Formules relatives aux probabilités qui dépendent de très grands nombers*

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"In the most important applications of the theory of probabilities, the chances of events are expressed by some fractions which have for numerator and for denominator some products of a great number of unequal factors; this which renders the calculation of these fractions completely impractical, either directly or by aid of logarithms. One is then obliged to recur to certain formulas of approximation of which Stirling has given the first example, that Euler has next considered, and that Laplace has made depend on a method of reduction in series proper to the quantities which he has named generally the *functions of large numbers*. These formulas have this of the singular that they contain the ratio of the circumference to the diameter, the base of the Naperian logarithms and other transcendents, which enter thus into the approximate values of quantities of which the exact values would be whole numbers or of ratios of similar numbers. Their usage is especially indispensable in the questions which have for object the chances of some future events deduced from observations of past events, that is to say in the most numerous questions of the calculus of probabilities; because it is rare that we know *a priori* the chances of events, and if one excepts the most simple games where it is possible to enumerate the favorable cases and the cases contrary to each event, we are nearly always obliged to substitute in this enumeration the knowledge of the numbers of times that the diverse events have taken place in some very great numbers of trials. But in this regard one must remark that the known rules of the theory of probabilities, and for example the theorem of Jacob Bernoulli, supposes implicitly that the chance of each event is the same in the series of trials already made and in the future trials, while on the contrary this chance varies the most often in an unknown and completely irregular manner, as well in the things of physical order as in those of the moral order. I have therefore sought to extend the rules of which there is concern to the general case of continually variable chances, and it is this which has led me to the demonstration of the *law of large numbers*, which one will find in the work of which I actually occupy myself. This law consists in this that if all the possible causes, knowns or unknowns, either of the arrival of an event, or of the magnitude of a thing, remain constantly the same in many sequences of a very great number of trials, whatever be besides the number and the nature of these causes, the ratio of the number of times that the event will arrive, to the number of experiences, thus as the sum of the magnitudes of

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. January 21, 2010 the thing which will be observed, divided by the number of observations, will remain also very nearly the same in the different sequences.

"In order to leave nothing vague and uncertain in the expression of this fundamental law, it is necessary to determine the probability that the difference between the results of two sequences of operations will be contained between some given limits; of such kind that if this probability neared much to certitude, and if the experience gives nevertheless a difference which exits from these limits, one is justified to conclude from it that the unknown causes of the events have changed in the interval of the two sequences. It is necessary thus to explicate, in a precise manner, this which one understands here by some causes which remain constantly the same. Now, when the arrival of an event or the magnitude of a thing are able to be attributed to different causes in any number, each of them has a determined probability, known or unknown, and gives to this arrival or to this magnitude a determined chance that one is able also to know or not know. This put, we say that a cause, whatever be its nature, is remained the same, when its particular probability, and the chances of the event, if its probability was certain, have not experienced any change. One makes besides no hypothesis on the magnitudes of this chance and of this probability; one eliminates them both, and the definitive formulas contain only some numbers given immediately by observations. It is for this, as I have said on numerous occasions, that these formulas arrive indistinctly to the things of each nature, physical or moral. In order to rise against their application to all the cases a difficulty which merited some attention, it would be necessary to show that their demonstration would not be satisfied, or else it would be necessary to cite some cases where the consequences which are deduced from it would have been denied by experience, that is to say some examples where the ratios which must be very nearly constants, would have varied notably, although no change had occurred in the causes of the events.

"Here is now a set of formulas susceptible of frequent and varied applications. Many are new; others have already been given in my preceding Memoirs, or were known before; all are demonstrated in a chapter in my work, where the delicate analysis on which they depend is exposed with all the necessary developments. The number of trials, supposed very great, is represented by μ; it is composed of two parts m and n which are supposed also very great numbers; the formulas are so much more close as this number μ is more considerable; and they would be completely exact if μ were infinite.

