THE ARITHMETIC TRIANGLE

BLAISE PASCAL

1. Introduction

The treatises related to arithmetic triangle appear to be dated near the end of 1654, which locates them near the same time as the exchange of letters with Fermat on the problem of points. These were discovered after his death and published at Paris by Guillaume Desprez in 1665 under the title: *Traité du Triangle arithmétique, avec quelques autres petits traités sur la même matière*. These treatises include *Divers usages du triangle arithmétique dont le générateur ist l'unité*. This last has four parts:

(1) Usage du triangle arithmétique pour les ordres numériques
(2) Usage du triangle arithmétique pour les combinations
(3) Usage du triangle arithmétique pour déterminaer les partis qu'on doit faire entre deux jouers qui jouent en plusiers parties
(4) Usage du triangle arithmétique pour trouver les puissances des binômes et des apotomes

Several other works are further associated with this treatise. These are *Traité des Ordres numériques*, *De numericis ordinibus Tractatus*, *De numerorum continuorum productis seu de numeris qui producuntur ex multiplicatione numerorum serie naturali procedentium*, *Numericarum potestatum generalis Resolutio*, *Combinationes*, and two other which seem to have been written later: *De numeris multiplicibus ex sola characterum numericorum additione agnoscendis* and *Potestatum numericarum summa*.

A comparison of the text as presented in Volume III of the complete works of Pascal printed by Hachette [1] to the same in the Pléide edition [2] shows some differences, generally of formatting, but sometimes of language. The translation below follows the latter.

With regard to the problem of points, one should refer to *Usage du triangle arithmétique pour déterminaer les partis qu'on doit faire entre deux jouers qui jouent en plusiers parties*. However, it is important to note that Pascal introduces the use of mathematical induction in a very clear form. For this see the Twelfth Consequence in *Traité du Triangle arithmétique*.

2. Treatise on the Arithmetic Triangle

DEFINITIONS

I call *Arithmetic Triangle*, a figure for which the construction is such.

I draw from any point, G, Fig. 1, two perpendicular lines the one to the other, GV, Gζ, from each of which I take as many as I wish of equal and contiguous parts, beginning with G, that I name 1, 2, 3, 4, etc.; and these numbers are *the exponents* of the divisions of the lines.

Date: Printed 1654, published 1665.

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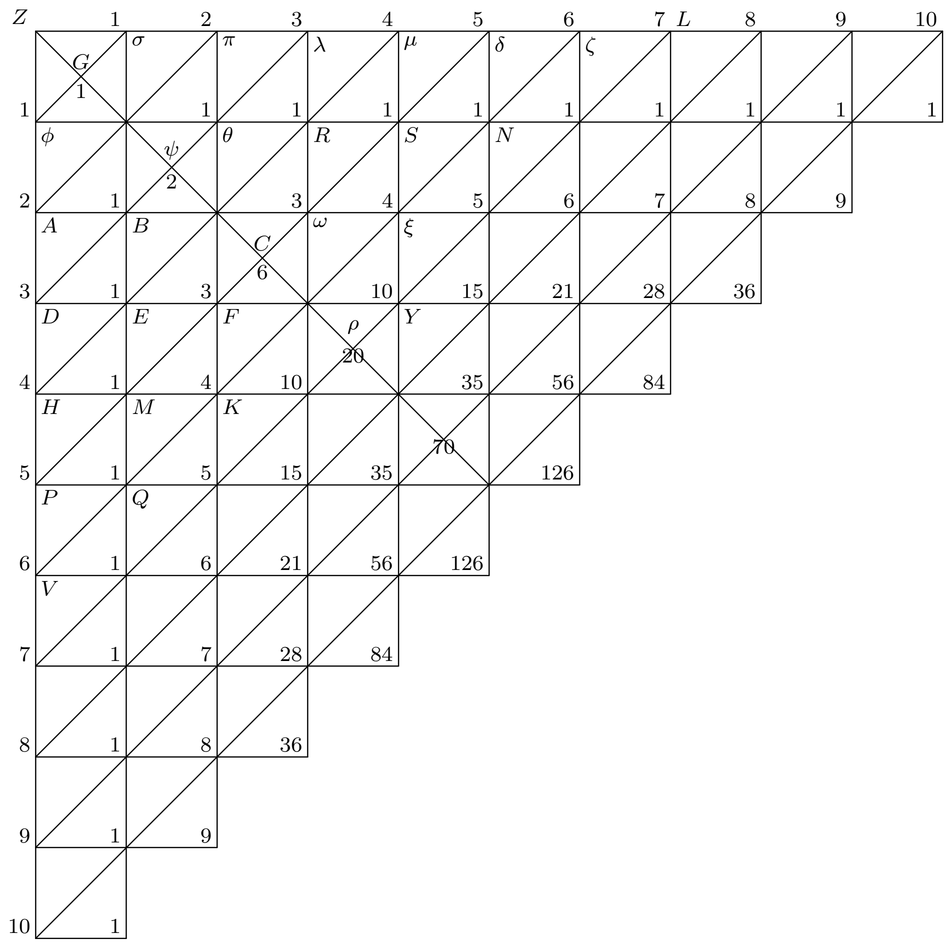

