]dx + \frac{\sqrt{-1}}{2\pi}\int_{-\pi}^{\pi} Y\sin[y-(m-n)x]dx.$$

For the values of x equal and of contrary sign, the values of r_i will be also, and those of ρ_i will be equal; consequently, the second definite integral will vanish, as being composed of elements equal two by two and of contrary signs; and that must be, indeed, since U is a real quantity. For some angles x supplements of one another, the angles r_i will be equally so, according to the expressions of $\cos r_i$ and $\sin r_i$; the sum of the two values of $y-(m-n)x$ which will correspond to them, will be therefore $\mu\pi - (m-n)\pi$ or $2n\pi$, and consequently the cosine of $y-(m-n)x$ will not change: it will be likewise in regard to the values of Y; so that the elements of the first definite integral, corresponding to x and $\pi - x$, will be equal, as well as those which correspond to x and $-x$. By suppressing therefore the second integral, reducing the limits of the first to zero and $\frac{1}{2}\pi$, and quadrupling the result, we will have simply

$$U = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} Y\cos[y-(m-n)x]dx. \tag{1}$$

The indicated integration will be executed always under finite form, by the ordinary rules. But when μ will not be a large number, this formula will not be able to be of any use in order to calculate the value of U; when, on the contrary, this number will be very great, one will deduce from this formula, as one just saw, a value of U as close as one will wish.

§95. Each of the factors Y are reduced to unity for $x = 0$, and is less than unity for each other value of x, comprehended between the limits of integration; it follows that when μ will be a very great number, this product will be generally a very small quantity, for all the values of x which will not be small, and that Y would vanish, for all the finite values of x, if μ became infinite. There would be one exception only if the factors of Y converged indefinitely toward unity; for one knows that the product of an infinite number of similar factors, are able to have for value a quantity of finite magnitude. Because of

$$\rho_i^2 = 1 - 4p_iq_i \sin^2 x,$$

this circumstance would suppose that one of the chances of the two events E and F, or their product p_iq_i decreased indefinitely during the series of trials. By excluding this particular case, one will be able therefore, in the case where μ is a very great number, to consider the variable x as a very small quantity, and to neglect the part of the preceding integral, which corresponds to the other values of x.

By developing then according to the powers of x^2, one will have, in a very convergent series,

$$\rho_i = 1 - 2p_iq_ix^2 + \left(\frac{2}{3}p_iq_i - 2p_i^2q_i^2\right)x^4 - \text{etc.},$$

and, consequently,

$$\log \rho_i = -2p_iq_ix^2 + \left(\frac{2}{3}p_iq_i - 4p_i^2q_i^2\right)x^4 - \text{etc.};$$

whence one concludes

$$\log Y = -\mu k^2x^2 + \mu\left(\frac{1}{3}k^2 - k'^2\right)x^4 - \text{etc.},$$

by making, for brevity,

$$2\Sigma p_iq_i = \mu k^2, \qquad 4\Sigma p_i^2q_i^2 = \mu k'^2, \qquad \text{etc.},$$

and extending the sum Σ from $i = 1$ to $i = \mu$. If one makes also

$$x = \frac{s}{\sqrt{\mu}},$$

that one considers the new variable z as a quantity very small with respect to $\sqrt{\mu}$, and that one neglects the quantities of order of smallness of $\frac{1}{\mu}$, there will result from it

$$Y = e^{-k^2z^2}.$$

According to the values of ρ_i and of sin r_i, one will have likewise

$$r_i = (p_i - q_i)x + \frac{4}{3}(p_i - q_i)p_iq_ix^2 + \text{etc.}$$

I will designate by p and q the mean chances of E and F during all the series of trials, so that one has

$$p = \frac{1}{\mu}\Sigma p_i, \qquad q = \frac{1}{\mu}\Sigma q_i, \qquad p + q = 1;$$

the sum Σ extending always from $i = 1$ to $i = \mu$. I will make also, for brevity,

$$\frac{4}{3\mu}\Sigma(p_i - q_i)p_iq_i = h.$$

By conserving only the quantities of order of smallness of $\frac{1}{\sqrt{\mu}}$, one will deduce from it first

$$y = z(p-q)\sqrt{\mu} + \frac{z^3 h}{\sqrt{\mu}},$$

and next

$$\cos[y-(m-n)x] = \cos(zg\sqrt{\mu}) - \frac{z^3 h}{\sqrt{\mu}} \sin(zg\sqrt{\mu}),$$

where one makes, for brevity,

$$p - \frac{m}{\mu} - \left(q - \frac{n}{\mu}\right) = g.$$

I substitute these values of Y and $\cos[y-(m-n)x]$ into formula (1), and I put $\frac{1}{\sqrt{mu}}dz$ instead of dx; there comes

$$U = \frac{2}{\pi\sqrt{\mu}} \left[\int e^{-k^2 z^2} \cos(zg\sqrt{\mu}) - \frac{h}{\sqrt{\mu}} \int e^{-k^2 z^2} z^2 \sin(zg\sqrt{\mu} dz \right].$$

The case where the values of p_i and q_i would decrease indefinitely having been excluded, k^2 is not able to be a very small quantity; for of the values of z comparable to $\sqrt{\mu}$, the exponential $e^{-k^2 z^2}$ will be therefore insensible; and although one must give to this variable only some very small values with respec to $\sqrt{\mu}$, one will be able now, without altering sensibly the integral, to extend beyond this limit, and to take it, if one wishes, from $z = 0$ to $z = \infty$. According to the known formula, one will have also

$$\int_0^\infty e^{-k^2 z^2} (\cos zg\sqrt{\mu}) dz = \frac{\sqrt{\pi}}{2k} e^{-\frac{\mu g^2}{4k^2}};$$

by differentiating successively with respect to g and to k, one deduces from it

$$\int_0^\infty e^{-k^2 g^2} z^3 (\sin zg\sqrt{\mu}) dz = \frac{g\sqrt{\pi\mu}}{8k^5} \left(3 + \frac{\mu g^2}{2k^2}\right) e^{-\frac{\mu g^2}{4k^2}};$$

and by means of these values, that of U becomes

$$U = \frac{1}{k\sqrt{\pi\mu}} e^{-\frac{\mu g^2}{4k^2}} - \frac{gh}{4k^5\sqrt{\pi\mu}} \left(3 + \frac{\mu g^2}{2k^2}\right) e^{-\frac{\mu g^2}{4k^2}}$$

By reason of the exponential $e^{-\frac{\mu g^2}{4k^2}}$, this probability will be insensible since g will not be of the order of a fraction $\frac{1}{\sqrt{\mu}}$; but because of $p+q=1$ and $m+n=\mu$, this quantity g is not able to be of this order of smallness, unless this holds separately for $p - \frac{m}{\mu}$ and $q - \frac{n}{\sqrt{\mu}}$, which are besides some quantities equal and of contrary sign; if therefore one makes

$$p - \frac{m}{\mu} = \frac{k\theta}{\sqrt{\mu}}, \qquad q - \frac{n}{\sqrt{\mu}} = -\frac{k\theta}{\sqrt{\mu}}, \qquad g = \frac{2k\theta}{\sqrt{\mu}},$$

the probability U will have sensible values only for the values of θ, positive, negative or zero, but very small with respect to $\sqrt{\mu}$, and there will result from it finally

$$U = \frac{1}{k\sqrt{\pi\mu}} e^{-\theta^2} - \frac{h\theta}{2k^4\mu\sqrt{\pi}} \left(3 + 2\theta^2\right) e^{-\theta^2}, \tag{2}$$

for the probability that the numbers m and n will have for values

$$m = p\mu - \theta k\sqrt{\mu}, \qquad n = q\mu + \theta k\sqrt{\mu},$$

that is, some values which will deviate very little from being proportional to the mean chances p and q and to the number μ of trials.