"I. Let p and q be the constant chances during all the duration of the trials, of two contrary events E and F, so that one has $p+q=1$. We name U the probability that in the number μ or $m+n$ trials, E will arrive m times and F will arrive n times. One will have

$$\mathrm{U} = \left(\frac{\mu p}{m}\right)^m \left(\frac{\mu q}{n}\right)^n \sqrt{\frac{\mu}{2\pi mn}} \tag{1}$$

π designating as ordinarily the ratio of the circumference to the diameter. This formula is reduced to

$$\mathrm{U} = \frac{1}{\sqrt{2\pi\mu pq}} e^{-v^2}$$

when one takes

$$m = \mu p - \nu\sqrt{2\mu pq}, \qquad n = \mu q + \nu\sqrt{2\mu pq};$$

ν being a given quantity, positive or negative, but very small with respect to $\sqrt{\mu}$, and e designating the base of the Naperian logarithms. And under this form, the expression of U subsists equally when the chances of E and F vary from one trial to another, by taking then for p and q the means of their values in the entire series of μ successive trials.

"II.[1] The events E and F having taken place effectively m and n times in the μ effective trials, and their constant chances p and q being unknowns, let U$'$ be the probability that they will arrive in μ' or $m'+n'$ future trials, of the number of times m' and n' proportionals to m and n, or such that one has

$$m' = \frac{\mu' m}{\mu} \qquad n' = \frac{\mu' n}{\mu}.$$

Whatever be the number μ', one will have

$$\mathrm{U}' = \sqrt{\frac{\mu}{\mu+\mu'}}\mathrm{U}_{\prime} \tag{2}$$

by representing by U the probability of the future event which will take place if the ratios $\frac{m}{\mu}$ and $\frac{n}{\mu}$ were certainly the chances of E and F, that is to say, by making for brevity

$$\frac{1.2.3\ldots\mu'}{1.2.3\ldots m'.1.2.3\ldots n'}\left(\frac{m}{\mu}\right)^{m'}\left(\frac{n}{\mu}\right)^{n'} = \mathrm{U}_{\prime}.$$

"III. The chances p and q of E and F were given, let P be the probability that in μ or $m+n$ trials, E will arrive at least m times and F at most n times. One will have

$$\left.\begin{aligned} \mathrm{P} &= \frac{1}{\sqrt{\pi}}\int_k^\infty e^{-t^2}dt + \frac{(\mu+n)\sqrt{2}}{3\sqrt{\pi\mu pq}}e^{-k^2}, \\ \mathrm{P} &= 1 - \frac{1}{\sqrt{\pi}}\int_k^\infty e^{-t^2}dt + \frac{(\mu+n)\sqrt{2}}{3\sqrt{\pi\mu pq}}e^{-k^2}; \end{aligned}\right\} \tag{3}$$

k being a positive quantity of which the square is

$$k^2 = n\log\frac{n}{q(\mu+1)} + (m+1)\log\frac{m+1}{p(\mu+1)},$$

where the logarithms are Naperian; and by employing the first or the second formula, according as one will have $\frac{q}{p} > \frac{n}{m+1}$, or $\frac{q}{p} < \frac{n}{m+1}$.

"IV. By naming R the probability that E and F will take place in μ trials, of the number of times which will not exit from the limits

$$\mu p \mp u\sqrt{2\mu pq}, \qquad \mu q \pm \sqrt{2\mu pq},$$

[1]*Translator's note*: At this point, Poisson introduces the prime as both subscript and superscript. The typesetter seems to have had difficulty with them. I have preserved their locations as it appears in the text rather than to attempt corrections. This problem occurs throughout.

where u is a positive and given quantity, but very small with respect to $\sqrt{u}$, one will have

$$\mathrm{R} = 1 - \frac{2}{\sqrt{\pi}}\int_u^\infty e^{-t^2}dt + \frac{1}{\sqrt{2\pi\mu pq}}e^{-n^2}; \tag{4}$$

and reciprocally if the chances p and q are unknowns, and if E and F are arrived the number of times m and n in μ or $m+n$ trials, one will have

$$\mathrm{R} = 1 - \frac{2}{\sqrt{\pi}}\int_n^\infty e^{-t^2}dt + \sqrt{\frac{\mu}{2\pi mn}}e^{-u^2}; \tag{5}$$

for the probability that the values of p and q will not exit from the limits

$$\frac{m}{\mu} \pm \sqrt{\frac{2mn}{\mu}}, \quad \frac{n}{\mu} \mp \sqrt{\frac{2mn}{\mu}}.$$