FIGURE 1. The Arithmetic Triangle

Next I join the points of the first division which are in each of the two lines by another line which forms a triangle of which it is *the base*.

I join thus the two points of the second division by another line, which forms a second triangle of which is *the base*.

And joining thus all the points of division which have one same exponent, I form from them as many *triangles and bases*.

I draw, through each of the points of division, lines parallel to the sides, which by their intersections form little squares, that I call *cells*.

And the cells which are between two parallels which go from left to right are called *cells of one same parallel rank*, as the cells G, σ, π, etc., or ϕ, ψ, θ, etc.

And cells which are between two lines which go from top to bottom are called *cells of one same perpendicular rank*, as the cells G, ϕ, A, D, etc., and these σ, ψ, B, etc.

And those that one same base traverse diagonally are so-called *cells of one same base*, as those which follow, D, B, θ, λ, and these A, ψ, π.

The cells of one same base equally distant from their extremities are so-called *reciprocals*, as these, E, R and B, θ, because the exponent of the parallel rank of the one is the same as the exponent of the perpendicular rank of the other, as it appears in this example, where E is in the second perpendicular rank and in the fourth parallel, and its reciprocal R is in the second parallel rank, and in the fourth perpendicular reciprocally; and it is quite easy to demonstrate that those which have their exponents reciprocally equal are in one same base and equally distant from their extremities.

It is also quite easy to demonstrate that the exponent of the perpendicular rank of any cell that it be, added to the exponent of its parallel rank, surpasses by unity the exponent of its base.

For example, the cell F is in the third perpendicular rank, and in the fourth parallel, and in the sixth base, and these two exponents of ranks $3 + 4$ surpass by unity the exponent of the base 6, that which comes from this that the two sides of the triangle are divided into an equal number of parts; but this is rather understood than demonstrated.

This remark is of similar nature, that each base contains one cell more than the preceding, and each as many as its exponent of units; thus the second $\phi\sigma$ has two cells, the third A$\psi\pi$ has three of them, etc.

Now, the numbers which are set in each cell are found by this method:

The number of the first cell which is at right angle is arbitrary; but that one being placed, all the others are forced; and for this reason it is called the *generator* of the triangle; and each of the others is specified by this single rule:

The number of each cell is equal to that of the cell which precedes it in its perpendicular rank, plus that of the cell which precedes it in its parallel rank. Thus the cell F, that is, the number of the cell F, equals the cell C, plus the cell E, and thus of the others.

Whence many consequences are drawn. Here are are the most important of them, where I consider the triangles of which the generator is unity; but that which will be said of them will be proper to all the others.

FIRST CONSEQUENCE

In every arithmetic triangle, all the cells of the first parallel rank and of the first perpendicular rank are equal to the generator.

For by the construction of the Triangle, each cell is equal to that which precedes it in its perpendicular rank, plus to that which precedes it in its parallel rank. Now, the cells of the first parallel rank have no other cells which precede them in their perpendicular ranks, nor cells of the first perpendicular rank in their parallel ranks: therefore they are all equal among them and therefore to the first number generator.

Thus ϕ equals G+zero, that is, ϕ equals G.

Thus A equals ϕ+zero, that is to say, ϕ.

Thus σ equals G+zero, and π equals σ+zero.

And thus of the others.

SECOND CONSEQUENCE

In every arithmetic triangle, each cell is equal to the sum of all cells of the preceding parallel rank, comprehended from its perpendicular rank to the first inclusively.

Let ω be any cell: I say that it is equal to R+$\theta + \psi + \phi$, which are cells of the superior parallel rank from the perpendicular rank of ω to the first perpendicular rank.

