Mémoire sur divers problèmes de probabilité*

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I give in this Memoir the solution of many questions concerning the probability that there is to bring forth a given sum, when one casts at random any number of polyhedrons of which the faces are marked by some positive and negative numbers. One knows that the theory of combinations offers a direct solution of the problems of this kind, by reducing them to the research of a certain term resulting from the development of a polynomial raised to a power. This research becomes so much more painful as the number of the dice that one considers is greater, so that if this number surpasses certain limits, the reduction of the formulas in numbers required some calculations of an excessive length. It is therefore principally in the case where the number of the polyhedra is very great that it is important to give some formulas susceptible of an easy application. The most general method for fulfilling this object is without doubt that which Mr. LAPLACE has given in the Mémoires de l'Académie de Paris (year 1782): It brings back the question of seeking a definite integral that one tries next to evaluate by a convergent series, by profiting from the circumstance of the great numbers that it contains.

By arresting oneself at the first enunciation of the problems of which there is question in this Memoir, one was able to believe that there are more curious than useful; but by examining the thing more closely, one does not delay to recognize that my principal object is the one to demonstrate in a manner at once simple and rigorous the principles relative to the mean that one must choose among the results of many observations, and it is without doubt under this relation that they must excite the attention of the Astronomer and the Physician. When one wishes to submit this theory to the analysis of hazards, it is first necessary, in order to better fix the ideas, to remove that which it appears to have vague, and it is for this reason that it has appeared to me simpler to present it under the form of problems concerning polyhedra. The mind is found thence habituated to reason on some simple and clear objects, that it knows with more quickness and clearness and passes next without efforts to the consequences of a greater utility.

One finds in the last Memoirs published by Mr. LAPLACE, very scholarly researches on this matter; my end will be fulfilled if the Academy comes to recognize that I have given some development to the ideas of this great man.

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Let us imagine a die composed of an even number of faces, expressed by $2n$. Let us suppose the first n faces respectively marked by the sequence of numbers 1, 2, 3,... n; and the n remaining faces marked with the same numbers taken negatively, that is, by the sequence -1, -2, -3, ... $-n$. One demands the probability that there is to bring forth a sum equal to zero, by casting at random a number P of similar polyhedras.

It is easy to see, by the theory of combinations, that the sought probability is found by raising to the power P the polynomial

$$x^{-n} + x^{-(n-1)} \cdots + x^{-2} + x^{-1} + x^{1} + x^{2} \cdots + x^{n-1} + x^{n} = X;$$

and by taking in the development the term independent of x. One would determine this coefficient by the method that LAGRANGE has given on page 206 of Volume V of the Mémoires de l'Académie de Turin; but the formula that one would find by operating thus would be so complicated for a considerable value of P that it would be nearly impossible to be able to reduce it to numbers. And in order to be convinced of it it suffices to note that in the very simple case where $n = 1$ and $P = 2q$, one has for the value of the sought coefficient

$$\frac{(q+1)(q+2)(q+3)\cdots 2q}{1.2.3\ldots q};$$

a formula of which the reduction into numbers is very painful, when q has a considerable value. One knows that STIRLING has crossed first this difficulty by reducing this formula into a descending series with respect to q, in a manner that one has, by naming π the semi-circumference of which the radius is unity,

$$x^{-n}+x^{-(n-1)} \cdots + x^{-2}+x^{-1}+x^{1}+x^{2} \cdots + x^{n-1}+x^{n} = \frac{2^{2q}}{\sqrt{q\pi}}\left(1 - \frac{1}{8q} + \frac{1}{128q^2} - \text{etc.}\right)$$

with so much more exactitude as q is a greater number.

2. By following the example of STIRLING we are going to attempt to reduce into a descending series, with respect to P, the term independent of x of the polynomial X^P. For these sorts of reduction, LAPLACE has given a general principle in the Mémoires de l'Académie de Paris. According to this principle, it is necessary to begin by expressing the function that there is concern to evaluate by a definite integral, and next it is necessary to develop this integral into a convergent series.

In order to know well the force of this principle it is necessary to apply it to many examples.

In order to find in our case the definite integral which is equal to the sought quantity, let us note first that since this is independent of the value of x, nothing prevents putting $x = e^{\varpi\sqrt{-1}}$ and considering the polynomial,

$$X^P = (2\cos\varpi + 2\cos 2\varpi + 2\cos 3\varpi \cdots + 2\cos n\varpi)^P \tag{1}$$

Let us suppose for an instant the second member of this equation developed; it is easy to comprehend that one will have a series of the form

$$2^P(A + A'\cos\varpi + A''\cos 2\varpi + \text{etc.}):$$

Now by multiplying this series by $d\varpi$, and integrating from $\varpi = 0$ to $\varpi = \pi$, it is clear that $2^P A\pi$ will be the result of the integration; therefore if one names Y the coefficient independent of ϖ in formula (1), one will have

$$Y = \frac{2^P}{\pi} \int d\varpi (\cos\varpi + \cos 2\varpi + \cos 3\varpi \cdots + \cos n\varpi)^P \tag{2}$$

the limits of the integral being $\varpi = 0$, $\varpi = 180^\circ$.

3. Now it is necessary to occupy ourselves with integrating this expression by a series descending with respect to P. As the greatest value of the function

$$\cos\varpi + \cos 2\varpi + \cos 3\varpi \cdots + \cos n\varpi$$

corresponds to $\varpi = 0$, in which case it is reduced to n, we will put

$$(\cos\varpi + \cos 2\varpi + \cos 3\varpi \cdots + \cos n\varpi)^P = n^P.e^{-t^2} \tag{3}$$

e designating the base of the hyperbolic logarithms. We will have therefore

$$Y = \frac{(2n)^P}{\pi} \int d\varpi.e^{-t^2}$$

where it is necessary to consider ϖ as a function of t which must be given by equation (3). In order to find the limits of t, let us note that by making $\varpi = 180^\circ$, equation (3) gives

$$0 = n^P.e^{-t^2}$$

if n is even; and

$$(-1)^P = n^P.e^{-t^2}$$

when n is odd. It follows thence that if n is *even* one will satisfy the equation $0 = n^P.e^{-t^2}$ by taking $t = \infty$, and that will be true, either by supposing P an even number, or by supposing P and odd number: But when n is *odd*, it is impossible to satisfy the equation $(-1)^P = n^P.e^{-t^2}$ by the real values of t at least unless P is an even number: Under this hypothesis one has $\frac{1}{n^P} = e^{-t^2}$, and as P is counted very great, and n greater than unity, it is evident that one will satisfy this equation by taking again $t = \infty$.

The limits of integration with respect to t are therefore $t = 0$, $t = \infty$. If one makes $\frac{1}{P} = \alpha$, equation (3) will give

$$\cos\varpi + \cos 2\varpi + \cos 3\varpi \cdots + \cos n\varpi) = n.e^{-\alpha t^2},$$

and by developing the first member according to the powers of ϖ one will have

$$n - \frac{\varpi^2}{1.2} \cdot S' + \frac{\varpi^4}{1.2.3.4} \cdot S'' - \frac{\varpi^6}{1.2.3.4.5.6} \cdot S''' + \text{etc.} = n.e^{-\alpha t^2}$$

by putting

$$S' = 1^2 + 2^2 + 3^2 + 4^2 \cdots + n^2;$$
$$S'' = 1^4 + 2^4 + 3^4 + 4^4 \cdots + n^4;$$
$$S' = 1^6 + 2^6 + 3^6 + 4^6 \cdots + n^6;$$
etc.

In order to give to this equation a simpler form, we will make

$$a = \frac{1}{1.2}\frac{S'}{n}; \quad b = \frac{1}{1.2.3.4}\frac{S''}{n}; \quad c = \frac{1}{1.2.3.4.5.6}\frac{S'''}{n}; \text{etc.}$$

so that one will have

$$\varpi^2(a - b\varpi^2 + c\varpi^4 - \text{etc.}) = 1 - e^{-\alpha t^2}$$

whence one draws

$$\begin{aligned}\varpi\sqrt{a}\left\{1 - \frac{b}{2a}\varpi^2 + \varpi^4\left(\frac{c}{2a} - \frac{b^2}{8a^2}\right) + \text{etc.}\right\}\\ = t\sqrt{\alpha}\left(1 - \frac{1}{4} + \frac{s}{96}\alpha^2 t^4 \text{ etc.}\right).\alpha t^2\end{aligned}$$

By applying to this equation the formula of NEWTON in order to return from the series one finds

$$\varpi = \sqrt{\frac{\alpha}{a}}.t\left(1 - \alpha t^2\left(\frac{1}{4} - \frac{b}{2a^2}\right)\right)$$

by neglecting the following terms. By drawing from this equation the value of $d\varpi$ one will conclude from it

$$y = \frac{(2n)^P}{\pi}\sqrt{\frac{\alpha}{a}}\int dt.e^{-t^2} + \frac{(2n)^P}{\pi}\frac{3\alpha\sqrt{\alpha}}{\sqrt{a}}\left(\frac{1}{4} - \frac{b}{2a^2}\right)\int dt.t^2 e^{-t^2};$$

Now between the prescribed limits one knows that

$$\int dt.e^{-t^2} = \frac{1}{2}\sqrt{\pi}; \qquad \int t^2 dt.e^{-t^2} = \frac{1}{4}\sqrt{\pi},$$

therefore

$$y = (2n)^P\sqrt{\frac{\alpha}{a\pi}}\left\{\frac{1}{2} - \frac{3\alpha}{4}\left(\frac{1}{4} - \frac{b}{2a^2}\right)\right\}.$$

One will have the values of a and of b by aid of the known formulas

$$\begin{aligned}S' &= \frac{n(n+1)(2n+1)}{1.2.3};\\ S'' &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{2.3.5},\end{aligned}$$

and by substituting them into that of y there will come

$$y = \frac{(2n)^P\sqrt{3}}{\sqrt{\pi P(n+1)(2n+1)}}\left\{1 - \frac{3}{8P}\cdot\frac{4n^2+9n+7}{5(n+1)(2n+1)}\right\}.$$

It is not necessary to neglect that this formula is true for all the positive integer values of n, when P is an even number; but if P is odd, it is necessary that n be an odd number.