"V. In two different sequences of large numbers μ and μ' of trials, let m and m' be the numbers of times that E has taken place or will take place, n and n' the number of times that F will arrive or is arrived; we designate by u a positive quantity, very small with respect to $\sqrt{\mu}$ and $\sqrt{\mu'}$; and let ϖ be the probability that the difference $\frac{m}{\mu} - \frac{m'}{\mu'}$ will not exit from the limits

$$\mp\frac{u\sqrt{2(\mu^3 m'n' + \mu'^3 m'n')}}{\mu\mu'\sqrt{\mu\mu'}},$$

no more than the difference $\frac{n}{\mu} - \frac{n'}{\mu'}$, from these same limits taken with the contrary signs. One will have

$$\varpi = 1 - \frac{2}{\sqrt{\pi}}\int_u^\infty e^{-t^2}dt + \sqrt{\frac{\mu\mu'}{2\pi m'n'(\mu+\mu')}}e^{-\frac{u^2(\mu^3 m'n' + \mu'^3 mn)}{\mu^3 m'n'(\mu+\mu')}} \tag{6}$$

As one will have also very nearly $\frac{m}{\mu} = \frac{m'}{\mu'}$ and $\frac{n}{\mu} = \frac{n'}{\mu'}$, one will be able without altering sensibly the value of ϖ, to replace in its last term, what will be always a small fraction, the letters μ', m', n', by μ, m, n, and reciprocally those here by those there. This formula, by setting aside at least its last term, will agree in the general case where the chances of E and F will vary from one trial to another, provided that, in the two sequences, of possible causes, of events, known or unknown, experience no change, that is to say, provided that each of these causes, thus as one has said above, conserve the same probability and give always the same chance to the arrival of E or of F.

"VI. The number of times that E and F are arrived in the μ trials relative to these events, being always m and n, are generally $m_\prime$ and $n_\prime$, the number of times that two other contrary events $\mathrm{E}_\prime$ and $\mathrm{F}_\prime$ have taken place in a number $\mu_\prime$ trials, also very great. We suppose that one has

$$\frac{m'}{\mu_\prime} - \frac{m}{\mu} = \delta;$$

δ being a small positive or negative fraction. We call p and $p_\prime$ the unknown and supposed constant chances of E and $\mathrm{E}_\prime$; and we designate by Q the probability that $p_\prime$

exceeds p by a small positive and given fraction ε. By representing by u a positive quantity, and making

$$u = \pm \frac{(\varepsilon-\delta)\mu\mu_{\prime}\sqrt{\mu\mu_{\prime}}}{2(\mu^3 m_{\prime}n_{\prime}+\mu_{\prime}^3 mn)},$$

according as the factor $\varepsilon-\delta$ will be positive or negative, one will have

$$Q = \frac{1}{\sqrt{\pi}}\int_u^\infty e^{-t^2}dt, \quad Q = 1-\frac{1}{\sqrt{\pi}}\int_u^\infty e^{-t^2}dt \tag{7}$$

the first expression is returning to the case where the difference $\varepsilon-\delta$ will be positive, and the second in the case where this difference will be negative. The same formulas will express also the probability that the unknown chance p of the arrival of E surpasses the ratio $\frac{m}{\mu}$ given by observation, of a fraction ω also given. For this, it will suffice to make

$$u = \pm\left(\omega-\frac{m}{\mu}\right)\frac{\mu\sqrt{\mu}}{\sqrt{2mn}},$$

and to take the first or the second formula, according as the difference $\omega-\frac{m}{\mu}$ will be positive or negative.

"VII. When the chances of the two contrary events E and F vary from one trial to another, let p_i and q_i be their values relative to the trial, of which the rank is marked by i, so that one has $p_i+q_i=1$, for all the indices i. The sums $\sum$ extending from $i=1$ to $i=\mu$, we make, for brevity,

$$\frac{1}{\mu}\sum p_i = p, \quad \frac{1}{\mu}\sum q_i = q, \quad \frac{2}{\mu}\sum p_i q_i = k^2.$$

Let always m and n be the number of times that E and F are arrived in the first trials. By designating by u a positive and given quantity, very small with respect to $\sqrt{\mu}$, one will have

$$\mathrm{R} = 1-\frac{2}{\sqrt{\pi}}\int_u^\infty e^{-t^2}dt+\frac{1}{k\sqrt{\pi\mu}}e^{-u^2}, \tag{8}$$

for the probability that the ratios $\frac{m}{\mu}$ and $\frac{n}{\mu}$ will not exit from the limits

$$p \mp \frac{uk}{\sqrt{\mu}}, \qquad q \pm \frac{\mu k}{\sqrt{\mu}}$$

this which coincides with formula (4) in the particular case of constant chances.