This is evident by the sole interpretation of the cells by those whence they are formed.

For

$$\omega \text{ equals R+}\underbrace{\text{C}}_{\theta + \underbrace{\text{B}}_{\psi + \underbrace{\text{A}}_{\phi}}}.$$

for A and ϕ are equal among them by the preceding.

Therefore ω equals R$+\theta + \psi + \phi$.

THIRD CONSEQUENCE

In every arithmetic triangle, each cell equals the sum of all cells of the preceding perpendicular rank, comprehended from its parallel rank to the first inclusively.

Let C be any cell: I say that it is equal to B$+\psi + \sigma$, which are cells of the preceding perpendicular rank, from the parallel rank of the cell C to the first parallel rank.

This appears similarly by the sole interpretation of the cells.

For

$$\text{C equals B+}\underbrace{\theta}_{\psi + \underbrace{\pi}_{\sigma,}}.$$

For π equals σ by the first.

Therefore C equals B$+\psi + \sigma$.

FOURTH CONSEQUENCE

In every arithmetic triangle, each cell diminished by unity is equal to the sum of all cells which are comprehended between its parallel rank and its perpendicular rank exclusively.

Let ξ be any cell: I say that $\xi-$G equals R$+\theta + \psi + \phi + \lambda + \pi + \sigma+$G, which are all the numbers comprehended between the rank $\xi\omega$CBA and the rank ξSμ exclusively.

This is apparent similarly by interpretation.

For

$$\xi \text{ equals } \lambda + \text{R+}\underbrace{\omega}_{\pi + \theta + \underbrace{\text{C}}_{\sigma + \psi + \underbrace{\text{B}}_{\text{G} + \phi + \underbrace{A}_{\text{G.}}}}}$$

Therefore ξ equals $\lambda+$R$+\pi + \theta + \sigma + \psi+G+\phi+$G.

NOTE

I have said in the enunciation: each cell diminished by unity, *because unity is the generator; but if it were another number, it would be necessary to say:* each cell diminished by the generator number.

FIFTH CONSEQUENCE

In every arithmetic Triangle, each cell is equal to its reciprocal.

For in the second base $\phi\sigma$, it is evident that the two reciprocal cells ϕ, σ, are equal to one another and to G.

In the third A, ψ, π, it is clear likewise that the reciprocals π, A, are equal to one another and to G.

In the fourth, it is clear that the extremes D, λ, are again equal to one another and to G.

And those from among two, B, θ, are clearly equals, since B equals A+ψ, and θ equals $\psi + \pi$; now $\pi + \psi$ are equal to A+ψ by that which is shown; therefore, etc.

Thus one will show in all the other bases that the reciprocals are equal, because the extremes are always equal to G, and that the others will be explained always by some equal others in the preceding base which are reciprocals to one another.

SIXTH CONSEQUENCE

In every arithmetic triangle, a parallel rank and a perpendicular which have one same exponent are composed of cells all equals the ones to the others.

Because they are composed of reciprocals.

Thus the second perpendicular rank $\sigma\psi$BEMQ is entirely equal to the second parallel rank $\phi\psi\theta$RSN.

SEVENTH CONSEQUENCE

In every arithmetic triangle, the sum of the cells of each base is double the cells of the base preceding.

Let DB$\theta\lambda$ be any base. I say that the sum of its cells is double of the sum of the cells of the preceding A$\psi\pi$.

Because the extremes	D,	λ,
equal the extremes	A,	π,
and each of the others	B,	θ,
equal two of the other base	A+ψ,	$\psi + \pi$.

Therefore D+λ+B+θ equals 2A+2$\psi + 2\pi$.

The same thing is demonstrated similarly of all the others.

EIGHTH CONSEQUENCE

In every arithmetic triangle, the sum of the cells of each base is a number of the double progression which begins with the unit of which the exponent is the same as that of the base.

Because the first base is unity.

The second is double of the first, therefore it is 2.

The third is double of the second, therefore it is 4.

And thus to infinity.

NOTE

If the generator were not the unit, but another number, as 3, the same thing will be true: but it would not be necessary to take the numbers of the double progression starting with the unit, namely: 1, 2, 4, 8, 16, etc., but those of another double progression starting with the generator 3, namely, 3, 6, 12, 24, 48, etc.

NINTH CONSEQUENCE

In every arithmetic triangle, each base diminished by unity is equal to the sum of all the preceding.