§96. In order that m and n be from the whole numbers, it will be necessary that θ be a multiple of $\frac{1}{k\sqrt{\mu}}$ or zero. By making $\theta = 0$ in formula (2), one will have $\frac{1}{k\sqrt{\pi\mu}}$ for the probability that m and n will be precisely between those as p and q. In designating by t a positive quantity, multiple of $\frac{1}{k\sqrt{\mu}}$; making successively in this formula $\theta = -t$ and $\theta = t$; and adding the two results, their sum $\frac{2}{k\sqrt{\pi\mu}} e^{-t^2}$ will express the probability that m will be one of the two numbers $p\mu \pm kt\sqrt{\mu}$, and n one of the two numbers $q\mu \pm kt\sqrt{\mu}$. Let

$$\frac{1}{k\sqrt{\mu}} = \delta;$$

we designate by u a given multiple of δ; we make successively, in the preceding sum, $t = \delta, = 2\delta, = 3\delta \ldots$ to $t = u$; we represent by R the sum of the results, increased by the value of U which corresponds to $\theta = 0$; we will have

$$R = \frac{1}{k\sqrt{\pi\mu}} + \frac{2}{\sqrt{\pi\mu}} \Sigma e^{-t^2},$$

for the probability that the numbers m and n would be comprehended between the limits

$$p\mu \mp uk\sqrt{k\mu}, \qquad q\mu \pm uk\sqrt{\mu},$$

or equal to one of them.

The sum Σ will be referred to the values of t comprehended from $t = \delta$ to $t = u$, and increasing by the differences equal to δ; but one will be able to replace it by the difference of the sums of e^{-t^2}, taken from $t = \delta$ to $t = \infty$, and from $t = u + \delta$ to $t = \infty$. By means of the formula of Euler, already employed in n° 91, this last sum, multiplied by δ, will have for value,

$$\int_u^\infty e^{-t^2} dt - \frac{\delta}{2} e^{-u^2},$$

to the degree of approximation where we ought to stop ourselves, that is by neglecting the square of δ. Consequently, if one subtracts from this last quantity the preceding, and if one divides by δ, one will have

$$\Sigma e^{-t^2} = \frac{1}{2\delta}\sqrt{\pi} - \frac{1}{\delta}\int_u^\infty e^{-t^2} dt - \frac{1}{2} + \frac{1}{2} e^{-u^2},$$

for the sum comprehended in the expression of R, and by having regard to the value of δ, this expression will become

$$R = 1 - \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-t^2} dt + \frac{1}{h\sqrt{\pi\mu}} e^{-u^2}. \tag{3}$$

When the chances p_i and q_i are constants and consequently equal to the means p and q, one has $h = \sqrt{2pq}$; this which makes this formula (3) coincide, and the preceding limits of m and n, with formula (17) of n° 79, and the limits to which it corresponds. This coincidence of two results obtained by some methods so different, would be able to serve, in case of need, of confirmation to our calculations.

By taking for u a number of little significance, such as three or four, one will render the value of R very little different from unity. It is therefore nearly certain that in a very great number μ of trials, the ratios $\frac{m}{\mu}$ and $\frac{n}{\mu}$ will deviate very little from the mean chances p and q, of which they will approach more and more, in measure as μ will increase yet further, and with which they would coincide rigorously if μ was able to be infinite; this which is already the first of the two general propositions of n° 52.

pp. 276–299

§104. We consider now, as in n° 52, an event E of any nature, of which the arrival is able to be due to a number of distinct causes, which are mutually exclusive and which are the only possible. We call the causes $C_1, C_2, C_3, \ldots, C_\nu$; let c_i be the chance that the cause C_i will give to the arrival of E, when it will be this cause which will intervene, and γ_i the probability of its intervention. The chance of E will be able to vary, in consequence, from one trial to another: it will be a thing susceptible of i different values, c_1, c_2, c_3, $\ldots c_\nu$, of which the respective probabilities will be γ_1, γ_2, γ_3, $\ldots \gamma_\nu$, and will remain the same as long as the causes C_1, C_2, $C_3, \ldots$, C_ν will not change. By taking therefore this chance for A, there will be the probability P, given by formula (13), that its mean value, in a very great number μ of trials, will be comprehended between the limits $k \mp \frac{2u\sqrt{h}}{\sqrt{\mu}}$, where one will put for k and h, their first values of the preceding number, applied in the case where the quantities c_1, c_2, c_3, etc. γ_1, γ_2, γ_3, etc., remain constants during the trials; this which will change those values into these here: Second theorem

$$k = \gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_\nu c_\nu,$$
$$h = \tfrac{1}{2}(\gamma_1 c_1^2 + \gamma_2 c_2^2 + \cdots + \gamma_\nu c_\nu^2) - \tfrac{1}{2}(\gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_\nu c_\nu)^2,$$

and render them, as one sees, independent of the number μ, whatever be besides the number and the inequality of the quantities that they contain. And as one is able to give to u a value of little significance, which render the probability P very close to certitude, it follows that the mean of the chances of E which will have place during the series of trials, will differ probably very little from the sum of the ν products $\gamma_1 c_1$, $\gamma_2 c_2$, etc.; of which it will be indefinitely closer in measure as the number μ will increase yet further, this which is the second of the two general propositions of n° 52, which remains for us to demonstrate.

In the two series composed of very great numbers μ and μ' of trials, if one represents by m and m' the numbers of times that the event E will arrive, the ratios $\frac{m}{\mu}$ and $\frac{m'}{\mu}$

will deviate probably very little (n^{o} 96) of the mean chances of E in these two series; it is therefore also very probable that they will differ very little from the preceding value of k, and, consequently, from one another, since that value of k will be common to the two series of trials, if, all the causes C_1, C_2, C_3, etc., have not changed in the interval. But what will be the probability of a small difference given between these ratios $\frac{m}{\mu}$ and $\frac{m'}{\mu}$? It is an important question of which we will occupy ourselves in one of the following sections.

§105. In the greater part of the questions to which formula (13) is applicable, the law of probability of the values of A is unknown, and, consequently, the quantities k and h contained within the limits of the mean value of A, are not able to be determined *a priori*. But by means of the values of A observed in a long series of trials, one will be able to eliminate the unknowns which would contain the limits of its mean value, in some other series equally composed of a great number of trials, and for which the diverse causes which are able to bring forth all the possible values of A, are the same as for the series of which one will have employed the results, in understanding by the same as for the series of which one will have employed the results, in understanding by the same causes, those which give the same chance to each of these values, and which have themselves an equal probability. The complete solution of this problem is the object of the following calculations.