By retaining only the first term of the preceding formula, that which suffices for the very great values of P, the sought probability will be equal to

$$\frac{\sqrt{3}}{\sqrt{\pi P(n+1)(2n+1)}}$$

If one supposes the number n considerable, this formula is reduced to

$$\frac{1}{n}\cdot\sqrt{\frac{3}{2\pi P}}.$$

4. It is not more difficult to resolve the same problem in the case where each die is composed of an odd number of faces expressed by $2n+1$, of which one is marked with zero. In fact, let y be the term independent of x resulting from the development of the polynomial.

$$(x^{-n}+x^{-(n-1)}\cdots+x^{-2}+x^{-1}+1+x^{1}+x^{2}\cdots+x^{n-1}+x^{n})^{P}$$

one will have here, by that which has been said previously,

$$y=\frac{1}{\pi}\int d\varpi(1+2\cos\varpi+2\cos2\varpi+cdots+2\cos n\varpi)^{P}$$

by integrating from $\varpi=0$ to $\varpi=180^{\circ}$.

Now if one puts

$$(1+2\cos\varpi+2\cos2\varpi+cdots+2\cos n\varpi)^{P}=(2n+1)^{P}.e^{-t^{2}} \tag{4}$$

one will have

$$y=\frac{(2n+1)^{P}}{\pi}\int d\varpi.e^{-t^{2}}$$

the limits of t being, whatever be n, $t=0$ and $t=\infty$.

By developing equation (4) as we have done in the preceding No., one will find,

$$1-e^{-\alpha t^{2}}=\frac{\varpi^{2}}{1.2}-\frac{2S'}{1+2n}-\frac{\varpi^{4}}{1.2.3.4}-\frac{2S''}{1+2n}+\text{etc.}$$

and thence it is quite easy to conclude from it, by aid of the preceding formulas,

$$y=\frac{(1+2n)^{P}\cdot\sqrt{3}}{\sqrt{2P\pi n(n+1)}}\left(1-\frac{3}{8P}\cdot\frac{(17n^{2}+17n+1)}{20n(n+1)}\right).$$

By conserving only the first term of this formula one will have

$$\frac{\sqrt{3}}{\sqrt{2P\pi n(n+1)}}$$

for the demanded probability: And if n is a very great number one will have

$$\frac{1}{n}\cdot\sqrt{\frac{3}{2P\pi}}$$

as in the preceding case.

5. One is able to render the enunciation of the problem of No. 1 more general, by demanding the probability that there is in order that the sum of the numbers marked on the face of each die be equal to a given quantity q. It is clear that this problem is reduced to determining the coefficient of x^q which is found in the development of the function X^P, or that which reverts to the same, to determining the coefficient of $\cos q\varpi$ of the function

$$(2\cos\varpi + 2\cos 2\varpi + cdots + 2\cos n\varpi)^P;$$

But $\cos q\varpi = \cos(-q\varpi)$; moreover it is evident that $\cos q\varpi$, and $\cos(-q\varpi)$ have the same coefficient, therefore it will be necessary to take only the half of the coefficient of $\cos q\varpi$ in order to have exactly the coefficient of x^q, or that which is yet simpler, it will suffice to take the term independent of ϖ of the function

$$\cos q\varpi(\cos\varpi + 2\cos 2\varpi + cdots + 2\cos n\varpi)^P.$$

It follows thence that if one names y the sought coefficient, one will have

$$y = \frac{2^P}{\pi}\int d\varpi.\cos q\varpi(\cos\varpi + 2\cos 2\varpi + cdots + 2\cos n\varpi)^P \tag{5}$$

the limiits of the integral being $\varpi = 0$, $\varpi = 180°$.

By substituting into this equation in place of

$$(\cos\varpi + 2\cos 2\varpi + cdots + 2\cos n\varpi)^P$$

its value found in No. 3, one will have

$$y = \frac{(2n)^P}{\pi}\cdot\frac{1}{\sqrt{qP}}\int dt.\cos q\varpi.e^{-t^2}\left(1 - \frac{3}{P}\left(\frac{1}{4} - \frac{b}{2q^2}\right)t^2\right).$$

Now we have

$$q\varpi = \frac{qt}{\sqrt{qP}} - \frac{q}{P}\frac{t^3}{\sqrt{aP}}\left(\frac{1}{4} - \frac{b}{2q^2}\right) + \text{etc.}$$

therefore all the time that q is very small with respect to P, and that P is a very great number, one will have by a sufficient approximation $\varpi = \frac{qt}{\sqrt{aP}}$, and

$$y = \frac{(2n)^P}{\pi}\cdot\frac{1}{\sqrt{aP}}\int dt.\cos\frac{qt}{\sqrt{aP}}\cdot e^{-t^2}$$

the limits of t being $t = 0$, $t = \infty$. But between the limits $x = 0$, $x = \infty$ it is demonstrate (See *Exercices de Calcul intégral de* LEGENDRE *p. 362*), that

$$\int dx.e^{-x^2}\cos ax = \frac{\sqrt{\pi}}{2}\cdot e^{-\frac{a^2}{4}}$$

therefore one will have

$$y = \frac{(2n)^P}{2\sqrt{\pi a P}} \cdot e^{-\frac{q^2}{4aP}},$$

or else

$$y = \frac{(2n)^P\sqrt{3}}{\sqrt{\pi P(n+1)(2n+2)}} \cdot e^{\frac{-3q^2}{P(n+1)(2n+1)}}$$

by substituting for a its value.

This formula shows us that the probability of bringing forth the sum q diminishes in measure as q increases. Besides, if one supposes $q = 0$, the value of y agrees with that found in No. 3 fo rthe same case.

Relatively to the case where n is also a very great number the probability of the sum q will be

$$\frac{1}{n} \cdot \sqrt{\frac{3}{2\pi P}} \cdot e^{-\frac{3q^2}{2n^2P}}.$$

6. We have supposed in the solution of the preceding probability $q < P$, but nothing prevents supposing $q > P$. In order to find under this hypothesis a convergent result, it is necessary to vary the process of integration in a manner to that which one is able to avoid the reduction into series of the factor $\cos q\varpi$.

By substituting into formula (5) in the place of

$$(\cos\varpi + 2\cos 2\varpi + cdots + 2\cos n\varpi)^P$$

its value

$$n^P(1 - a\varpi^2 + b\varpi^4 - \text{etc.})^P$$

found in No. 3, we will have

$$y = \frac{(2n)^P}{\pi}\int d\varpi.\cos q\varpi(1 - a\varpi^2 + b\varpi^4 - \text{etc.})^P.$$

In order to avoid the elevation to the power P of the polynomial, we note that one has

$$(1 - a\varpi^2 + b\varpi^4 - \text{etc.})^P = e^{P\log(1 - a\varpi^2 + b\varpi^4 - \text{etc.})}.$$

and by developing the logarithmic function

$$(1 - a\varpi^2 + b\varpi^4 - \text{etc.})^P = e^{-Pa\varpi^2}e^{P\varpi^4\left(\frac{2b-a^2}{2}\right)}$$

or else

$$(1 - a\varpi^2 + b\varpi^4 - \text{etc.})^P = e^{-Pa\varpi^2}\left(1 + P\frac{(2b-a^2)}{2}\varpi^4 + \text{etc.}\right)$$

It follows thence that by making $x = \varpi\sqrt{aP}$, one will have

$$y = \frac{(2n)^P}{\pi\sqrt{aP}}\int dx.\cos\frac{qx}{\sqrt{aP}} \cdot e^{-x^2}\left(1 + \frac{(2b-a^2)}{2}\frac{x^4}{a^2P} + \text{etc.}\right)$$

and since P is supposed very great, one will be able to take $x = 0$, $x = \infty$ for the limits of the integral, that which will give (See *Exercices de Calcul Intégral p.* 363)

$$y = \frac{(2n)^P}{2\sqrt{\pi a P}} \cdot e^{-\frac{q^2}{4aP}} \cdot \left\{1 + \frac{(2b - a^2)}{8Pa^2}\left(3 - \frac{3q^2}{aP} + \frac{q^4}{4a^2P^2}\right)\right\}$$

If one conserves only the first term of this formula one will have for y the same value that we have found previously.

7. An analogous reasoning to that which we have done in No. 7, proves that one has

$$y = \frac{1}{\pi}\int d\varpi \cos q\varpi(\cos\varpi + 2\cos 2\varpi + cdots + 2\cos n\varpi)$$

when the die is composed of $2n + 1$ faces among which there is one marked with zero; the limits of the integral being always $\varpi = 0$, $\varpi = 180°$.

By an analysis exactly conformed to that of the preceding No. one finds

$$y = \frac{(1 + 2n)^P}{2\sqrt{\pi P a'}} \cdot e^{-\frac{q^2}{4a'P}} \cdot \left\{1 + \frac{(2b' - a'^2)}{8Pa'^2}\left(3 - \frac{3q^2}{a'P} + \frac{q^4}{4a'^2P^2}\right)\right\}$$

$$a' = \frac{1}{1.2}\frac{2S'}{1 + 2n}; \quad b' = \frac{1}{1.2.3.4} - \frac{2S''}{1 + 2n}.$$

The first term of this formula gives

$$\frac{\sqrt{3}}{\sqrt{2\pi Pn(n+1)}} \cdot e^{-\frac{3q^2}{2Pn(n+1)}}$$

for the demanded probability: And if n is a very great number, it is reduced to

$$\frac{1}{n}\sqrt{\frac{3}{2\pi P}} \cdot e^{\frac{-3q^2}{2Pn^2}} \tag{A}$$

as in the case of No. 5.