Because it is a property of the double progression.

NOTE

If the generator were other than unity, it would be necessary to say: each base diminished by the generator.

TENTH CONSEQUENCE

In every arithmetic Triangle, the sum of as many contiguous cells as one will wish from its base, beginning with an extremity, is equal to as many cells of the preceding base, plus again as many except one.

Let be taken the sum of as many cells as one will wish from the base Dλ, for example, the first three, D+B+θ.

I say that it is equal to the sum of the first three of the preceding base A+$\psi + \pi$, plus to the first two of the same base A+ψ.

Because	D,	B.	θ.
equal	A.	A+ψ	$\psi + \pi$.

Therefore D+B+θ equals 2A+2$\psi + \pi$.

DEFINITION

I call cells of the divide *those that the line which divides the right angle in half across diagonally, as the cells G, ψ, C, ρ, etc.*

ELEVENTH CONSEQUENCE

Each cell of the divide is double of that which precedes it in its parallel or perpendicular rank.

Let C be a cell of the divide. I say that it is double of θ, and also of B.

For C equals θ+B, and θ equals B, by the fifth consequence.

NOTE

All these consequences are on the subject of the equalities which are encountered in the arithmetic Triangle. We are going to see now the proportions, of which the following proposition is the foundation.

TWELFTH CONSEQUENCE

In every arithmetic Triangle, two contiguous cells being in one same base, the superior is to the inferior as the number of cells from the superior to the top of the base to the number of cells from the inferior to the bottom inclusively.

Let E, C be any two contiguous cells of one same base: I say that:

E	is to	C	as	2	to	3
inferior		*superior*		*because there are two cells from* E *to the bottom: namely E, H*		*because there are three cells from* C *to the top: namely C, R, μ.*

Although this proposition has an infinite number of cases, I will give a quite short demonstration, in supposing 2 lemmas.

The first, that it is evident by itself, that this proportion is encountered in the second base; for it is quite clear that ϕ is to σ as 1 to 1.

The second, that if this proportion is found in any base, it will be found necessarily in the base following.

Whence it is seen that it is necessarily in all the bases: for it is in the second base by the first lemma; therefore by the second it is in the third base, therefore in the fourth, and to infinity.

It is necessary therefore only to prove the second lemma, in this manner. If this proportion is encountered in any one base, as in the fourth Dλ, that is, if D is to B as 1 to 3, and B to θ as 2 to 2, and θ to λ as 3 to 1, etc.; I say that the same proportion will be found in the following base, Hμ, and that, for example, E is to C as 2 to 3.

For D is to B as 1 to 3, by hypothesis.

$$\begin{array}{lcccc} \text{Therefore} & \underbrace{D+B} & \text{is to B as} & \underbrace{1+3} & \text{to 3.} \\ & E & \text{to B as} & 4 & \text{to 3.} \end{array}$$

Similarly B is to θ as 2 to 2, by the hypothesis.

$$\begin{array}{lcccc} \text{Therefore} & \underbrace{B+\theta} & \text{to B, as} & \underbrace{2+2} & \text{to 2.} \\ & C & \text{to B, as} & 4 & \text{to 2.} \\ \text{But} & B & \text{to E, as} & 3 & \text{to 4.} \end{array}$$

Therefore, by the disturbed proportion, C is to E as 3 to 2.

That which it was necessary to demonstrate.

One will prove it similarly in all the rest, since this proof is based only on this that this proportion is found in the preceding base, and that each cell is equal to its preceding, plus to its superior, that which is true everywhere.

THIRTEENTH CONSEQUENCE

In every arithmetic Triangle, two contiguous cells being in the same perpendicular rank, the inferior is to the superior as the exponent of the base of this superior to the exponent of its parallel rank.

Let F, C be any two cells in the same perpendicular rank. I say that

F	is to	C	as	5	to	3
the inferior,		the superior		exponent of the base of C,		*exponent of parallel rank of C.*

Because E is to C as 2 to 3.

$$\begin{array}{lcccc} \text{Therefore} & \underbrace{E+C} & \text{is to C as} & \underbrace{2+3} & \text{to 3.} \\ & F & \text{is to C as} & 5 & \text{to 3.} \end{array}$$

FOURTEENTH CONSEQUENCE

In every arithmetic Triangle, two contiguous cells being in the same parallel rank, the greatest is to its preceding as the exponent of the base of that preceding to the exponent of its perpendicular rank.