I make $c = \varepsilon$ in formula (12); there results from it

$$P = \frac{1}{\pi}\int_0^\infty e^{-\theta^2}\sin(\mu kx)\frac{d\theta}{\theta} + \frac{1}{\pi}\int_0^\infty e^{-\theta^2}\sin(2\varepsilon x - \mu kx)\frac{d\theta}{\theta}$$
$$- \frac{g}{\pi h\sqrt{\mu h}}\int_0^\infty e^{-\theta^2}\cos(\mu kx)\theta^2 d\theta + \frac{g}{\pi h\sqrt{\mu h}}\int_0^\infty \cos(2\varepsilon x - \mu kx)\theta^2 d\theta,$$

for the probability that the sum s of the μ values of A will be comprehended between zero and 2ε. One concludes from it that the differential of P with respect to ε, namely:

$$\frac{dP}{d\varepsilon}d\varepsilon = \frac{2d\varepsilon}{\pi}\int_0^\infty e^{-\theta^2}\cos(2\varepsilon x - \mu kx)\frac{xd\theta}{\theta} - \frac{2gd\varepsilon}{\pi h\sqrt{\mu h}}\int_0^\infty e^{-\theta^2}\sin(2\varepsilon x - \mu kx)x\theta^2 d\theta,$$

will express the infinitely small probability that s will have precisely 2ε for value. I make also

$$2\varepsilon = \mu k + 2\nu\sqrt{\mu h}, \quad \varepsilon = \sqrt{\mu h}d\nu;$$

I designate by $\varpi d\nu$ the value corresponding to $\frac{dP}{d\varepsilon}d\varepsilon$, in which I neglect the quantities of order of smallness of $\frac{1}{\mu}$, this which will permit to reduce x in the first term $\frac{\theta}{\sqrt{\mu h}}$ from its value in series (n^{o} 101); there comes

$$\varpi d\nu = \frac{2d\nu}{\pi}\int_0^\infty e^{-\theta^2}\cos(2\nu\theta)d\theta - \frac{2gd\nu}{\pi h\sqrt{\mu h}}\int_0^\infty e^{-\theta^2}\sin(2\nu\theta)\theta^3 d\theta,$$

and because of

$$\int_0^\infty e^{-\theta^2}\cos(2\nu\theta)d\theta = \frac{1}{2}\sqrt{\pi}e^{-\nu^2},$$
$$e^{-\theta^2}\sin(2\nu\theta)\theta^3 d\theta = \frac{1}{4}\sqrt{\pi}(3\nu - 4\nu^3)e^{-\nu^2},$$

this value of ϖdv will take the form

$$\varpi dv = \sqrt{1}\sqrt{\pi}\left(1 - \frac{1}{\sqrt{\pi}}V\right)e^{-v^2}dv;$$

V designating a polynomial which contains only some odd powers of v, and which will not influence, whatever it be besides, on the result of our calculations. This expression of ϖdv will be therefore the probability of the sum s equal to the preceding value of 2ε, or else by dividing, by μ, it will be the probability of the equation

$$\frac{s}{\mu} = k + \frac{2v\sqrt{h}}{\sqrt{\mu}},$$

in which v is a positive or negative quantity, but very small with respect to $\sqrt{\mu}$.

I will name now C_1, C_2, C_3,$\ldots C_v$, all the causes, known or unknown, which are mutually exclusive, and which are able to give to A one of the values of which this thing is susceptible; and I will designate by γ_1, γ_2, γ_3, $\ldots \gamma_v$, their respective probabilities, of which the sum will be equal to unity, and of which each will have a infinitely small value, if the number of these possible causes were infinite. The possible values of A being all those which are comprehended between a and b, and, consequently, in number infinite, the chance of each of them, arising from each of these causes, will be infinitely small. I will represent by $Z_i dz$ the chance that C_i would give, if this cause was certain, to the value z of A. The integral $\int_a^b zf_nzdz$, relative to the n^{th} trial, will be therefore a thing susceptible of v values $\int_a^b zZ_1dz$, $\int_a^b zZ_2dz$, $\ldots \int_a^b zZ_vdz$, of which the probabilities will be those of the corresponding causes; so that γ_i will express, in any trial, the chance of the value $\int_a^b zZ_idz$. Consequently, the infinitely small probability of a value of the mean $\frac{1}{\mu}\Sigma\int_a^b zf_nzdz$, will be determined by the preceding rule, which corresponds to the mean $\frac{s}{\mu}$ of the values of any thing, in a very great number μ of trials: s will be then the sum of the μ unknown values of $\int_a^b zf_nzdz$, which will have place in this series of trials, and the quantities that one must take for k and h, will be determined according to the v possible values of this intergral.

Now, by taking these v values $\int_a^b zZ_1dz$, $\int_a^b zZ_2dz$, $\ldots \int_a^b zZ_vdz$, for those that one has designated by c_1, c_2, $\ldots c_v$, in the n° 103, by making, for brevity,

$$\gamma = S\gamma_i\int_a^b zZ_1dz, \quad \beta = \frac{1}{2}S\gamma_i\left(\int_a^b zZ_idz\right)^2 - \frac{1}{2}\left(\S\gamma_i\int_a^b zZ_idz\right)^2,$$

where the characteristic S indicates a sum which extends to all the indices i from $i = 1$ to $i = v$, these are, according to the formula of this section, the quantities γ and β, independent of μ, that it will be necessary to take for k and h. If therefore one designates by $v_\prime$ a positive or negative quantity, very small with respect to $\sqrt{\mu}$; if $V_\prime$ is a polynomial which contains only some odd powers of $v_\prime$; and if one makes

$$\varpi_\prime dv_\prime = \frac{1}{\sqrt{\pi}}\left(1 - \frac{1}{\sqrt{\mu}}V_\prime\right)e^{-v_\prime^2}dv_\prime,$$

this infinitely small $\varpi dv_\prime$ will be the probability of the equation

$$\frac{1}{\mu}\Sigma\int_a^b zf_ndz = \gamma + \frac{2v_\prime\sqrt{\beta}}{\sqrt{\mu}}.$$

By considering likewise the quantity

$$\frac{1}{2}\int_a^b z^2 f_n z dz - \frac{1}{2}\left(\int_a^b z f_n z dz\right)^2,$$

as a thing susceptible of the ν values corresponding to the causes C_1, C_2,... C_ν and of which the probabilities, at each trial, will be those of these same causes; designating by $v_{\prime\prime}$ a positive or negative quantity, such that the ratio $\frac{v_{\prime\prime}}{\sqrt{\nu}}$ is a very small fraction, and by $V_{\prime\prime}$ a polynomial which contains only some odd powers of $v_{\prime\prime}$; making next

$$\varpi_{\prime\prime} dv_{\prime\prime} = \frac{1}{\sqrt{\pi}}\left(1 - \frac{1}{\sqrt{\mu}}V_{\prime\prime}\right)e^{-v_{\prime\prime}^2}dv_{\prime\prime},$$

and, for brevity,

$$\alpha = \frac{1}{2}S\gamma_i \int_a^b z^2 Z_i dz - \frac{1}{2}S\gamma_i\left(\int_a^b z Z_i dz\right)^2,$$

this expression of $\varpi_{\prime\prime} dv_{\prime\prime}$ will be the probability that the mean of the μ values of the quantity of which there is concern, namely

$$\frac{1}{2\mu}\Sigma\left[\int_a^b z^2 f_n z dz - \left(\int_a^b z f_n z dz\right)^2\right],$$