8. In order to make an application of this formula, let us imagine a center of attraction placed in a fixed point which acts on a number P of bodies of which each has received an impulse. One knows that these bodies will describe some plane curves differently inclined with respect to a plane of determined position, and by supposing the impulses given at random, all the inclinations will be equally probable. Under this hypothesis it is curious to seek the probability that there is in order that the sum of the inclinations of the orbits are contained between the given limits $-\psi$ and $+\psi$.

If one takes the supplement of the inclinations which are greater than a right angle, all the orbits would be able to be considered as contained between two planes which are cut at right angle: Let us imagine now this right angle divided into tow equal parts, and let us fix the origin of the numeration of the angles at the point which corresponds to 50^g; thence an inclination of 57^g, for example, will become 7^g, according to this manner of counting, and an inclination of 40^g will be expressed by -10^g. Whence it follows that all the inclinations will be comprehended between -50^g and $+50^g$. Now, if one supposes the 50^g positive as well as the 50^g negative divided into an infinitely great number of equal parts expressed by n, it is clear that formula (A) of the preceding No. will give the probability in order that the sum of the inclinations is q, since the problem of which we speak returns to the one of a number P of dice having $2n+1$ faces.

The probability in order that the sum of the inclinations are contained between zero and $+\psi$ will be therefore

$$\frac{1}{n}\cdot\frac{\sqrt{3}}{\sqrt{2\pi P}}S\cdot e^{-\frac{3q^2}{2Pn^2}}.$$

The sign S of the finite integrals extending to all the values of q from $q=0$ to $q=+\psi$: And as this sum remains the same for the negative values of q comprehended between zero and $-\psi$ one will have

$$\frac{2}{n}\frac{\sqrt{3}}{\sqrt{2\pi P}}S\cdot e^{-\frac{3q^2}{2Pn^2}}$$

for the probability that the sum of the inclinations is comprehended between $-\psi+\psi$. Let $\frac{q}{n}=x$ and $\frac{\psi}{n}=B$: The change of the successive values of x being the infinitely small fraction $\frac{1}{n}$, one will be able to suppose $\frac{1}{n}=dx$, and to change the sing S of the finite integrals into the one of the infinitely small integrals, so that one will have

$$\sqrt{\frac{6}{\pi P}}\cdot\int dx.e^{-\frac{3x^2}{2P}} \qquad (b)$$

for the probability demanded, the limits of integration with respect to x being $x=0$ to $x=B$.

Let us apply this formula to the Comets. The number of those that one has observed to 1807 inclusively is 97. The sum of the inclinations of all these orbits each counted from 0^q to 100^q, is raised to 5032.033^g: And this same sum counted, as we have said, will be

$$5032.033^g - 97\times 50 = 182.033^g.$$

We have therefore $P=97$; $\psi=182.033^g$; $B=\frac{\psi}{n}=\frac{182.033}{50}=3.6406$. With these numbers formula (b) becomes

$$\frac{2}{\sqrt{\pi}}\int dx'.e^{-x'^2}$$

by putting $x'=x\cdot\sqrt{\frac{3}{2P}}$. The limits of x' are $x'=0$, $x'=0.45273$; Substituting this value of x' into the series

$$\int dx'.e^{-x'^2}=x-\frac{1}{1.2}\cdot\frac{x^3}{3}+\frac{1}{2.3}\cdot\frac{x^5}{5}-\text{etc.}$$

one will find

$$\frac{2}{\sqrt{\pi}}\int dx'.e^{-x'^2}=0.4934.$$

The division of 182.033 by 97 gives $1.87663°$. The fraction 9.4934 expresses therefore the probability that the mean inclination of the 97 observed Comets will be comprehended between the limes $50^g\pm 1.87663^g$, by admitting all the inclinations equally probable. It is therefore very likely that the hypothesis of an equal facility of inclination for these stars is that of nature, since the fraction 0.4934 is not small enough in order to reject it. The preceding result agrees with the one that LAPLACE has given in the Mémoires de l'Institut, year 1809, pag. 374.

9. Let us pass actually to the solution of a much more general problem than the preceding. Let $2n$ be the number of faces of each die, and p the number of those dice that one has cast at random. Let us name

$$\mathcal{E}'; \mathcal{E}''; \mathcal{E}'''; \dots \mathcal{E}^{(p)}$$

the numbers marked on the respective faces of these dice, and let us suppose each of these numbers multiplied by the one which corresponds in the sequence

$$q'; q''; q'''; \dots q^{(p)};$$

one demands the probability that there is in order that the sum

$$q'\mathcal{E}' + q''\mathcal{E}'' + q'''\mathcal{E}''' \cdots + q^{(p)}\mathcal{E}^{(p)} \tag{B}$$

of these products is equal to a given quantity q.

Let us suppose q', q'', q'''... $q^{(p)}$ whole numbers.

Let us designate by X' that which the polynomial X put into No. 1 becomes when one raises each of these terms to the power q', one will have

$$\begin{aligned}X' = x^{-q'n} + x^{-q'(n-1)} \cdots + x^{-2q'} + x^{-q'} + x^{q'} + x^{2q'} \cdots \\ + x^{q'(n-1)} + x^{q'n}.\end{aligned}$$

Let X'', X'''... $X^{(p)}$ be the successive values that this polygon takes by changing from q' to q''; q'' to q''' and so forth until $q^{(p)}$. It is clear, by the theory of combinations, that the problem of which there is concern is reduced to determining the coefficient of x^q which is found in the development of the function $X'.X''.X''' \dots X^{(p)}$. Now by putting $x = e^{\varpi\sqrt{-1}}$ one has

$$X' = 2\cos q'\varpi + 2\cos 2q'\varpi \cdots + 2\cos nq'\varpi,$$

or else

$$X' = 2S.\cos nq'\varpi$$

by extending the sign S of the finite integrals to all the values of n from 1 to n inclusively; therefore the coefficient of x^q will be equal to the half of the coefficient of $\cos q\varpi$ resulting from the development of the function

$$2^p.S\cos nq'\varpi.S\cos nq''\varpi \dots S\cos nq^{(p)}\varpi,$$

or, that which reverts to the same, it will be equal to the term independent of ϖ of the function

$$2^p.S\cos q\varpi.S\cos nq'\varpi.S\cos nq''\varpi \dots S\cos nq^{(p)}\varpi.$$

It follows thence that by naming y the coefficient of x^q, one will have

$$y = \frac{2^p}{\pi}\int d\varpi \cos q\varpi . S\cos nq'\varpi . S\cos nq''\varpi \dots S\cos aq^{(p)}\varpi$$

by integrating from $\varpi = 0$ to $\varpi = \pi$.

This posed, if one develops the function subjected to the sign S according to the powers of ϖ, one will have

$$\begin{aligned} S\cos nq' &= n.\{1 - aq'^2\varpi^2 + bq'^4 - \text{etc.}\}; \\ S\cos nq'' &= n.\{1 - aq''^2\varpi^2 + bq''^4 - \text{etc.}\}; \\ &\dots\dots \\ S\cos nq^{(p)} &= n.\{1 - aq^{(p)2}\varpi^2 + bq^{(p)4} - \text{etc.}\}; \end{aligned}$$

where the values of a, b, etc. are known by No. 3.

Now, if one forms the sum of the logarithms of the second members of these equations, one will have

$$\begin{aligned} &\log S\cos nq'\varpi + \log X\cos nq''\varpi \cdots + \log S\cos nq^{(p)}\varpi \\ &\qquad = \log n^p - a\varpi^2 P + \frac{(2b-a^2)}{2}\varpi^4 P' - \text{etc. } e \end{aligned}$$

by making

$$\begin{aligned} P &= q'^2 + q''^2 + q'''^2 \cdots + q^{(p)2}; \\ P' &= q'^4 + q''^4 + q'''^4 \cdots + q^{(p)4}. \end{aligned}$$

The value of y will be able therefore to be set under this form

$$y = \frac{(2n)^p}{\pi}\int d\varpi \cos q\varpi . e^{-a\varpi^2 P} . e^{\frac{(2b-a^2)}{2}P'\varpi^4}$$

or else, under this here,

$$y = \frac{(2n)^p}{\pi}\int d\varpi \cos q\varpi . e^{-a\varpi^2 P}\left\{1 + \frac{(2b-a^2)}{2}P'\varpi^4 + \text{etc.}\right\}.$$

This value of y is similar to that which we have found in No. 6, consequently one will be able to integrate by the same process, that which will give

$$y = \frac{(2n)^p}{2\sqrt{a\pi P}} . e^{\frac{-q^2}{4aP}}\left\{1 + \frac{(2b-a^2)}{8a^2}\cdot\frac{P'}{P^2}\left(3 - \frac{3q^2}{aP} + \frac{q^4}{4a^2P^2}\right)\right\},$$

or simply

$$y = \frac{(2n)^p\sqrt{3}}{\sqrt{\pi P(n+1)(2n+1)}}\cdot e^{\frac{3q^2}{P(n+1)(2n+1)}}$$

by taking only the first term of this formula. By changing arbitrarily the signs of the multipliers q', q'', etc. the preceding value of y will remain always the same, since P, P' are formed by the even powers of these multipliers.

According to that which one has seen in the preceding cases, one will comprehend without difficulty that if the number of faces is odd, one must have

$$y = \frac{(2n)^p}{2\sqrt{a'\pi P}}.e^{\frac{-q^2}{4a'P}}\left\{1 + \frac{(2b' - a'^2)}{8a^2}\cdot\frac{P'}{P^2}\left(3 - \frac{3q^2}{a'P} + \frac{q^4}{4a'^2P^2}\right)\right\},$$

the values of a' and b' being those that one finds in No. 7. The first term of this formula gives

$$\frac{\sqrt{3}}{\sqrt{2\pi Pn(n+1)}}\cdot e^{\frac{-3q^2}{2n(n+1)P}}$$

for the demanded probability, which is reduced to

$$\frac{1}{n}\cdot\sqrt{\frac{3}{2\pi P}}\cdot e^{\frac{-3q^2}{2Pn^2}}$$

under the very great hypothesis. Here, if one makes $\frac{q}{n} = x$; $\frac{1}{n} = dx$; $\frac{\psi}{n} = k$ one will have

$$\sqrt{\frac{6}{\pi P}}\int dx.e^{\frac{-3x^2}{2P}}$$

for the probability that the value of the function (B) will be comprehended between the limits $\mp\psi$. The limits of the integral being $x = -k$; $x = +k$.