Let F, E be two cells in one same parallel rank: I say that

F	is to	E	as	5	to	2
the greatest,		the preceding		exponent of the base of E,		*exponent of perpendicular rank of E.*

Because E is to C as 2 to 3.

$$\begin{array}{lcccc} \text{Therefore} & \underbrace{E+C} & \text{is to C as} & \underbrace{2+3} & \text{to 2.} \\ & F & \text{is to C as} & 5 & \text{to 2.} \end{array}$$

FIFTEENTH CONSEQUENCE

In every arithmetic Triangle, the sum of the cells of any parallel rank is to the last of this rank as the exponent of the triangle is to the exponent of the rank.

Let there be any triangle, for example the fourth GDλ: I say that any rank that we take there, as the second parallel, the sum of its cells, namely $\phi + \psi + \theta$, is to θ as 4 to 2. For $\phi + \psi + \theta$ equals C, and C is to θ as 4 to 2, by the thirteenth consequence.

SIXTEENTH CONSEQUENCE

In every arithmetic Triangle, any parallel rank is to the inferior rank as the exponent of the inferior rank to the number of its cells.

Let there be any triangle, for example the fifth μGH: I say that, whatever rank that we take there, for example the third, the sum of its cells is to the sum of those of the fourth, that is, A+B+C is to D+E as 4, exponent of the fourth rank, to 2 which is the exponent of the number of its cells, for it contains 2 of them.

For A+B+C equals F, and D+E equals M.

Now F is to M as 4 to 2, by the twelfth consequence.

NOTE

On is able to state it also in this way:

Each parallel rank is to the inferior rank, as the exponent of inferior rank to the index of superior rank.

For the exponent of a triangle, less the exponent of one of its ranks, is always equal to the number of the cells of the inferior rank.

SEVENTEENTH CONSEQUENCE

In every arithmetic Triangle, any cell that is added to all cells of its perpendicular rank, is to the same cell added to all cells of its parallel rank, as the number of the cells taken in each rank.

Let B be any cell; I say that B+$\psi + \sigma$ is to B+A, as 3 to 2.

I say 3, because there are three cells added in the antecedent, and 2, because there are two in the consequent.

For, B+$\psi + \sigma$ equals C, by the thirteenth consequence; and B+A equals E, by the second consequence.

Now C is to E as 3 to 2, by the twelfth consequence.

EIGHTEENTH CONSEQUENCE

In every arithmetic Triangle, two parallel ranks equally distant from the extremities, are between them as the number of their cells.

Let GVζ be any triangle, and two of its ranks equally distant from the extremes, as the sixth P+Q, and the second $\phi + \psi + \theta$+R+S+N: I say that the sum of the cells of one is to the sum of the cells of the other, as the number of the cells of the one is to the number of the cells of the other.

For, by the sixth consequence, the second parallel rank $\phi\psi\theta$RSN is the same as the second perpendicular rank $\sigma\psi$BEMQ, from which we have just proved this proportion.

NOTE

One is able to state also:

In every arithmetic Triangle, two parallel ranks, of which the exponents added together exceed by unity the exponent of the triangle, are between them as their exponents reciprocally.

For this is only one same thing as that which has just been enunciated.

LAST CONSEQUENCE

In every arithmetic Triangle, two contiguous cells being in the divide, the inferior is to the superior taken four times, as the exponent of the base of that superior to a number greater by the unit.

Let ρ, C be two cells of the divide: I say that ρ is to 4C as 5, exponent of the base of C, to 6.

Because ρ is double of ω, and C of θ; therefore 4θ equals 2C.

Therefore 4θ is to C as 2 to 1.

Now ρ is to 4C as ω to θ, or by reason composed of *by the preceding consequences*

$$\underbrace{\underbrace{\omega \text{ to C}}_{5 \text{ to } 3} + \underbrace{\text{C to } 4\theta}_{\substack{1 \text{ to } 2 \\ \text{or } 3 \text{ to } 6}}}_{5 \text{ to } 6}$$

Therefore ρ is to 4C as 5 to 6.

That which it was necessary to demonstrate.

NOTE

We are able to draw from there many other proportions that I suppress, because each one is easily able to conclude them, and that those who will wish to have an interest in them will find perhaps better than those that I am able to give. I end therefore with the following problem, which makes the fulfillment of the treatise.