will differ from α by a determined quantity, of the order of smallness of $\frac{1}{\sqrt{\mu}}$, and which will be useful for us to known. Besides this mean is nothing other than the quantity h of n° 101; if therefore one neglects the quantities of order of $\frac{1}{\mu}$, it will suffice to put α instead of h, in the second term of the preceding value of $\frac{s}{\mu}$, which is already of order of $\frac{1}{\sqrt{\mu}}$: in this manner, one will have

$$\frac{s}{\mu} = k + \frac{2\nu\sqrt{\alpha}}{\sqrt{\mu}};$$

and the probability of this equation will be yet $\varpi d\nu$, if the value of h that one has employed was certain. But this value having only a probability $\varpi_{\prime\prime} dv_{\prime\prime}$, depends on the variable $v_{\prime\prime}$ which does not enter into the value of $\frac{s}{\mu}$, it follows that the probability of that will have for complete expression, the product of $\varpi d\nu$ and of the sum of the values of $\varpi_{\prime\prime} dv_{\prime\prime}$, corresponding to all those that one is able to give to $v_{\prime\prime}$. Now, although these values must be very small with respect to $\sqrt{\mu}$, one will be able nevertheless, by reason of the exponential $e^{-\frac{2}{\prime\prime}}$ factor of $\varpi_{\prime\prime} dv_{\prime\prime}$, extends the integral of $\varpi_{\prime\prime} dv_{\prime\prime}$ without altering it sensibly, from $v_{\prime\prime} = -\infty$ to $v_{\prime\prime} = \infty$; the part dependent on $V_{\prime\prime}$ will vanish as being composed of elements, two by two equal and of contrary signs; and one will have simply $\int_a^b \varpi_{\prime\prime} dv_{\prime\prime} = 1$. Consequently, the probability of the preceding equation will be always $\varpi d\nu$, as if the approximate value of h of which one has made use, had been certain.

One is able also to note that the mean $\frac{1}{\mu}\Sigma\int_a^b z f_n z dz$ is nothing other than the quantity k of n°; the expression of $\varpi_{\prime} dv_{\prime}$ is therefore the probability that the value of this quantity will be

$$k=\gamma+\frac{2\nu_{\prime}\sqrt{\beta}}{\sqrt{\mu}};$$

therefore by substituting this value into that of $\frac{s}{\mu}$, this which gives

$$\frac{s}{\mu}=\gamma+\frac{2\nu_{\prime}\sqrt{\beta}}{\sqrt{\mu}}+\frac{2\nu\sqrt{\alpha}}{\sqrt{\mu}},$$

the probability of this last equation, for each pair of values of ν and $\nu_{\prime}$ will be the product of $\varpi d\nu$ and $\varpi_{\prime} d\nu_{\prime}$, that I will represent by σ, so that one has

$$\sigma=\frac{1}{\pi}\left[1-\frac{1}{\sqrt{\mu}}(V+V_{\prime})\right]e^{-\nu^2-\nu_{\prime}^2}d\nu d\nu_{\prime},$$

by neglecting the term which would have μ for divisor.

We designate by θ a positive or negative variable, very small, as ν and $\nu_{\prime}$, with respect to $\sqrt{\mu}$; one will be able to make

$$\nu_{\prime}\sqrt{\beta}+\nu\sqrt{\alpha}=\theta\sqrt{\alpha+\beta};$$

and if one wishes to replace $\nu_{\prime}$ by this new variable, in the preceding differential formula, it will be necessary to put, instead of $\nu_{\prime}$ and $d\nu_{\prime}$, the values

$$\nu_{\prime}=\frac{\theta\sqrt{\alpha+\beta}}{\sqrt{\beta}}-\frac{\nu\sqrt{\alpha}}{\sqrt{\beta}},\quad d\nu_{\prime}=\frac{\sqrt{\alpha+\beta}}{\sqrt{\beta}}d\theta;$$

this which will change it into this here

$$\sigma=\frac{1}{\pi}\left(1-\sqrt{1}\sqrt{\mu}T\right)e^{-\left(\frac{\nu\sqrt{\alpha+\beta}}{\sqrt{\beta}}-\frac{\theta\sqrt{\alpha}}{\sqrt{\beta}}\right)^2-\theta^2}\frac{\sqrt{\alpha+\beta}d\nu d\theta}{\sqrt{\beta}},$$

in which T is a polynomial resulting from V and $V_{\prime}$, and of which each term contains an odd power of ν and of θ. The equation

$$\frac{s}{\mu}=\gamma+\frac{2\theta\sqrt{\alpha+\beta}}{\sqrt{\mu}}, \tag{14}$$

containing no longer but the variable θ, it follows that its total probability will be the sum of the values of σ, relative to all the positive or negative values that one is able to give to the other variable ν. Moreover, by reason of the exponential that contains the expression of σ, it will be permitted to extend this integral, without altering sensibly the value of it, from $\nu=-\infty$ to $\nu=\infty$. Then, by making

$$\frac{\nu\sqrt{\alpha+\beta}}{\sqrt{\beta}}-\frac{\theta\sqrt{\alpha}}{\sqrt{\beta}}=\theta_{\prime},\qquad \frac{d\nu\sqrt{\alpha+\beta}}{\sqrt{\beta}}=d\theta_{\prime},$$

and designating by T', that which T will become in function of θ and $\theta_{\prime}$, we will have

$$\sigma = \frac{1}{\pi}\left(1 - \frac{1}{\sqrt{\mu}}T'\right)e^{-\theta^2-\theta_{\prime}^2}d\theta d\theta_{\prime}:$$

the limits of the integral relative to the new variable $\theta_{\prime}$ will be again $\pm\infty$; by representing therefore by $\eta d\theta$ its infinitely small value, there will result from it

$$\eta d\theta = \frac{1}{\sqrt{\pi}}e^{-\theta^2}d\theta - \frac{1}{\sqrt{\pi\mu}}\Theta e^{-\theta^2}d\theta,$$

for the probability of equation (14); Θ being a polynomial which contains only some odd powers of θ.

The concern is actually to eliminate the unknown $\alpha+\beta$ of this equation (14); this which will be possible, as one just saw, because the expression of $\alpha+\beta$ is reduced to

$$\alpha+\beta = \frac{1}{2}S\gamma_i\int_a^b z^2 Z_i dz - \frac{1}{2}\left(S\gamma_i\int_a^b zZ_i dz\right)^2,$$

and is found independent of the sum $S\gamma_i\left(\int_a^b zZ_i dz\right)^2$, which was contained in each of the quantities α and β.