"VIII. Any thing A being susceptible to all the values contained between the limits $h \mp g$, all these values being equally possible and the only possible, let P be the probability that in any number i of trials, the sum of the values of A which would take place, will be contained between the limits given also $c \mp \varepsilon$. One will have

$$2(2g)^i P = \frac{\Gamma-\Gamma^i}{1.2.3\ldots i}, \tag{9}$$

by making, for brevity,

$$\begin{aligned}
r = &\pm (ih+ig-c+\varepsilon)^i \mp i(ih+ig-2g-c+\varepsilon)^i \\
&\pm \frac{i.i-1}{1.2}(ih+ig-4g-c+\varepsilon)^i \mp \frac{i.i-1.i-2}{1.2.3}(ih+ig-6g-c+\varepsilon)^i \pm \text{etc.}, \\
r_{\prime} = &\pm (ih+ig-c-\varepsilon)^i \mp i(ih+ig-2g-c-\varepsilon)^i \\
&\pm \frac{i.i-1}{1.2}(ih+ig-4g-c-\varepsilon)^i \mp \frac{i.i-1.i-2}{1.2.3}(ih+ig-6g-c-\varepsilon)^i \pm \text{etc.},
\end{aligned}$$

by taking, in each term, the superior sign or the inferior sign, according as the quantity which is found raised to the power i is positive or negative: g and ε are some positive quantities, h and c are able to be some positive or negative quantities.

"IX. Whatever be the law of probability of the possible values of the thing A in each trial, and the manner of which this law will vary from one trial to another, if one calls s the sum of the values of A which will take place in a very great number μ of trials, one will have

$$\mathrm{P} = 1 - \frac{2}{\sqrt{\pi}} \int_u^{\infty} e^{-t^2} dt, \tag{10}$$

for the probability that the mean $\frac{s}{\mu}$ of the values of A will fall between the limits

$$k \mp \frac{2u\sqrt{h}}{\sqrt{\mu}};$$

u designating a positive quantity and very small with respect to $\sqrt{\mu}$; k and h being some quantities of which the second is positive, and which depends on the probabilities of the values of A during all the duration of the trials. When these probabilities will be constants, equals for all the possible values between the given limits a and b, and null beyond these limits, one will have

$$k = \frac{1}{2}(a+b), \quad h = \frac{b-a}{2\sqrt{6}}.$$

When A will have only a finite number of possible values $c_1, c_2, c_3, \ldots c_\nu$, and when these constant values will be all equally probable, one will have

$$\begin{gathered}
k = \frac{1}{\nu}(c_1 + c_2 + c_3 \cdots + c_\nu), \\
h = \frac{1}{2\nu^2}\left[\nu\left(c_1^2 + c_2^2 + c_3^2 \cdots + c_\nu^2\right) - (c_1 + c_2 + c_3 \cdots + c_\nu)^2\right].
\end{gathered}$$

"X. Let λ_n be the value of A which takes place at the n^{th} trial; and we make

$$\frac{1}{\mu}\sum \lambda_n = \lambda, \qquad \frac{1}{\mu}\sum(\lambda_n - \lambda)^2 = \frac{1}{2}l^2;$$

the sums $\sum$ extending from $n = 1$ to $n = \mu$. We suppose that the causes of all the possible values of A experience no change, either in their respective probabilities, or in the chances which they give to each of these values. There will be then a special quantity γ of which the mean $\frac{s}{\mu}$ of the values of A will approach indefinitely in measure as μ will increase more and more, and that it will attain, if μ became infinite. Now, formula (10) will express the probability that this quantity γ is contained between the limits

$$\frac{s}{\mu} \mp \frac{ul}{\sqrt{\mu}},$$

which contain nothing unknown.