10. The problem that we just resolved is able to be rendered more general by demanding the probability that there is in order to satisfy at the same time the two equations

$$q'\mathcal{E}' + q''\mathcal{E}'' + q'''\mathcal{E}''' \cdots + q^{(p)}\mathcal{E}^{(p)} = Q;$$
$$q_{\prime}\mathcal{E}' + q_{\prime\prime}\mathcal{E}'' + q_{\prime\prime\prime}\mathcal{E}''' \cdots + q_{(p)}\mathcal{E}^{(p)} = Q;$$

Let

$$\begin{aligned}X' &= x^{-nq'}\cdot y^{-nq_{\prime}} + x^{-(n-1)q'}\cdot y^{-(n-1)q_{\prime}}\cdots\\ &\quad + x^{-2q'}\cdot y^{-2q_{\prime}} + x^{-q'}\cdot y^{-q_{\prime}} + x^{q'}\cdot y^{q_{\prime}} + x^{2q'}\cdot y^{2q_{\prime}}\cdots\\ &\quad + x^{(n-1)q'}\cdot y^{(n-1)q_{\prime}} + x^{nq'}\cdot y^{nq_{\prime}};\\ X'' &= x^{-nq''}\cdot y^{-nq_{\prime\prime}}\cdots + x^{-2q''}\cdot y^{-2q_{\prime\prime}} + x^{-q''}\cdot y^{-q_{\prime\prime}} + x^{+q''}\cdot y^{+q_{\prime\prime}}\\ &\quad + x^{2q''}\cdot y^{2q_{\prime\prime}}\cdots + x^{nq''}\cdot y^{nq_{\prime\prime}};\\ &\quad \text{etc.}\end{aligned}$$

If one supposes the function $X'.X''.X'''\ldots X^{(p)}$ developed it is clear that the probability demanded will be given by the coefficient of $x^Q.y^{Q'}$, which is found in the development. But if one makes

$$x = e^{\varpi\sqrt{-1}}; \qquad y = e^{\varpi\sqrt{-1}}$$

one has

$$X' = 2S\cos n(q'\varpi + q_{\prime}\varpi'); \quad X'' = 2S\cos n(q''\varpi + q_{\prime\prime}\varpi'); \quad \text{etc.}$$

therefore the coefficient of $x^Q.y^{Q'}$ is equal to the term independent of ϖ and of ϖ' which is found in the product

$$\cos(Q\varpi + Q'\varpi')X'.X''.X''' \dots X^{(p)}.$$

To the exclusion of the term independent of ϖ and of ϖ' it is evident that any one term of this product must have one or the other of these three forms

$$A\cos(\alpha\varpi + \beta\varpi'), \quad B\cos M\varpi, \quad C\cos N\varpi'.$$

Now, by multiplying the first two of these functions by $d\varpi$ and integrating them from $\varpi = -\pi$ to $\varpi = \pi$ one has zero for result; likewise by multiplying the third by $d\varpi'$ and integrating between the same limits one has again zero; therefore if one names z the coefficient of $x^Q.y^{Q'}$ one will have between the prescribed limits

$$z = \frac{1}{4\pi^2}\int d\varpi' \int d\varpi \cos(Q\varpi + Q'\varpi').X'.X''.X''' \dots X^{(p)},$$

since by this double integration all the terms vanish, except the one which is independent of ϖ and of ϖ'. Now by a calculation analogous to the one of the preceding No., one will find

$$X'.X''.X''' \dots X^{(p)} = (2n)^p.e^{-aP}.e^{\frac{(2b^2-a^2)P}{2}}$$

by putting

$$\begin{aligned} P &= (q'\varpi + q_{\prime}\varpi')^2 + (q''\varpi + q_{\prime\prime}\varpi')^2 \dots + (q^{(p)}\varpi + q_{(p)}\varpi')^2; \\ P' &= (q'\varpi + q_{\prime}\varpi')^4 + (q''\varpi + q_{\prime\prime}\varpi')^4 \dots + (q^{(p)}\varpi + q_{(p)}\varpi')^4; \\ &\text{etc.} \end{aligned}$$

And if one makes

$$\begin{aligned} A &= q'^2 + q''^2 + q'''^2 \dots + q^{(p)2}; \\ B &= q'q_{\prime} + q''q_{\prime\prime} + q'''q_{\prime\prime\prime} \dots + q^{(p)2}q_{(p)}; \\ C &= q_{\prime}^2 + q_{\prime\prime}^2 + q_{\prime\prime\prime}^2 \dots + q_{(p)}^2, \end{aligned}$$

one will have

$$z = \frac{(2n)^p}{4\pi^2}\int d\varpi' \int d\varpi \cos(Q\varpi + Q'\varpi').e^{-(A\varpi^2 + 2aB\varpi\varpi' + Ca'\varpi'^2)}$$

by omitting the factor $\frac{(2b^2-a^2)}{2}P'$ which produce only some very small terms in the result of the integration.

If one makes

$$x = \varpi\sqrt{ap}; \qquad x' = \varpi'\sqrt{ap}$$

the preceding value of z becomes

$$z = \frac{(2n)^p}{4a\pi^2}\int dx' \int dx \cos\left(\frac{Qx}{\sqrt{ap}} + \frac{Q'x}{\sqrt{ap}}\right).e^{-\frac{Ax^2}{p} - \frac{2aBxx'}{p} - \frac{Ca'x'^2}{p}}$$

the limits of x and of x' being $-\infty$ and $+\infty$ since one supposes p very great.

In order to render possible the double integration by known methods, let us substitute in place of the cosine it exponential value, we will have

$$z = \frac{(2n)^p}{2.4ap\pi^2}\int dx' \int dx.e^{-(\alpha x^2+\beta x'^2+\gamma xx'+\delta x+\mathcal{E}x^2)}$$
$$+\frac{(2n)^p}{2.4ap\pi^2}\int dx' \int dx.e^{-(\alpha x^2+\beta x'^2-\gamma xx'-\delta x-\mathcal{E}x^2)}$$

by making, for more simplicity,

$$\alpha = \frac{A}{p}; \quad \beta = \frac{C}{p}; \quad \gamma = \frac{2B}{p}; \quad \delta = \frac{Q\sqrt{-1}}{\sqrt{ap}}; \quad =\frac{Q'\sqrt{-1}}{\sqrt{ap}}.$$

Now it is necessary to transform the exponent

$$\alpha x^2 + \beta x'^2 + \gamma xx' + \delta x + \mathcal{E}x^2 = Y$$

of the number e into another containing only the squares of the two variables. For that one will put

$$x = u - \frac{\gamma}{2a}u' + f; \qquad x' = u' - h;$$
$$f = \frac{\gamma\mathcal{E} - 2\beta\delta}{4\alpha\beta - \gamma^2}; \qquad h = \frac{2\alpha\mathcal{E} - \gamma\delta}{4\alpha\beta - \gamma^2};$$

and one will have

$$Y = \alpha u^2 + \left(\frac{4\alpha\beta - \gamma^2}{4\alpha}\right).u'^2 + H;$$
$$H = \frac{\alpha\mathcal{E}^2\gamma^2 + \beta\gamma^2\delta^2 - \mathcal{E}\delta\gamma^3 - 4\alpha\beta^2\delta^2 - 4\alpha^2\mathcal{E}^2 + r\alpha\beta\gamma\delta\mathcal{E}}{(4\alpha\beta - \gamma^2)^2}.$$

Substituting in the place of α, β, γ, δ, $\mathcal{E}$ their values one will find, after the reductions,

$$Y = \frac{A}{p}\cdot u^2 + \left(\frac{AC - B^2}{pA}\right)\cdot u'^2 + H;$$

$$H = \frac{CQ^2 - 2BQQ' + AQ'^2}{4a(AC - B^2)}.$$

It follows thence that by putting

$$E = AC - B^2$$

one will have

$$\int dx' \int dx.e^{-Y} = \int du' \int du.e^{\frac{-A.u^2}{p}}.e^{\frac{-E.u'^2}{Ap}}.e^{-H}$$

the limits of u and of u' being the same as those of x and of x'. Now one knows that from $x = -\infty$ to $x = +\infty$, one has

$$\int dx.e^{-x^2} = \sqrt{\pi},$$

therefore

$$\int dx' \int dx.e^{-Y} = e^{-H} \cdot \frac{\pi p}{\sqrt{E}}.$$

If one examines in the least the first transformation of Y one will comprehend that in regard to the function

$$\alpha x^2 + \beta x'^2 - \gamma x x' - \delta x - \mathcal{E} x^2 = Y'$$

one must yet have

$$\int dx' \int dx.e^{-Y'} = e^{-H} \cdot \frac{\pi p}{\sqrt{E}}.$$

By reuniting these two integrals, one will have finally

$$z = \frac{(2n)^p}{4a\pi\sqrt{E}} \cdot e^{\frac{-1}{4aE}(CQ^2 - 2BQQ' + AQ'^2)} \qquad (\alpha')$$

or else

$$z = \frac{3(2n)^p}{\pi(n+1)(2n+1)\sqrt{E}} \cdot e^{\frac{-3(CQ^2 - BQQ' + AQ'^2)}{E(n+1)(2n+1)}}$$

by substituting for a its value (No. 3).