§106. In applying to $\frac{1}{2}\int_a^b z^2 f_n dz$ the same reasoning as to the diminished quantity, as in the preceding section, of $\frac{1}{2}\left(\int_a^b zf_n zdz\right)^2$, and designating by $\frac{1}{2}\phi$ its mean value, so that one has

$$\frac{1}{\mu}\Sigma\int_a^b z^2 f_n zdz = \phi,$$

there will be the probability $\varpi_{\prime\prime}d\nu_{\prime\prime}$ that $\frac{1}{2}S\gamma_i\int_a^b z^2 Zdz$ will differ from $\frac{1}{2}\phi$, only be a determined quantity and of the order of smallness of $\frac{1}{\sqrt{\mu}}$. Moreover, by neglecting always the terms which have $\frac{1}{\mu}$ for divisor, one will see also, as in this section, that it will be permitted to employ, in equation (14), $\frac{1}{2}\phi$ instead of this part $\frac{1}{2}S\gamma_i\int_a^b z^2 Z_i dz$ of the preceding value of $\alpha+\beta$, without anything changing in the probability $\eta d\theta$ of this equation. The other part of the value of $\alpha+\beta$ being exactly the quantity $\frac{1}{2}\gamma^2$, one will have therefore

$$\alpha+\beta = \frac{1}{2}\phi - \frac{1}{2}\gamma^2;$$

by means of which equation (14) will become first

$$\frac{s}{\mu} = \gamma + \frac{\theta}{\sqrt{\mu}}\sqrt{2\phi - 2\gamma^2}.$$

This put, let Z be a given function of z. The analysis of n^{os} 97 and 101, and hence, the expression of $\varpi d\nu$ of the preceding section, will be extended without difficulty to the sum of the values of Z which will have place in the μ trials that we will consider. It will suffice to take instead of A, another thing A$_{\prime}$ of which the values are those of this function Z. the infinitely small probability of any value of A$_{\prime}$ will be the same as that of the corresponding value of z, and will be expressed, consequently, by $f_n z dz$ to the n^{th} trial; and if one designates by $k_\prime$, $h_\prime$, $g_\prime$, etc., that which the quantities k, h, g etc. of n° 101 which correspond to A become relatively to A$_\prime$, one will have

$$\mu k_\prime = \Sigma \int_a^b Z f_n z dz, \quad \mu h_\prime = \Sigma \left[\int_a^b Z^2 f_n z dz - \left(\int_a^b Z f_n z dz \right)^2 \right], \text{etc.}$$

Therefore, by naming $s_\prime$, the sum of the μ values of A$_\prime$ which will have place in the series of trials, the infinitely small $\varpi d\nu$ will be the probability that one will have precisely

$$\frac{s_\prime}{\mu} = k_\prime + \frac{2\nu\sqrt{h_\prime}}{\sqrt{\mu}}.$$

Now, if we make $Z = z^2$, we will have

$$k_\prime = \frac{1}{\mu} \Sigma \int_a^b z^2 f_n z dz = \phi;$$

to the degree of approximation where we stop ourselves, one will be able therefore to take $\frac{s_\prime}{\mu}$ for the value of ϕ, in the preceding expression of $\frac{s}{\mu}$; and one will be assured, as in the preceding section, that the probability of this expression will not change; so that $\eta d\theta$ will be always the infinitely small probability of the equation

$$\frac{s}{\mu} = \gamma + \frac{\theta}{\sqrt{\mu}} \sqrt{\frac{2s_\prime}{\mu} - 2\gamma^2},$$

or of this

$$\frac{s}{\mu} = \gamma + \frac{\theta}{\sqrt{\mu}} \sqrt{\frac{2s_\prime}{\mu} - 2\frac{2s^2}{\mu^2}},$$

which is deduced from the preceding, by neglecting always the quantities of the order of smallness of $\frac{1}{\mu}$.

I represent by λ_n the value of A which has had or which will have place in the n^{th} trial; and I make, for brevity,

$$\frac{s_\prime}{\mu} = \frac{1}{\mu}\Sigma\lambda_n^2, \quad \frac{s}{\mu} = \frac{1}{\mu} = \frac{1}{\mu}\Sigma\lambda_n, \quad \frac{s_\prime}{\mu} - \frac{s^2}{\mu^2} = \frac{1}{\mu}\Sigma(\lambda_n - \lambda)^2;$$

by means of which, the preceding equation will become

$$\frac{s}{\mu} = \gamma + \frac{\theta l}{\sqrt{\mu}}.$$

Now, one concludes thence that one designates by u a quantity positive and given, the integral of the probability $\eta d\nu$ of this equation, taken from $\theta = u$ to $\theta = -u$, will express the probability that the value of $\frac{s}{\mu}$ will fall between the limits

$$\gamma \mp \frac{ul}{\sqrt{\mu}}.$$

By calling Γ this last probability, and having regard to the expression of $\eta d\theta$, one will have

$$\Gamma = \frac{1}{\sqrt{\pi}} \int_{-u}^{u} e^{-\theta^2} d\theta - \frac{1}{\sqrt{\pi\mu}} \int_{-u}^{u} e^{-\theta^2} \Theta d\theta;$$

and as Θ is a polynomial which contains only some odd powers of θ, the second integral will be null, and one will have simply

$$\Gamma = \frac{1}{\sqrt{\pi}} \int_{-u}^{u} e^{-\theta^2} d\theta;$$

a result which coincides with the probability P given by formula (13).

Thus, this formula expresses the probability that the limits $\mp\frac{ul}{\sqrt{\pi}}$, which no longer contains anything unknown after the trials, will comprehend the difference between the mean $\frac{s}{\mu}$ of the values of A and the special quantity γ, to which this mean approaches indefinitely, and that it will attain if μ became infinite, unless the causes $C_1, C_2, C_3, \ldots C_\nu$, of the possible values of A never change.

§107. Suppose actually that one makes two series of a great number of trials, which will be represented by μ in one of these series and by μ' in the other. Let s and s' by the sums of the values of A in these two series; let also λ_n and λ_n' be the values of A which will have or which have had place in the n^{th} trial; and we make

$$\begin{array}{ll} \frac{1}{\mu}\Sigma\lambda_n = \lambda, & \frac{1}{\mu}\Sigma(\lambda_n - \lambda)^2 = \frac{1}{2}l^2, \\ \frac{1}{\mu'}\Sigma\lambda_n' = \lambda', & \frac{1}{\mu}\Sigma(\lambda_n' - \lambda')^2 = \frac{1}{2}l'^2; \end{array}$$

the sums Σ being extended to all the trials of each series, that is, the first two from $n = 1$ to $n = \mu$, and the last two from $n = 1$ to $n = \mu'$. If the causes $C_1, C_2, C_3, \ldots C_\nu$, do not change from one series of trials to another, the quantity γ of n$^{\text{o}}$ 105 will not change any longer; by designating then by θ and θ' positive and negative variables, but very small with respect to $\sqrt{\mu}$ and $\sqrt{\mu'}$, the equations relative to the mean values of A in these two series, will be

$$\frac{s}{\mu} = \gamma + \frac{\theta l}{\sqrt{\mu}}, \qquad \frac{s'}{\mu'} = \gamma + \frac{\theta' l'}{\sqrt{\mu'}} \tag{15}$$

and their respective probabilities $\eta d\theta$ and $\eta' d\theta'$ will have for expressions

$$\eta d\theta = \frac{1}{\sqrt{\pi}}\left(1 - \frac{1}{\sqrt{\mu}}\Theta\right) e^{-\theta^2} d\theta, \quad \eta' d\theta' = \frac{1}{\sqrt{\pi}}\left(1 - \frac{1}{\sqrt{\mu'}}\Theta'\right) e^{-\theta'^2} d\theta';$$

Θ and Θ' being some polynomials which contain only odd powers of θ and θ'. Moreover, if the series is composed of different trials, one will be able to consider these values of $\frac{s}{\mu}$ and $\frac{s'}{\mu'}$ as some events independent from one another; and by the rule of n$^{\text{o}}$5, the probability of their simultaneous arrival will be the product of $\eta d\theta$ and $\eta' d\theta'$. This will be also the probability of any combination of the two equations (15), and, for example, of the equation that one obtains by subtracting them from one another, namely

$$\frac{s'}{\mu'} - \frac{s}{\mu} = \frac{\theta' l'}{\sqrt{\mu'}} - \frac{\theta l}{\sqrt{mu}}.$$

Thus, by designating by ψ the product $\eta\eta' d\theta d\theta'$, and neglecting the term which would have $\sqrt{\mu\mu'}$ for divisor, we will have

$$\psi = \frac{1}{\pi}\left(1 - \frac{1}{\sqrt{\mu}}\Theta - \frac{1}{\sqrt{\mu'}}\Theta'\right) e^{-\theta^2 - \theta'^2} d\theta d\theta',$$

for the probability of the preceding equation, relatively to each pair of values of θ and θ'.