PROBLEM

Being given the exponents of the perpendicular and parallel ranks of a cell, to find the number of the cell, without using the arithmetic Triangle.

Let, for example, it be proposed to find the number of the cell ξ of the fifth perpendicular rank and the third parallel rank.

Having taken all the numbers which precede the exponent of the perpendicular 5, namely 1, 2, 3, 4, let there be taken as many natural numbers, starting with the exponent of the parallel 3, namely 3, 4, 5, 6.

Let the first ones be multiplied by one another, and let the product be 24. Let the others be multiplied by one another, and let the product be 360, which, divided by the other product 24, gives for quotient 15. This quotient is the number sought.

For ξ is the first of its base V by reason composed of all the ratios of the cells in-between, that is to say, ξ is to V,

by reason composed of

or by the twelfth consequence

$$\underbrace{\underbrace{\xi \text{ to } \rho}_{3 \text{ to } 4} + \underbrace{\rho \text{ to K}}_{4 \text{ to } 3} + \underbrace{\text{K to Q}}_{5 \text{ to } 2} + \underbrace{\text{Q to V}}_{6 \text{ to } 1}}$$

Therefore ξ is to V as 3 by 4 by 5 by 6, to 4 by 3 by 2 by 1.

But V is unity; therefore ξ is the quotient of the division of the product of 3 by 4 by 5 by 6 by the product of 4 by 3 by 2 by 1.

NOTE

If the generator is not unity, it would be necessary to multiply the quotient by the generator.

3. VARIOUS USAGES OF THE ARITHMETIC TRIANGLE OF WHICH THE GENERATOR IS UNITY

After having given the proportions which are encountered among the cells and the ranks of the arithmetic triangles, I pass to various usages of those of which the generator is unity; this is that which we will see in the following treatises. But I allow much more than I give; it is a strange thing how it is fertile in properties! Each is able to be practiced; I caution here only that, in all the following, I intend to speak only of the arithmetic Triangle of which the generator is unity.

I
USAGE OF THE ARITHMETIC TRIANGLE
FOR THE NUMERIC ORDERS

We have considered in arithmetic the numbers of different progressions; we have also considered those of different powers and of different degrees; but we have not, it seems to me, examined enough those of which I speak, although they be of a very great usage: and similarly they have no name; thus I have been obliged to give to them; and because those of progression, of degree and of power are already employed, I myself serve with the one of *orders*.

I call therefore *numbers of the first order* the simple units:

$$1, 1, 1, 1, 1, \text{etc.}$$

I call *numbers of the second order* the naturals which are formed by addition of the units:

$$1,\ 2,\ 3,\ 4,\ 5,\ \text{etc.}$$

I call *numbers of the third order* those which are formed by the addition of the naturals, which are called triangular,

$$1,\ 3,\ 6,\ 10,\ \text{etc.}$$

That is, that the second of the triangular, namely 3, equals the sum of the first two naturals, which are 1, 2; thus the third triangular 6 equals the sum of the first three naturals 1, 2, 3, etc.

I call *numbers of the fourth order* those which are formed by addition of the triangular, which are called *pyramidal*:

$$1,\ 4,\ 10,\ 20,\ \text{etc.}$$

I call *numbers of the fifth order* those which are formed by addition of the preceding, to which we have not given expressed name, and which we could name triangular-triangular:

$$1,\ 5,\ 15,\ 35,\ \text{etc.}$$

I call *numbers of the sixth order* those which are formed by addition of the preceding:

	1,	6,	21,	56,	126,	252,	etc.
And thus to infinity,	1,	7,	28,	84,	etc.		
	1,	8,	36,	120,	etc.		

Now, if we makes a table of all the orders of the numbers, where we mark to the side the exponents of the orders, and above the roots, in this way:

			Roots					
			1	2	3	4	5	etc.
Units	....	Order 1	1	1	1	1	1	etc.
Naturals	....	Order 2	1	2	3	4	5	etc.
Triangular	....	Order 3	1	3	6	10	15	etc.
Pyramidal	....	Order 4	1	4	10	20	35	etc.
		etc.						

we will find this table similar to the arithmetic Triangle.

And the first order of numbers will be the same as the first parallel rank of the triangle. The second order of numbers will be the same as the second parallel rank; and thus to infinity.

For in the arithmetic Triangle, the first rank is all units, and the first order of the numbers is similarly all units.