Relative to the case where the number of faces of each die would be equal to equal to $2n+1$ one would have

$$z = \frac{3(1+2n)^p}{\pi n(n+1)\sqrt{E}} \cdot e^{\frac{-3(CQ^2 - BQQ' + AQ'^2)}{En(n+1)}}$$

11. The same method is applied to the case where the concern is to determine the probability that there is to satisfy at the same time the three following equations,

$$q'\mathcal{E}' + q''\mathcal{E}'' + q'''\mathcal{E}''' \cdots + q^{(p)}\mathcal{E}^{(p)} = Q;$$
$$q_{\prime}\mathcal{E}' + q_{\prime\prime}\mathcal{E}'' + q_{\prime\prime\prime}\mathcal{E}''' \cdots + q_{(p)}\mathcal{E}^{(p)} = Q;$$
$$r'\mathcal{E}' + r''\mathcal{E}'' + r'''\mathcal{E}''' \cdots + r^{(p)}\mathcal{E}^{(p)} = Q''.$$

By some considerations absolutely similar to those of the preceding No., one would find that in this case one must have

$$z = \frac{1}{8\pi^3} \int d\varpi'' \int d\varpi' \int d\varpi \cos(Q\varpi + Q'\varpi' + Q''\varpi'') X'X'' \ldots X^{(p)}.$$
$$X' = 2S \cos n(q'\varpi + q_{\prime}\varpi' + r'\varpi'');$$
$$X'' = 2S \cos n(q''\varpi + q_{\prime\prime}\varpi' + r''\varpi'');$$

etc.

the limits of these integrals being always $\varpi = \varpi' = \varpi'' = -\pi$; $\varpi = \varpi' = \varpi'' = +\pi$. Now if one transforms the product $X'X''X''' \dots X^{(p)}$ in the ordinary manner one will have, by retaining only the first term,

$$z = \frac{(2n)p}{8\pi^3} \int d\varpi'' \int d\varpi' \int d\varpi \cos(Q\varpi + Q'\varpi' + Q''\varpi'').e^{-aP}$$
$$P = (a'\varpi + q_{\prime}\varpi' + r'\varpi'')^2 + (q''\varpi + q_{\prime\prime}\varpi' + r''\varpi'')^2 \cdots$$
$$\cdots\cdots + (q^{(p)}\varpi + q_{(p)}\varpi' + r^{(p)}\varpi'')^2.$$

Let us make

$$x = \varpi\sqrt{ap}; \quad x' = \varpi'\sqrt{ap}; \quad x'' = \varpi''\sqrt{ap}$$
$$A = q'^2 + q''2 + q'''^2 \cdots + q^{(p)2};$$
$$B = q_{\prime}^2 + q_{\prime\prime}^2 + q_{\prime\prime\prime}^2 \cdots + q_{(p)}^2;$$
$$C = r'^2 + r''^2 + r'''^2 \cdots + r^{(p)2}$$
$$D = q'q_{\prime} + q''q_{\prime\prime} + q'''q_{\prime\prime\prime} \cdots + q^{(p)}q_{(p)};$$
$$E = q'r' + q''r'' + q'''r''' \cdots + q^{(p)}r^{(p)};$$
$$F = q_{\prime}r' + q_{\prime\prime}r'' + q_{\prime\prime\prime}r''' \cdots q_{(p)}r^{(p)};$$

one will have

$$P = Ax2 + Bx'^2 + Cx'^2 + 2Dxx' + 2Exx'' + 2Fx'x'';$$
$$z = \frac{(2n)^p}{8\pi^3} \int dx'' \int dx' \int dx \cos\left(\frac{Qx}{\sqrt{ap}} + \frac{Q'x}{\sqrt{ap}}\right).e^{-\frac{P}{p}}$$

If one makes $\alpha = \frac{A}{p}$; $\beta = \frac{B}{p}$; $\gamma = \frac{C}{p}$; $\delta = \frac{2D}{p}$; $\mathcal{C} = \frac{2E}{p}$; $\zeta = \frac{2F}{p}$; $\varsigma = \frac{Q\sqrt{-1}}{\sqrt{ap}}$; $\theta = \frac{Q'\sqrt{-1}}{\sqrt{ap}}$; $\tau = \frac{Q''\sqrt{-1}}{\sqrt{ap}}$;

$$X = \alpha x2 + \beta x'^2 + \gamma x''2 + \delta xx' + \mathcal{E}xx'' + \zeta x'x'' + \varsigma x + \theta x' + \tau x'';$$
$$Y = \alpha x2 + \beta x'^2 + \gamma x''2 + \delta xx' + \mathcal{E}xx'' + \zeta x'x'' - \varsigma x - \theta x' - \tau x'';$$

and if in the place of the cosine one substitutes its exponential value, one will have

$$z = \frac{(2n)^p}{2.8\pi^3} \int dx'' \int dx' \int dx.e^{-X} + \frac{(2n)^p}{2.8\pi^3} \int dx'' \int dx' \int dx.e^{-Y}.$$

In order to render possible these integrations, it is necessary to transform the functions X, Y into some others which contain only the squares of the variables. Here is the indication of this calculation for X. One will put

$$x = u + Ku' + gu'' + h;$$
$$x' = u' + mu'' + f;$$
$$x'' = u'' - b$$

and one will have in order to determine the coefficients k, g, h, m, f, b the following equations;

$$2\alpha k + \delta = 0$$

$$g = \frac{\delta\zeta - 2\beta\mathcal{E}}{4\alpha\beta - \delta^2};$$

$$m = \frac{\mathcal{E}\delta - 2\alpha\zeta}{4\beta\alpha - \delta^2};$$

$$b = \frac{\varsigma\delta\zeta + \theta\mathcal{E}\delta - \tau\delta^2 - 2\beta\varsigma\mathcal{E} - 2\alpha\zeta\theta + 4\alpha\beta\tau}{2\mathcal{E}\delta\zeta - 2\gamma\delta^2 - 2\beta\mathcal{E}^2 - 2\alpha\zeta^2 + 8\alpha\beta\gamma};$$

$$f(4\alpha\beta - \delta^2) - b(2\alpha\zeta - \delta\mathcal{E}) + 2\alpha\theta - \varsigma\delta = 0; \tag{I}$$

$$2\alpha h + \delta f - b\mathcal{E} + \varsigma = 0 \tag{II}$$

according to which the value of X is reduced to

$$X = \alpha u^2 + Gu'^2 + Hu''2 + N$$

by putting

$$G = \frac{4\alpha\beta - \delta^2}{4\alpha};$$

$$H = \frac{\left\{\begin{array}{c} \mathcal{E}\zeta\delta^3 + \gamma\delta^4 - \beta\mathcal{E}\delta^2 - \alpha\delta^2\zeta 2 - 8\alpha\beta\gamma\delta^2 - 4\alpha\beta\delta\mathcal{E}\zeta \\ +r\beta\alpha^2\zeta^2 + 4\alpha\beta^2\mathcal{E}^2 + 16\gamma\alpha^2\beta^2 - 2\alpha\zeta^2 - 2\beta\mathcal{E}^2 \end{array}\right\}}{(4\alpha\beta - \delta^2)^2}$$

$$N = \alpha h^2 + \beta f^2 + \gamma b^2 + \delta fh - \mathcal{E}bh - \zeta bf + \varsigma h + \theta f - \tau h.$$

By transforming Y in the same manner one will have the same result that we just obtained for X; thus by effecting the integrations from negative infinity to positive infinity, one will have

$$z = \frac{(2n)^p \cdot e^{-N}}{8\pi\sqrt{\pi}\sqrt{\alpha GH}}.$$

In order to better understand the form of the function N we develop further the preceding equations.

By substituting into the value of b in the place of α, β, etc. their values one finds

$$b = \frac{1}{2} \cdot \sqrt{\frac{p}{a}} \cdot \sqrt{-1} \cdot \left\{\frac{IQ'' + I'Q' + I''Q}{IC + I'F + I''E}\right\}$$

by making

$$I = AB - D^2; \quad I' = DE - AF; \quad I'' = DF - BE.$$

The same substitutions change equations (I) and (II) into these here;

$$If + I'b + \frac{1}{2}\sqrt{\frac{p}{a}} \cdot \sqrt{-1} \cdot (AQ' - DQ) = 0$$

$$Ah + Df - Eb + \frac{1}{2}\sqrt{\frac{p}{a}} \cdot \sqrt{-1}Q = 0.$$

If follows thence that if one puts

$$M = \frac{IQ'' + I'Q" + I''Q}{IC + I'F + I''E};$$
$$M' = AQ' - DQ + MI';$$
$$M'' = BQ - Q'D + MI''$$

one will have

$$b = \frac{1}{2}\sqrt{\frac{p}{a}} \cdot \sqrt{-1} \cdot M; \qquad f = -\frac{1}{2}\sqrt{\frac{p}{a}} \cdot \sqrt{-1} \cdot \frac{M'}{1};$$
$$h = -\frac{1}{2}\sqrt{\frac{p}{a}} \cdot \sqrt{-1} \cdot \frac{M''}{1}.$$

Now we have

$$PN = Ah^2 + Bf^2 + Cb^2 - 2Fbf + \sqrt{\frac{p}{a}} \cdot \sqrt{-1} \cdot Q'f - \sqrt{\frac{p}{a}} \cdot \sqrt{-1} \cdot Q''h$$

therefore by the substitution of the values of b, f, h, one will have

$$4aIN = AQ'^2 + BQ^2 - 2DQQ' + 2DQ'Q'' - 2BQQ'' = 2M(QI' - Q''I'' + QI'') - M^2(CI + FI' + EI'')$$

or else

$$raIN = AQ'^2 + BQ2 - 2DQQ' + 2DQ'Q'' - BQQ'' + \frac{2Q'Q''I'' - 2QQ''I''^2 + 2QQ'' + Q'^2I'^2 + Q2I''^2 - Q''^2(I^2 + 2I'' + I''^2)}{IC + I'F + I''E}$$

by replacing M with its value.

One sees by this equation that the value fo N is a homogeneous function of the second dimension with respect to Q, Q', Q'', this which is analogous to that which holds for H in the problem of the preceding No.