In order to follow here, the same march as in n° 105, I make

$$\frac{\theta' l'}{\sqrt{\mu'}} - \frac{l\theta}{\sqrt{\mu}} = \frac{t\sqrt{l'^2 + l^2\mu'}}{\sqrt{\mu\mu'}};$$

this which changes that equation into this here:

$$\frac{s'}{\mu'} - \frac{s}{\mu} = \frac{t\sqrt{l'^2 + l^2\mu'}}{\sqrt{\mu\mu'}}.$$

I replace θ in Ψ, by the new varialbe t; and for this, I make

$$\theta' = \frac{t\sqrt{l'^2\mu + l^2\mu}}{l'\sqrt{\mu}} + \frac{\theta l\sqrt{\mu'}}{l'\sqrt{\mu}}, \qquad d\theta' = \frac{\sqrt{l'^2\mu + l^2\mu'}}{l'\sqrt{\mu}} dt;$$

whence there results

$$\psi = \frac{dt d\theta\sqrt{l'^2\mu + l^2\mu'}}{\pi l'\sqrt{\mu}}(1 - \Pi) e^{-\left(\frac{t\sqrt{l'^2\mu + l^2\mu}}{l'\sqrt{\mu}} + \frac{\theta l\sqrt{\mu'}}{l'\sqrt{\mu}}\right) - t^2};$$

Π being a polynomial of which each term contains an odd power of t or of θ. The value of $\frac{s'}{\mu} - \frac{s}{\mu'}$ containing no more than the variable t, its probability will be the integral of ψ extended to all the values that one will be able to give to the other variable θ; and because of the exponential contained in ψ, this integral will be able to be extended, without altering sensibly the value, from $\theta = -\infty$ to $\theta = \infty$. By making then,

$$\frac{t\sqrt{l'^2\mu + l^2\mu}}{l'\sqrt{\mu}} + \frac{\theta l\sqrt{\mu'}}{l'\sqrt{\mu}} = t', \qquad \frac{\sqrt{l'^2\mu + l^2\mu}}{l'\sqrt{\mu}} dt = dt',$$

and designating by Π' that which Π will become, we will have

$$\psi = \frac{1}{\pi}(1 - \Pi') e^{-t'^2 - t^2} dt' dt;$$

the limits of the integral relative to t' will be further $t' = \mp\infty$; and if one represents by ζdt the infinitely small probability of the preceding value of $\frac{s'}{\mu'} - \frac{s}{\mu}$, one will have

$$\zeta dt = \frac{1}{\sqrt{\pi}}(1 - T) e^{-t^2} dt;$$

T being a polynomial which contains only some odd powers of t. Finally, if we represent by u a quantity positive and given, and by Δ the probability that this difference $\frac{s'}{\mu'} - \frac{s}{\mu}$ will fall between the limits

$$\mp \frac{u\sqrt{l'^2\mu + l^2\mu}}{\sqrt{\mu\mu'}},$$

we will have

$$\Delta = \frac{2}{\sqrt{\pi}}\int_0^u e^{-t^2}dt;$$

this which coincides with the value of P given by formula (13). Consequently, this quantity P is the probability that the difference between the mean values of A in two long series of trials, will fall between these limits which contain nothing unknown.

After having taken for u a value sufficient in order to render that of P very little different from unity, if the observation gives for this difference $\frac{s'}{\mu'} - \frac{s}{\mu}$, a quantity which falls beyond the preceding limits, one will be established to conclude from it that the causes $C_1, C_2, C_3, \dots C_\nu$, of the possible values of A, are not remained the same in the interval of the two series of trials, that is there will be occurring some change, either in the probabilities γ_1, γ_2, γ_3, $\dots \gamma_\nu$, of these causes, or in the chances that they give to the different values of A.

According to that which one has seen in the preceding section, each of the quantities l and l' must differ very probably quite little from one same quantity $2\sqrt{\alpha+\beta}$, unknown and the same in the two series of trials; it is therefore also very probable that the quantities l and l' will differ very little from one another; and without changing sensibly, neither the magnitude of the preceding limits, nor their probability, one will be able to make $l' = l$. In a series of future trials, there will be therefore the probability P, given by formula (13), that the mean $\frac{s'}{\mu'}$ of the values of A, will fall between the limits

$$\frac{s}{\mu} \mp \frac{ul\sqrt{\mu+\mu'}}{\sqrt{\mu\mu'}};$$

which depend, for each given value of u, only on the results of the first series of trials already made.

For one same value of u, that is to equal degree of probability, one sees that the amplitude of these limits is greater than that of the limits of the difference $\gamma - \frac{s}{\mu}$, in the ratio of $\sqrt{\mu+\mu'}$ to $\sqrt{\mu'}$, and that these two amplitudes coincide very nearly, when μ' is a very large number with respect to the very large number μ.

§108. If the two series of μ and μ' trials have for object the measure of one same thing, and are made with some different instruments, for each of which the errors equal and contrary are equally probable; the mean values $\frac{s}{\mu}$ and $\frac{s'}{\mu'}$, resultants of these two series, will converge indefinitely toward one same quantity which will be the true value of A (n$^{\text{o}}$ 60). In this case, the unknown γ will be therefore the same for the two series of observations, and the means $\frac{s}{\mu}$ and $\frac{s'}{\mu'}$ will differ very probably quite little from one another; but, for these two series, the unknown $\alpha+\beta$ will be able to be very different; this which will render very unequal the quantities l and l'. The values of these quantities being known, one is able to demand what is the most advantageous manner to combine the means $\frac{s}{\mu}$ and $\frac{s'}{\mu'}$, in order to deduce from it the limits of γ, or of the true value of A.

In order to find this combination, I designate by g and g' the indeterminate quantities of which the sum is unity, and I add equations (15), after having multiplied the first by g and the second by g', this which gives

$$\gamma = \frac{gs}{\mu} + \frac{g's'}{\mu'} - \frac{gl\theta}{\sqrt{\mu}} - \frac{g'l'\theta'}{\sqrt{\mu'}};$$

an equation of which the probability is equal to ψ, according to that which one has said above, for all the pairs of values of θ and θ'. Now, by a calculation similar to the one that we just effected, one will conclude from it that the quantity P, given by formula (13), will express the probability that the unknown value of γ is comprehended between the limits

$$\frac{gs}{\mu} + \frac{g's'}{\mu'} \mp \frac{u\sqrt{g'^2l'^2\mu + g^2l^2\mu'}}{\sqrt{\mu\mu'}}.$$

If therefore one wishes that for one same probability P, that is, for each given value of u, the amplitude of these limits, is the smallest that is possible, it will be necessary to determine g and g' by equating to zero the differential of the coefficient of u, with respect to these quantities: because of $g + g' = 1$ and $dg' = -dg$, one will deduce from it

$$g = \frac{l'^2\mu}{l'^2\mu + l^2\mu'}, \qquad g' = \frac{l^2\mu'}{l'^2\mu + l^2\mu'};$$

and the narrowest limits of γ will be these here

$$\frac{sl'^2 + s'l^2}{l'^2\mu + l^2\mu'} \mp \frac{ull'}{\sqrt{l'^2u + l^2\mu'}},$$

of which formula (13) will express always the probability.