"XI. In a second series of a very great number μ' of trials, let s' be the sum of the values of A, and l' that which the quantity l will become which is related in the first series. Formula (10) will express equally the probability that the difference $\frac{s'}{\mu'} - \frac{s}{\mu}$ of the two means will be contained between the limits

$$\frac{s}{\mu} \pm \frac{ul\sqrt{\mu l'^2 + \mu' l^2}}{\sqrt{\mu\mu'}},$$

or else, because one will have very nearly $l' = l$, this will be the probability that the mean $\frac{s'}{\mu}$, relative to the second series, will fall between the limits

$$\frac{s}{\mu} \pm \frac{ul\sqrt{\mu + \mu'}}{\sqrt{\mu\mu'}},$$

which depend only on the results of the first and on the given quantity u, and which are so much more narrow as μ' is greater with respect to μ.

"XII. In order to determine the value of one same thing A, one has made many sequences of trials, which comprehend very great numbers μ, μ', μ'', etc. The sums of the values of A, which one has obtained in these successive sequences, are s, s', s'', etc.; the preceding quantity l is related always to the first series, and one designates by l', l'', etc., this which it becomes in regard of the following sequences. One supposes that the causes of errors in the measures vary from one sequence to another, but that nevertheless all the means $\frac{s}{\mu}$, $\frac{s'}{\mu'}$, $\frac{s''}{\mu''}$, etc., converge indefinitely in measure as μ, μ', μ'', etc., increase more and more, toward one same unknown quantity γ, which would be the true value of A, in the most ordinary case where these causes do not render unequally probable, in any of these sequences of observations, the errors equal and of contrary signs. This put, formula (10) will express further the probability that the quantity γ is contained between the limits:

$$\frac{sq}{\mu} + \frac{s'q'}{\mu'} + \frac{s''q''}{\mu''} + \text{etc.} \mp \frac{u}{D},$$

in which one makes, for brevity,

$$\frac{\mu}{l^2} + \frac{\mu'}{l'^2} + \frac{\mu''}{l''^2} + \text{etc.} = D^2,$$

$$\frac{\mu}{D^2 l^2} = q, \quad \frac{\mu'}{D^2 l'^2} = q', \quad \frac{\mu''}{D^2 l''^2} = q'', \text{ etc.}$$

Moreover, the part $\frac{sq}{\mu}+\frac{s'q'}{\mu'}+\frac{s''q''}{\mu''}$+etc., that is to say the sum of the means $\frac{s}{\mu}$, $\frac{s'}{\mu'}$, $\frac{s''}{\mu''}$, etc., multiplied respectively by the quantities q, q', q'', etc., will be the approximate value of γ, the most advantageous that one is able to deduce from the concurrence of all the sequences of observations, that is to say the value of this unknown of which the limits of error $\mp\frac{u}{D}$ will have the least extent that is possible for a given value of u, or else, in equal degree of probability.

"XIII. Finally, the causes of the arrival of an event E remain the same during the trials, the ratio $\frac{m}{\mu}$ of the number of times that E will take place to the total number of trials, will converge indefinitely toward a special quantity r, which it would attain rigorously, if μ became infinite. Now, formula (6), by neglecting its last term, or else again formula (10), will be the probability that the unknown value r falls between the limits

$$\frac{m}{\mu}\pm\frac{u\sqrt{2m(\mu-m)}}{\mu\sqrt{\mu}}.$$

One finds at the end of the *Analyse des réfractione astronomique* of Kramp, a table of numerical values of $\int_n^\infty e^{-t^2}dt$, which extends from $u=0$ to $u=3$. This integral decreases very rapidly when the value of u increases; it is equal to $\frac{1}{2}\sqrt{\pi}$, for $u=0$, and its value falls below two hundred thousandths, for $u=3$. If one takes $u=0.4765$, one will have, very nearly,

$$\frac{2}{\sqrt{\pi}}\int_u^\infty e^{-t^2}dt=\frac{1}{2};$$

this which will reduce also to $\frac{1}{2}$ the probability expressed by formula (10). By giving to u a value such that $u=5$ or $u=4$, which without being considerable renders extremely small that of the integral $\int_u^\infty e^{-t^2}dt$, the results which one just announced contains the *law of large numbers* in all its generality."