12. In all the problems resolved until now we have supposed that the polyhedron which has served us as example has a number of faces marked with each of the numbers of the sequence 0; ± 1; ± 2; $\pm 3 \ldots \pm n$. But one is able to generalize the question by enunciating it thus: Let h be the total number of faces of the polyhedron; let us name a the number of thes faces marked with a zero; $2a'$ the number of those marked, half with positive unity, and half with negative unity; $2a''$ the number of those marked half $+2$, and half -2; by continuing in the same manner one will form the equation

$$a + 2a' + 2a'' + 2a''' \cdots + 2a(n) = h \qquad \text{(I)}$$

This put, let us propose to resolve with this change of circumstances the same problem that we have enunciated at the beginning of No. 9.

It is clear that here it will be necessary to consider the polynomial

$$X' = a^{(n)}.x^{-nq'} + a^{(n-1)}.x^{-(n-1)q'} \cdots + a''x^{-2q'} + a'x^{-q'} + ax^0$$
$$+a'x^{q'} + a''x^{2q'} \cdots + a^{(n)}x^{nq'}$$

and to determine the coefficient of xq resulting from the development of the function $X'.X''.X''' \ldots X^{(p)}$; X''; $X''' \ldots X^{(p)}$ being the successive values, that take X' with the change from q' into q'', $q''' \ldots q(p)$. By making, as in the preceding case, $x = e^{\pi\sqrt{-1}}$, one will have

$$X' = a + 2a'.\cos q'\varpi + 2a''.\cos 2q'\varpi \cdots + 2a^{(n)}.\cos nq'\varpi.$$

It is not necessary to repeat here the reasoning that we have already made in order to comprehend that by naming y the sought coefficient one must have

$$y = \frac{1}{\pi}\int d\varpi \cos q\varpi.X'.X''.X''' \ldots X^{(p)}$$

by integrating from $\varpi = 0$ to $\varpi = \pi$.

Developing X' according to the powers of ϖ one will have

$$X' = h\left(1 - \frac{h'}{k}\varpi^2 + \frac{h''}{k}\varpi^4 + \text{etc.}\right)$$

by putting

$$a' + a''.22 + a'''.3^2 \cdots + a^{(n)}.n^2 = h' \qquad \text{(II)}$$
$$\frac{1}{12}\left(a' + a''.24 + a'''.34 \cdots + a^{(n)}.n^4\right) = h''.$$

Now the transformation used will give us

$$X'.X''.X''' \ldots X^{(p)} = h^p e^{\frac{-h'}{h}P\varpi^2}\left\{1 + \frac{\left(\frac{2h''}{h} - \frac{h'^2}{h2}\right)}{2}P'\varpi^4\right\}$$

by putting

$$P = q'^2 + q''^2 + q'''^2 \cdots + q^{(p)2};$$
$$P' = q'^4 + q''^4 + q'''^2 \cdots + q^{(p)4}.$$

If one retains only the first term of this series, one will have

$$y = \frac{hp}{\pi}\int d\varpi.\cos q\varpi.e^{-\frac{h'}{h}P\varpi^2};$$

whence one concludes by the preceding formulas

$$y = \frac{h^p}{2\sqrt{\pi P \cdot \frac{h'}{h}}} \cdot e^{-\frac{hq^2}{4Ph'}}.$$

This value of y divided by h^p, which expresses the total number of combinations of a number p of polyhedra such as the one that we have described, will give

$$y = \frac{1}{2\sqrt{\pi P \frac{h'}{h}}} \cdot e^{-\frac{hq^2}{4h'P}} \qquad (\alpha)$$

for the probability of satisfying the equation

$$q'\mathcal{E}' + q''\mathcal{E}'' + q'''\mathcal{E}''' \cdots + q^{(p)}\mathcal{E}^{(p)} = q.$$

The quantities h and h' are counted known by equations (I) and (II).

Before going further I would make here a remark which will be useful for us in the following. If one takes only the first term of the value of y found in No. 9 one has

$$\frac{1}{2\sqrt{\pi a P}} . e^{\frac{-q^2}{4aP}}$$

for the expression of the probability. This function is of the same form as that designated by (α) and differs from it only by the value of the constant a which in that one is expressed by $\frac{h'}{h}$. By departing from that consideration, one would have obtained first the solution of the problem.

In the case where the law of the probability of each of the numbers of the sequence 0; ± 1; ± 2; $\pm 3 \ldots \pm n$ will be expressed by a function of one variable, one would be able to obtain the values of h and of h' by the calculation of the finite differences. In fact, let $F\left(\frac{x}{2n}\right)$ be a function such that one has

$$F\left(\frac{x}{2n}\right) = F\left(\frac{-x}{2n}\right),$$

and that by making successively $x = 0, 1, 2, 3, 4, \ldots n$ one had for result $\frac{a}{h}$, $\frac{a'}{h}$, $\frac{a''}{h} \ldots \frac{a^{(n)}}{h}$. The equations (I) and (II) would become

$$1 = \frac{a}{h} + 4n\frac{1}{2n} SF\left(\frac{x}{2n}\right);$$

$$\frac{h'}{h} = 8n^3$$

$$cdot \frac{1}{2n} S\left(\frac{x}{2n}\right) 2F\left(\frac{x}{2n}\right),$$

the sign S of the finite integral extending to all the values from $x = 1$ to $x = n$.

But if the number n is very great, then one is able to suppose $\frac{x}{2n} = \frac{x'}{b}$; $\frac{1}{2n} = \frac{dx'}{b}$, and to change the sign S into the one of the infinitely small integrals, so that by neglecting the very small fraction $\frac{a}{h}$ the two preceding equations will give

$$1 = \frac{4n}{b} \int dx' . F\left(\frac{x'}{b}\right)$$

$$\frac{h'}{h} = \frac{8n^3}{b} \int dx' . \frac{x'^2}{b^2} F\left(\frac{x'}{b}\right)$$

by integrating from $x' = 0$ to $x = \frac{1}{2}b$.

Let there be, for more simplicity,

$$2\int dx'.F\left(\frac{x'}{b}\right) = K; \quad \int dx'.\frac{x'^2}{b^2}.F\left(\frac{x'}{b}\right) = K',$$

one will have

$$\frac{h^p}{h} = 4n^2 \cdot \frac{K'}{K}$$

and formula (α) will become, by multiplyin it by 2 and putting $\frac{q}{2n} = \frac{d}{b}$,

$$\frac{1}{\frac{1}{2n}\sqrt{\pi P \cdot \frac{K'}{K}}}.e^{\frac{-K}{4K'P}\cdot\frac{a'^2}{b^2}} \qquad (\beta)$$

Such is the probability for satisfying the equation

$$q'\mathcal{E}' + q''\mathcal{E}'' + q'''\mathcal{E}''' \cdots + q^{(p)}\mathcal{E}^{(p)} = \pm q.$$

when n is infinitely great, and when p is a considerable number.

13. Let us replace in formula (β) $\frac{a'}{b}$ by $\frac{q}{2n}$; we will have

$$\frac{1}{2n} \cdot \frac{1}{\sqrt{\pi P \cdot \frac{K'}{K}}}.e^{\frac{-K}{4K'P}\cdot\frac{q^2}{4n^2}}.$$

Let E; E'; E''; E''', etc. be the values that this formula takes by making successively $q = 0$, 1, 2, 3, etc. This put, let us imagine a player subject to the following condition: If the sum designated by q is equal to zero, the player will pay nothing; if the value $q$is ± 1, the player will pay a certain sum; but he will pay the double, the triple, the quadruple, etc.; if the value brought forth from q is ± 2; ± 3; ± 4 etc. One demands the sum that this player must pay in supposing that he not wish to be exposed to such a game.

The only probability favorable to the player is E; all the others E'; E''; E'''; etc. are contrary to him; and although the probabilities are decreasing, they do not permit from increasing the disadvantage of the player by reason of the greatest sum that he must pay. Because it is clear that the probability E'' is equivalent to the probability $2E'$ provided that the sum to pay is the same as that which corresponds to E'; likewise the probability E''' is equivalent to $3E''$ if the sum to pay remains the same as for E'; and so forth. Thence one concludes that the state of the player is the same as if he had against him the probabilities E', $2E''$, $3E'''$ etc., and in his favor the sole probability E, with the condition of always losing the same sum, whatever be the value of q that he will bring forth. Therefore the lot of the player, or that which reverts to the same, the mean value of q, will be given by the sum

$$E' + 2E'' + 3E''' + 4E'''' + \text{etc.} = \frac{1}{2n\sqrt{\pi P \cdot \frac{K'}{K}}} Sq.e^{\frac{-K}{4K'P}\cdot\frac{q^2}{4n^2}}. \qquad (\gamma)$$

14. This formula gives the solution of the problem concerning the mean that it is necessary to choose among the observations. Let us suppose that one has a number p of observations in order to correct an element already very nearly known. Let u be the correction of this element and α' the quantity given by observation: This quantity must be considered as the result of a function of the element, in which one would have substituted in the place of the element its approximate value increased by u, so that, by neglecting the powers of u superior to the first, one will have the equation

$$\alpha' = \beta' + \delta'.u$$

β' and δ' being some quantities that one knows to determine.

This equation would be exact if the observation were, and it would suffice to know u; but because of the inevitable errors of the observations, one would have exactly

$$\mathcal{E}' = \delta'.u - \gamma'$$

by making $\gamma' = \alpha' - \beta'$, and naming $\mathcal{E}$ the error of the observation. Each observation will furnish a similar equation, and one will form thus the following equations:

$$\left.\begin{array}{lll} \mathcal{E}' & = \delta'\,. & u - \gamma'; \\ \mathcal{E}'' & = \delta''\,. & u - \gamma''; \\ \mathcal{E}^{(p)} & = \delta^{(p)}\,. & u - \gamma^{(p)}; \end{array}\right\} \qquad \text{(C)}$$

In order to determine the most advantageous combination of these equations, multiply them respectively by q', q'', $q'''\ldots q^{(p)}$, and taking their sum there will come,

$$S.q^{(p)}\mathcal{E}^{(p)} = u.Sq^{(p)}.\delta^{(p)} - Sq^{(p)}.\gamma^{(p)} \qquad (\delta)$$

If it were possible to choose the multipliers q', q'', etc., in a manner to render $S.q^{(p)}\mathcal{E}^{(p)} = 0$, this equation would give exactly

$$u = \frac{Sq^{(p)}.\gamma^{(p)}}{S.q^{(p)}gamma^{(p)}},$$

but as this is impractical, we try to make so that this value of u differs from the truth as little as is possible.