One is able easily to generalize this result, and to extend to any number of series of a great number of observations, made with different instruments in order to measure one same thing A. the three quantities μ, s, l, correspond to the first series, if one designates the analogous quantities by μ', s', l', in the second series; by μ'', s'', l'', in the third; etc.; and if one makes, first

$$\frac{\mu}{l^2} + \frac{\mu'}{l'^2} + \frac{\mu''}{l''^2} + \text{etc.} = D^2,$$

and next

$$\frac{\mu}{D^2l^2} = q, \qquad \frac{\mu'}{D^2l'^2} = q', \qquad \frac{\mu''}{D^2l''^2} = q'', \text{ etc.},$$

formula (13) will express the probability that the unknown value of A is comprehended between the limits

$$\frac{sq}{\mu} + \frac{s'q'}{\mu'} + \frac{s''q''}{\mu''} + \text{ etc.} \mp \frac{u}{D},$$

resultants of the most advantageous combination of the observations. And as one will be able to render this formula (13) very little different from unity, by taking for u a number of little consequence, it follows that the value of A will differ very probably quite little from the sum of the means $\frac{s}{\mu}$, $\frac{s'}{\mu'}$, $\frac{s''}{\mu''}$, etc., multiplied respectively by the quantities q, q', q'', etc. The result of each series of observations will influence so much more on that approximate value of A and on the amplitude $\mp\frac{u}{D}$ of these limits, than the one of the quotients $\frac{u}{l^2}$, $\frac{u'}{l'^2}$, $\frac{u''}{l''^2}$, etc., which is referred to this series, will have a greater value.

When all the series of observations will have been made with one same instrument, one will be able to consider them as a single series, composed of a number of observations equal to $\mu+\mu'+\mu''+\text{etc.}$. Thus as we have said above, the quantities l, l', l'', etc., will be very nearly and very probably equal; by extending the sums Σ to the total series, or from $n=1$ to $n=\mu+\mu'+\mu''+\text{etc.}$, and making

$$\frac{1}{\mu+\mu'+\mu''+\text{etc.}}\Sigma\lambda_n=\lambda,\quad \frac{1}{\mu+\mu'+\mu''+\text{etc.}}\Sigma(\lambda_n-\lambda)^2=\frac{1}{2}l_1^2,$$

one will be able to take l_1 for the common value of l, l', l'', etc.; by means of which the preceding limits of the unknown γ, and of which formula (13) expresses the probability, will become

$$\frac{s+s'+s''+\text{etc.}}{\mu+\mu'+\mu''+\text{etc.}}\mp\frac{ul_1}{\sqrt{\mu+\mu'+\mu''+\text{etc.}}};$$

this which coincides with the result of n° 106, relative to a single series of trials.

§109. The question indicated at the end of n° 104 will be resolved by some considerations similar to those of which we just made use.

Let m be the number of times that the even E, of any nature, will arrive in a very great number μ of trials. The chance of E varying from one trial to another, let p_n be that which will take place at the n^{th} trial. Make

$$\frac{1}{\mu}\Sigma p_n=p,\qquad \frac{1}{\mu}\Sigma p_n^2=q;$$

designate by ν a positive or negative quantity, but very small with respect to $\sqrt{\mu}$; and represent by U the probability of the equation

$$\frac{m}{\mu}=p-\frac{\nu}{\sqrt{\mu}}\sqrt{2p-2q}.$$

By neglecting, in order to simplify the calculations, the second term of formula (2); having regard to that which prepresents the quantity k which renders it; and by putting ν in place of θ, one will have

$$U=\frac{1}{\sqrt{2\pi\mu(p-q)}}e^{-\nu^2}.$$

As in n° 104, we name $C_1, C_2, \dots C_\nu$, all the possible causes of the event E, which is able to be a finite or infinite number; γ_1, γ_2, $\dots\gamma_\nu$, their respective probabilities; c_1,

$c_2, \ldots c_\nu$, the chances that they give to the arrival of E. By considering p_n as a thing susceptible of these ν values $c_1, c_2, \ldots c_\nu$, of which $\gamma_1, \gamma_2, \ldots \gamma_\nu$, are the probabilities; making

$$\gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_\nu c_\nu = r,$$
$$\gamma_1 c_1^2 + \gamma_2 c_2^2 + \cdots + \gamma_\nu c_\nu^2 = \rho;$$

and by designating by $\nu_\prime$ a positive or negative value, very small with respect to $\sqrt{\mu}$, the infinitely small probability that one will have precisely

$$p = r + \frac{\nu_\prime \sqrt{2\rho - 2r^2}}{\sqrt{\mu}},$$

will be the quantity $\varpi_\prime d\nu_\prime$ of n° 105, or simply $\frac{1}{\sqrt{\pi}} e^{-\nu_\prime{}^2} d\nu_\prime$, by neglecting the second term of its expression. If one designates further by $\nu_{\prime\prime}$ a variable very small with respect to $\sqrt{\mu}$, there will be also the probability $\varpi_{\prime\prime} d\nu_{\prime\prime}$ of this same section, or simply $\frac{1}{\sqrt{\pi}} e^{-\nu_{\prime\prime}^2} d\nu_{\prime\prime}$, that the quantity $p - q$ will differ from $r - \rho$ only by a determined quantity, proportional to $\nu_{\prime\prime}$, and of the order of smallness of $\frac{1}{\sqrt{\mu}}$; and one will see moreover that by neglecting the quantities of the order of $\frac{1}{\mu}$, one will be able, without altering the probability U of the preceding value of $\frac{m}{\mu}$, putting $r - \rho$ instead of $p - q$; this which will change this value into this here

$$\frac{m}{\mu} = p - \frac{\nu \sqrt{2r - 2\rho}}{\sqrt{\mu}}.$$

Besides, if one makes

$$\frac{1}{\sqrt{2\mu(r-\rho)}} = \delta,$$

it will be necessary, in order that m be a whole number, to take for ν only the positive or negative multiples of δ, which must, besides, be very small with respect to μ.

This posed, I add the preceding values of p and $\frac{m}{\mu}$; this which gives

$$\frac{m}{\mu} = r + \frac{\nu_\prime \sqrt{2\rho - 2r^2}}{\sqrt{\mu}} - \frac{\nu \sqrt{2r - 2\rho}}{\sqrt{\mu}};$$

an equation of which the probability, for each pair of values of ν and of $\nu_\prime$, will be the product of U and of $\frac{1}{\sqrt{\pi}} e^{-\nu_\prime{}^2 d\nu_\prime}$ that I will represent by ε and which will have for value

$$\varepsilon = \frac{1}{\pi\sqrt{2\mu(r-\rho)}} e - \nu^2 - \nu_\prime{}^2 d\nu_\prime,$$

by putting $r - \rho$ instead of $p - q$ in the expression of U. I make

$$\nu_\prime = \theta \sqrt{\frac{r - r^2}{\rho - r^2}} + \nu \sqrt{\frac{r - \rho}{\rho - r^2}}, \quad d\nu_\prime \sqrt{\frac{r - r^2}{\rho - r^2}} d\theta;$$

there results from it

$$\frac{m}{\mu} = r + \frac{\theta\sqrt{2r - 2r^2}}{\sqrt{\mu}};$$

whence one draws

$$r = \frac{m}{\mu} - \frac{\theta\sqrt{2m(\mu - m)}}{\mu\sqrt{\mu}},$$

by neglecting the terms of order of smallness of $\frac{1}{\mu}$. One will have, at the same time,

$$\varepsilon = \frac{\delta d\theta}{\pi}\sqrt{\frac{r - r^2}{\rho - r^2}}e^{-\frac{[\nu^2(r-r^2)+2\nu\theta\sqrt{(r-r^2)(r-\rho)}+\theta^2(r-r^2)]}{\rho - r^2}},$$

by having regard to that which δ represents. But the expression of r not containing ν, its probability is independent of it also; it is equal to the sum of the values of ε corresponding to all those that one is able to give to ν, and which must increase by some differences equal to δ, of which ν is a multiple; because of the smallness of δ, one will obtain an approximate value of this sum by putting $d\nu$ instead of δ in ε, and replacing the sum by an integral: this value will be exact for the quantities near of the order of δ or of $\frac{1}{\sqrt{\mu}}$. Although the variable ν must be a very small quantity with respect to $\sqrt{\mu}$, one will be able, by reason of the exponential contained in ε, to extend the integral, without altering sensibly the value, from $\nu = -\infty$ to $\nu = \infty$. Then, if one makes

$$\nu\sqrt{\frac{r - r^2}{\rho - r^2}} + \theta\sqrt{\frac{r - \rho}{\rho - r^2}} = \theta_{\prime}, \qquad \sqrt{\frac{r - r^2}{\rho - r^2}}d\nu = d\theta_{\prime},$$

the limits of the integral relative to $\theta_{\prime}$ will be also $\pm\infty$; and by designating by $\zeta d\theta$ the infinitely small probability of the expression of r, one will have

$$\zeta d\theta = \frac{d\theta}{\pi}e^{-\theta^2}\int_{-\infty}^{\infty} e^{-\theta_{\prime}^2} = \frac{1}{\sqrt{\pi}}e^{-\theta^2}d\theta.$$

Therefore u being a positive and given quantity, the probability that the unknown value of r will fall between the limits

$$\frac{m}{\mu} \mp \frac{u\sqrt{2m(\mu - m)}}{\mu\sqrt{\mu}},$$

will coincide with the quantity P given by formula (13), since this probability will be

$$\int_{-u}^{u}\zeta d\theta = \frac{2}{\sqrt{\pi}}\int_0^u e^{-\theta^2}d\theta.$$

Thus, P is the probability that the special quantity r by which the ratio $\frac{m}{\mu}$ is approached indefinitely, in measure as the great numberμ increases yet further, differs from this ratio only by a quantity comprehended by the limits

$$\mp\frac{u\sqrt{2m(\mu - m)}}{\mu\sqrt{\mu}},$$

which contains nothing unknown.

In a second series composed of a very great number μ' of trials, let m' be the number of times that the event E will arrive. By designating by θ' a positive or negative variable, but very small with respect to $\sqrt{\mu'}$, the infinitely small probability of the equation

$$r = \frac{m'}{\mu'} - \frac{\theta'\sqrt{2m'(\mu'-m')}}{\mu'\sqrt{\mu'}},$$

will be $\frac{1}{\sqrt{\pi}}e^{-\theta\prime^2}d\theta'$; that from the equation

$$\frac{m'}{\mu'} - \frac{m}{\mu} = \frac{\theta'\sqrt{2m'(\mu'-m')}}{\mu'\sqrt{\mu'}} - \frac{\theta\sqrt{2m(\mu-m)}}{\mu\sqrt{\mu}},$$

that one obtains by subtracting this value from r, of the preceding, will be therefore the product of $\frac{1}{\sqrt{\pi}}e^{-\theta'^2}d\theta'$ and of $\frac{1}{\sqrt{\pi}}e^{-\theta^2}d\theta$ for all the pairs of values of θ and θ'; and if one makes first

$$\frac{\theta'\sqrt{m'(\mu'-m')}}{\mu'\sqrt{\mu'}} - \frac{\theta\sqrt{m(\mu-m)}}{\mu\sqrt{\mu}} = \frac{t\sqrt{\mu^3m'(\mu'-m')+\mu'^3m(\mu-m)}}{\mu\mu'\sqrt{\mu\mu'}},$$

$$d\theta' = frac\sqrt{\mu^3m'(\mu'-m')+\mu'^3m(\mu-m)}\mu\sqrt{\mu m'(\mu'-m')}dt,$$

and next

$$\frac{\theta\sqrt{\mu^3m'(\mu'-m')+\mu'^3m(\mu-m)}}{\mu\sqrt{\mu m'(\mu'-m')}} + \frac{t\mu'\sqrt{\mu'm(\mu-m)}}{\mu\sqrt{\mu m'(\mu'-m')}} = t',$$

$$\frac{\sqrt{\mu^3m'(\mu'-m')+\mu'^3m(\mu-m)}}{\mu\sqrt{\mu m'(\mu'-m')}}d\theta = dt',$$

that is, if one replaces first the variable θ' by t without changing θ, and next θ by t' without changing t, this probability of the preceding equation will become

$$\frac{1}{\pi}e^{-t^2-t'^2}dtdt'.$$

This equation becomes, at the same time,

$$\frac{m'}{\mu'} - \frac{m}{\mu} = \frac{t\sqrt{2\mu^3m'(\mu'-m')+2\mu'^3m(\mu-m)}}{\mu\sqrt{\mu\mu'}},$$

and containing no longer but the variable t, its total probability will be the integral relative to t' of this differential expression; an integral that one will be able to extend, without altering sensibly the value of it, from $t' = -\infty$ to $t' = \infty$, this which will give $\frac{1}{\sqrt{\pi}}e^{-t^2}dt$; whence one will conclude finally that $\frac{1}{\sqrt{\pi}}\int_{-u}e^{-t^2}dt$, or the quantity P given by formula (13), will express the probability that the difference $\frac{m'}{\mu'} - \frac{m}{\mu}$ is comprehended between the limits

$$\mp\frac{u\sqrt{2\mu^3m'(\mu'-m')+2\mu'^3m(\mu-m)}}{\mu\sqrt{\mu\mu'}},$$

in which u will be a positive and given quantity, and which contains only known numbers.

These limits coincide with those that we have found in n$^{\text{o}}$, in a manner much more simple, but for the case only where the chance of the event E is constant and the same in the two series of trials. However formula (24) of this section contains a term of order of $\frac{1}{\sqrt{\mu}}$ or $\frac{1}{\sqrt{\mu}}$', which is not found in formula (13); this which holds in this that, in the calculation we just made, we have neglected the terms of the probabilities that we have considered, which would be of this order of smallness.