Let us name u' the error of this result, we will have

$$u = u' + \frac{Sq^{(p)}.\gamma^{(p)}}{S.q^{(p)}\delta^{(p)}},$$

substituting this value in equation (δ) one will find

$$S.q^{(p)}\mathcal{E}^{(p)} = u'.Sq^{(p)}.\delta^{(p)}.$$

Now, if one adopts the rather natural hypothesis that the positive errors are equally probable as the negative errors of like value, and if one imagines that the interval comprehended among the extreme errors are divided into an infinitely great number of equal parts, represented by $2n$; it is clear that one will be able to apply here all that which has been said in No. 12 in order to determine the probability relative to any value of $S.q^{(p)}\mathcal{E}^{(p)}$. Moreover, if one adapts to the case that we treat the considerations made in No. 13, one will understand that, if one makes

$$q = u'.S.q^{(p)}\delta^{(p)} = u'Q$$

in formula (γ), the function

$$\frac{Q\sqrt{K}}{2n\sqrt{\pi PK'}}.Su'e^{\frac{-K}{4K'P}\cdot\frac{Q^2u'^2}{4n2}}$$

which results from it, expresses the most probable value of the error u', whence it follows that if one makes $\frac{u'}{2n} = \frac{x'}{b}$; $\frac{1}{2n} = \frac{dx'}{b}$ one will have

$$\frac{2nQ\sqrt{K}}{\sqrt{\pi K'P}}\int\frac{x'dx'}{b^2}.e^{\frac{-Q^2K.a'^2}{4K'P.b^2}}$$

for the most probable correction of u. To rigor it would be necessary to take for limits of this integral the value of x' corresponding to the greatest value of the error u', and that corresponding to $u' = 0$; but the rapidity with which the exponential function decreases, permits taking $x' = 0$, $x' = \infty$ for limits of the integral, that which gives

$$\frac{4n}{Q}\cdot\sqrt{\frac{PK'}{\pi K}}$$

for the correction of u relative to any system of multipliers q', q'', $q'''\ldots q^{(p)}$. We have supposed the interval $2n$, which comprehends the positive and negative errors, equal to b, thus by replacing $4n$ by $2b$, one will have

$$\frac{2b}{Q}\cdot\sqrt{\frac{K'P}{\pi K}} \tag{B}$$

for the correction of u expressed by some units of same kind as those which measure the interval b.

It is clear actually that the better system of multipliers is the one which renders *minimum* formula (B). Now we have

$$\frac{\sqrt{P}}{Q} = \frac{\sqrt{q'^2 + q''^2 + q'''^2\cdots + q^{(p)2}}}{q'\delta' + q''\delta'' + q'''\delta'''\cdots + q^{(p)}\delta^{(p)}};$$

therefore, if we suppose $1 = q' + q'' = \cdots q^{(p)}$, it will be necessary, in order that the correction of u be smaller, to prepare the equations (C), in a manner that in each of them the coefficient of u has the positive sign. One knows that the celebrated astronomer Tobie MAYER is the first who has invented this rule, and the he has made use of it in order to perfect the tables of the Moon. Following this method one would have

$$u = \frac{S.\gamma^{(p)}}{S.\delta^{(p)}},$$

and formula (B) gives

$$\frac{2b\sqrt{PK'}}{S.\delta^{(p)}\sqrt{\pi K}}$$

for the correction of this value. But this correction is not the smallest possible. In order to find that here it is necessary to determine the multipliers q', $q'' \ldots q^{(p)}$, by equating to zero the partial differential of the function $\frac{\sqrt{P}}{Q}$ taken with respect to all the variables q', q'', $q''' \ldots q^{(p)}$, this which will give

$$\frac{q^{(p)}}{\delta^{(p)}} = \frac{q'^2 + q''^2 + q'''^2 \cdots + q^{(p)2}}{q'\delta' + q''\delta'' + q'''\delta''' \cdots + q^{(p)}\delta^{(p)}},$$

where the first member must take successively all the values $\frac{q'}{\delta'}$; $\frac{q''}{\delta''}$; $\ldots \frac{q^{(p)}}{\delta^{(p)}}$, and the second remains invariable. It is clear that one satisfies the preceding equation by taking

$$q' = \mu\delta'; \quad q'' = \mu\delta''; \quad q''' = \mu\delta'''; \ldots q^{(p)} = \mu\delta^{(p)},$$

this which gives $\mu = \frac{P}{Q}$.

It follows thence that one has

$$u = \frac{\gamma'\delta' + \gamma''\delta'' + \gamma'''\delta''' \cdots + \gamma^{(p)}\delta^{(p)}}{\delta'^2 + \delta''^2 + \delta'''^2 \cdots + \delta^{(p)2}},$$

and formula (B) gives

$$\frac{2b\sqrt{K'}}{\sqrt{\pi K(\delta'^2 + \delta''^2 + \delta'''^2 \cdots + \delta^{(p)2})}}$$

for the correction of this value, which is effectively smaller than that which takes place by supposing

$$1 = q' = q'' = \cdots q^{(p)}.$$

The comparison of the preceding value of u with the equations (C) show that that here enjoys the remarkable property of rendering *minimum* the function

$$(\delta'.u' - \gamma')^2 + (\delta''.u - \gamma'')^2 \cdots + (\delta^{(p)}.u - \gamma^{(p)})^2$$

which is equal to the sum of the squares of the errors $\mathcal{E}'$, $\mathcal{E}''$, $\mathcal{E}''' \ldots \mathcal{E}^{(p)}$. The calculus of the probabilities establishes thence the principle of least squares, discovered by LEGENDRE and GAUSS in these last times.

The value of the correction depends on the ratio of K' to K, which is not able to be determined *a priori* because one does not know nearly always the form of the function $F\left(\frac{x'}{b}\right)$, whence depends the law of probability of errors, but LAPLACE demonstrates that one is able in all cases to suppose $\frac{K}{K'} > 6$.

15. Let us take the formula

$$\frac{1}{2n} \cdot \frac{\sqrt{K}}{\sqrt{\pi K'P}}.e^{\frac{-Kq^2}{4K'P4n2}}$$

found in No. 13. We have seen (No. 14) that $q = u'Q$, therefore if one names c the interval which comprehends the positive and negative errors of q, by making $\frac{q}{2n} = \frac{u'Q}{c}$; $\frac{1}{2n} = \frac{Qdu'}{c}$, the integral

$$\frac{2Q\sqrt{K}}{\sqrt{\text{1}PK'}} \int \frac{du'}{2c}.e^{\frac{-Ku'^2Q^2}{4K'Pc^2}}$$

taken from $u' = 0$ to $u' = u'$ will give the probability in order that the error of u is comprehended between $\pm u'$. In order to express this integral more simply, it suffices to put

$$u' = \frac{2ct\sqrt{K'P}}{Q\sqrt{K}},$$

that which will change into

$$\frac{2}{\sqrt{\pi}} \int dt.e^{-t^2}. \qquad (\mathcal{E})$$

Following the method of least squares of the errors of the observations we have seen in the preceding No. that one has

$$Q = \mu(\delta'^2 + \delta''^2 \cdots + \delta^{(p)2})$$
$$P = \mu^2(\delta'^2 + \delta''^2 \cdots + \delta^{(p)2})$$

therefore one will have

$$u' = \frac{2ct\sqrt{K'}}{\sqrt{K(\delta'^2 + \delta''^2 \cdots + \delta^{(p)2})}} \qquad (\theta)$$

According to the method of MAYER one has $P = p$;

$$Q = \delta' + \delta'' + \delta''' \cdots + \delta^{(p)},$$

whence one concludes

$$u' = \frac{2ct\sqrt{K'P}}{(\delta' + \delta'' \cdots + \delta^{(p)})\sqrt{K}}. \qquad (\theta')$$

Now, if one observes that the coefficient of t, which enters into equation (θ), is precisely of the same form as the expression of the correction relative to the method of least squares, and if the coefficient of t of formula (θ') is of the same form as the expression of the correction relative to the method of MAYER, one will conclude from it that for one same value of t the value of u' given by equation (θ') must be greater than that given by equation (θ). But the integral ($\mathcal{E}$) remains the same for these two values of u', therefore with equal probability the limits of the errors are more narrow by following the principle of least squares than by following the ordinary method.

16. If one had to determine more than one unknown, according to a system of equations of which the number would exceed by much the one of the unknowns, the method of least squares of errors of the observations would be yet that which it would be necessary to follow in order to diminish as much as possible the most probable the most probable correction relative to each unknown. In order to establish this principle,

let us consider first the case where one would have to correct two elements already very nearly known.

In naming u and z the corrections of the two elements, it would be easy to form the following system of equations

$$\left.\begin{array}{ll} \mathcal{E}' & = \delta'.u + \beta'.z - \gamma'; \\ \mathcal{E}'' & = \delta''.u + \beta''.z - \gamma''; \\ \ldots & \ldots \\ \mathcal{E}^{(p)} & = \delta^{(p)}.u + \beta^{(p)}.z - \gamma^{(p)} \end{array}\right\} \qquad (\mathrm{C}')$$

by some considerations analogous to those which we have employed in No. 14.

In order to determine the most advantageous combination of these equations. Multiply them respectively by q'; q'''; $q'''\ldots q^{(p)}$; their sum, after having thus multiplied them, will be

$$Q = Mu + Nz - L \qquad (1)$$

by making

$$Q = Sq^{(p)}\mathcal{E}^{(p)}; \quad M = Sq^{(p)}\mathcal{I}^{(p)}; \quad N = Sq^{(p)}\beta^{(p)}; \quad L = Sq^{(p)}\gamma^{(p)}.$$

The same equations multiplied respectively by $q_{\prime}$; $q_{\prime\prime}$; $q_{\prime\prime\prime}$;$\ldots q_{(p)}$ give

$$Q' = M + N'z - L' \qquad (2)$$

by making

$$Q' = Sq_{(p)}\mathcal{E}^{(p)}; \quad M' = Sq_{(p)}\delta^{(p)}; \quad N' = Sq_{(p)}\beta^{(p)}; \quad L' = Sq_{(p)}\gamma^{(p)}.$$

By admitting the possibility of choosing the multipliers such that one has $Q = 0$, $Q' = 0$, equations (1) and (2) would give exactly

$$\left.\begin{array}{ll} (NM' - MN').z & = LM' - L'M; \\ (NM' - MN').u & = NL' - LN'; \end{array}\right\} \qquad (\mathrm{B})$$

But, if the conditions $Q = 0$, $Q' = 0$ do not hold, the values of z and of u given by these equations will have need each of correction, so that if one names z' the correction of z, u' that of u, equations (1) and (2) give

$$Q = Mu' + Nz'; \qquad Q' = M'u + N'z'.$$

Now, if one supposes the positive errors equally probable as the negative errors, it is clear that one would have the probability relative to any values of Q and of Q' by resolving a problem analogous to the one that we have resolved in No. 10. But one is able to dispense of undertaking the solution of this problem by being aided of the remark made in No. 12, in the similar case, where one has seen that the unequal probability of errors does not change the form of the function that one seeks. In consequence of this, it will suffice to divide by $(2n)^p$ the formula (α') put in No. 10, and one will have

$$\frac{1}{4\pi a\sqrt{E}}.e^{-\frac{1}{4aE}(CQ^2 - 2BQQ' + AQ'^2)}$$

for the expression of the probability that Q and Q' are the values of the sums $Q = Sq^{(p)}\mathcal{E}^{(p)}$, $Q = Sq_{(p)}\mathcal{E}^{(p)}$. There is no need to avert that the constant a which enters into this formula, must have a different value from that which it had in No. 10. In the analogous problem treated in No. 12, we have seen that the constant a was of the form $2nK$, but in the actual case, where there is concern to satisfy a double condition, the probability must be infinitely smaller, thus it will be necessary to suppose $a = 4n2K$, that which changes the preceding probability into this here:

$$\frac{1}{4\pi K.4n^2\sqrt{E}}.e^{-\frac{1}{4EK.4n2}(CQ^2-2BQQ'+AQ'^2)}$$

We name c the interval $2n$, which comprehends the positive and negative values of Q and of Q'. One will be able to suppose $\frac{Q}{2n} = \frac{x}{c}$; $\frac{Q'}{2a} = \frac{y}{c}$; $\frac{1}{2n} = \frac{dx}{c}$; $\frac{1}{2a} = \frac{dy}{c}$, this which changes the preceding formula into this one here,

$$\frac{dxdy}{4\pi Kc^2\sqrt{E}}.e^{-\frac{1}{4c^2E}(Cx^2-2Bxy+Ay^2)} \tag{3}$$

Now one will have

$$x = Mu' + Nz'$$
$$y = M' + N'z$$

therefore one will have, in accordance with a known theorem of the Integral Calculus,

$$dxdy = (MN' - M'N)du'dz'.$$

Let for more simplicity,

$$F = CM^2 - 2BMM' + AM'^2;$$
$$G = CMN - B(M'N + MN') + AM'N';$$
$$H = CN^2 - 2BNN' + AN'^2;$$
$$I = MN' - M'N$$

formula (3) will become

$$\frac{Idu'dz}{4\pi Kc^2\sqrt{E}}.e^{-\frac{1}{4Kc^2E}(Fu'^2+2Gu'z'+Hz'^2)}$$

This expression gives the probability that the errors of u and of z will be u' and z'; thus by supposing z' constant and given to u' all the values that it is able to receive between its limits, one will have a sequence of probabilities of which the sum will be evidently equal to the probability that there is in order that the error of z is z'; therefore the integral

$$\frac{Idz}{4\pi Kc^2\sqrt{E}}.e^{-\frac{1}{4Kc^2E}(Fu'^2+2Gu'z'+Hz'^2)}$$

will be the probability of the error z'. In regard to the limits of this integral one would be able to take $u' = -\infty$, $u' = +\infty$, because the exponential function decreases rapidly.

It is easy to see that the preceding integral is able to be put under this form

$$\frac{Idz'}{4\pi Kc^2\sqrt{E}}.e^{-\frac{(HG-G^2)z'^2}{4Kc^2EF}}\int du'.e^{-\frac{F}{4c^2KE}\left(\frac{u'+Gz'}{F}\right)^2}$$

and by taking it from negative to positive infinity, the result will be

$$\frac{Idz'}{2c\sqrt{\pi KF}}.e^{-\frac{(FG-G^2)z'^2}{4Kc^2KEF}}.$$

Now we have

$$HF-G^2=(AC-B^2)(M'N-MN')^2,$$

but $E=AC-B^2$...(No. 10), therefore

$$\frac{HF-G2}{E}=I^2.$$

It follows thence that the probability of the error z' will be expressed by

$$\frac{Idz'}{2c\sqrt{\pi KF}}.e^{-\frac{I^2z'^2}{4c^2KF}}. \qquad (\beta)$$

It is not useless to observe that this function, like that which in No. 14 gave the probability of the error u' are each of the form

$$\frac{hdx}{\sqrt{\pi}}.e^{-h^2x^2}.$$

This function does not change by changing the sign of x: Its greatest value corresponds to $x=0$, and it diminishes rapidly in measure as x increases; moreover its integral taken from negative infinity, to positive infinity, is equal to unity. These properties are precisely those that each proper function must have to represent the law of errors of observations.

If one multiplies the expression (β) by z', the integral

$$\frac{Idz'}{2c\sqrt{\pi KF}}.e^{-\frac{I^2z'^2}{4c^2KF}}$$

will give the most probable correction of z'...(13).

Conformably to that which has been said in No. 14, one will be able to take $z'=0$, $z'=\infty$ for the limits of this integral; that which will give

$$\frac{c}{1}\sqrt{\frac{KF}{\pi}} \qquad (4)$$

for the correction of z relative to any system of multipliers q', $q''\ldots q^{(p)}$; $q_{\prime}$; $q_{\prime\prime}\ldots q_{(p)}$.

It is clear that it suffices to change F into H in order to have the corresponding corresponding to u which will be consequently

$$\frac{c}{1}\sqrt{\frac{KH}{\pi}} \qquad (5)$$

Actually it is clear that the better system of multipliers is the one which will render *minimum* the functions (4) and (5). Now it is easy to prove by the known rules of the differential Calculus that one fulfills this double condition by taking

$$\begin{array}{lllll} q' = \mu\delta'; & q'' = \mu\delta''; & q''' = \mu\delta'''; & \dots & q^{(p)} = \mu\delta^{(p)}; \\ q_{\prime} = \mu\beta'; & q_{\prime\prime} = \mu\beta''; & q_{\prime\prime\prime} = \mu\beta'''; & \dots & q_{(p)} = \mu\beta^{(p)}. \end{array}$$

Substituting these values into equations (D) there will result from it for u and z the following values:

$$z = \frac{S.\gamma^{(p)}.\delta^{(p)}.S\beta^{(p)}.\delta^{(p)} - S.\beta^{(p)}.\gamma^{(p)}.S\delta^{(p)2}}{(S.\delta^{(p)}.\beta^{(p)})^2 - S\delta^{(p)2}.S\beta^{(p)2}};$$

$$u = \frac{S.\delta^{(p)}.\beta^{(p)}.S\beta^{(p)}.\gamma^{(p)} - S.\beta^{(p)}.\gamma^{(p)}.S\delta^{(p)2}}{(S.\delta^{(p)}.\beta^{(p)})^2 - S\delta^{(p)2}.S\beta^{(p)2}}.$$

By comparing these values with equations (C′) one recognized that they coincide with those which one would find in order to render *minimum* the function

$$(\delta'.u + \beta'z - \gamma')^2 + (\delta''.u + \beta''z - \gamma'')^2 \cdots + (\delta^{(p)}.u + \beta^{(p)}z - \gamma^{(p)})^2$$

that is the sum of the squares of the errors of the observations.

17. In one had to determine three or a greater number of unknowns according to a number of equations superior to the one of the unknowns one would find, by following the preceding analysis that the principle of least squares always holds. But it is necessary to swear that the calculation of it would be extremely long, even for the case where there are three unknowns alone.

However, if one adopts the function

$$\frac{hdx}{\sqrt{\pi}}.e^{-h^2x^2}$$

in order to express the law of probability of any error $\pm x$, it becomes quite easy to demonstrate the principle of least squares for any number of unknowns. In fact, let us name x', x'', x''',… $x^{(p)}$ the errors of any number p of observations; the probability that this system of errors is the one which will take place, is equal, as one knows, to the product of the probabilities relative to each error, that is the function

$$\frac{h^p}{(\pi)^{\frac{p}{2}}}dx'\,dx''\,dx'''\dots dx^{(p)}.e^{-h^2(x'^2+x''^2+x'''^2\cdots+x^{(p)2})}.$$

Now it is clear that the better system of errors that one is able to choose is the one which is the most probable.

But the *maximum* of the preceding probability corresponds to the *minimum* of the sum of squares

$$x'^2 + x''^2 + x'''^2 \cdots + x^{(p)2}$$

of the errors of observations, therefore it will be necessary to determine the unknowns in accordance with this principle. It is in this manner that the celebrated GAUSS has established the principle of least squares in his excellent work entitled: (*Theoria motus corpourm cælestium*).

If one pays attention that the function

$$\frac{hdx}{\sqrt{\pi}}.e^{-h^2x^2}$$

is presented in the solution of all the problems that we have traversed, one will encounter that it is rather natural to suppose that the probability of the errors of observations is represented by this function.