Thus in the arithmetic Triangle, each cell, as the cell F, equals C+B+A, that is, that it equals its superior, plus all cells which precede this superior in its parallel rank; as it has been proved in the 2nd consequence of the treatise on this triangle. And the same thing is found in each of the orders of the numbers. Because, for example, the third of the pyramidals 10 equals the first three of the triangulars $1 + 3 + 6$, because it is formed by their addition.

Whence it is seen manifestly that the parallel ranks of the triangle are nothing other than the orders of the numbers, and that the exponents of the parallel ranks are the same as the exponents of the orders, and that the exponents of the perpendicular ranks are the same as the roots. And thus the number, for example, 21, which in the arithmetic Triangle is found in the third parallel rank, and in the sixth perpendicular rank, being considered among the numerical orders, will be of the third order, and the sixth of its order, or of the sixth root.

This shows that all that which has been said of the ranks and of the cells of the arithmetic Triangle agree exactly with the orders of the numbers, and that the same equalities and the same proportions which have been noticed in the one, will be found also in the others; it will be necessary only to change the statements, by substituting the terms which are proper to the numerical orders, as those of root and of order, for those which are proper to the arithmetic Triangle, as the parallel and perpendicular rank. I will give a small treatise apart, where some examples which are stated there will make it easy to see all the others.

USAGE OF THE ARITHMETIC TRIANGLE
FOR COMBINATIONS

The word *Combination* has been taken in many different senses, so that, in order to remove the ambiguity, I am obliged to speak as I intend it.

When among many things we give the choice of a certain number, all the ways of taking from them as many as is permitted among all those which are presented, are called here the *different combinations*.

For example, if from four things expressed by these four letters, A, B, C, D, we permit to take from them, for example, any two, all the ways of taking from them two different in the four which are proposed, are called *Combinations*.

Thus we will find, by experience, that there are six different ways of choosing two in four; we are able to take A and B, A and C or A and D, or B and C, or B and D, or C and D.

I do not count A and A as one of the ways of taking two; for these are not different things; this is only a repetition.

Thus I do not count A and B and next B and A as two different ways; for we take in the one and in the other manner only the same two things, but of a different order only; and I take no notice at all of the order; so that I am able to explain myself in a word to those who are accustomed to consider the combinations, by saying simply that I speak only of the combinations which are made without changing the order.

We will find likewise, by experience, that there are four ways to take three things in four; because we are able to take ABC, or ABD, or ACD, or BCD.

Finally we will find that we are able to take four from four but in one way, namely, ABCD.

I will speak therefore in these terms:

1 in 4 is combined 4 times.
2 in 4 is combined 6 times.
3 in 4 is combined 4 times.
4 in 4 is combined 1 time.

Or thus:

The number of combinations of 1 in 4 is 4.
The number of combinations of 2 in 4 is 6.
The number of combinations of 3 in 4 is 4.
The number of combinations of 4 in 4 is 1.

But the sum of all the combinations, in general, that we are able to make in 4, is 15, because the number of combinations of 1 in 4, of 2 in 4, of 3 in 4, of 4 in 4, being added altogether, make 15.

Next from this explication, I will give those consequences in the form of lemmas.

LEMMA I

A number does not combine at all in a smaller; for example, 4 is not combined at all in 2.

LEMMA II

1 *in* 1 *is combined* 1 *time*.
2 *in* 2 *is combined* 1 *time*.
3 *in* 3 *is combined* 1 *time*.

And generally any number is combined one time only in its equal.

LEMMA III

1 *in* 1 *is combined* 1 *time*.
1 *in* 2 *is combined* 2 *times*.
1 *in* 3 *is combined* 3 *times*.

And generally the unit is combined in any number that it be as many times as it contains unity.

LEMMA IV

If there are any four numbers, the first such as we will wish, the second greater by unity, the third such as we will wish, provided that it is not smaller than the second, the fourth greater by unity than the third: the number of combinations of the first in the third, added to the number of combinations of the second in the third, equals the number of combinations of the second in the fourth.

Let there be four numbers such as I have said:

The first such as we will wish, for example, 1.

The second greater by the unit, namely, 2.

The third such as we will wish, provided that it is not smaller than the second, for example, 3.

The fourth greater by the unit, namely 4.

I say that the number of combinations of 1 in 3, plus the number of combinations of 2 in 3, equals the number of combinations of 2 in 4.

Let there be any three letters, B, C, D.

Let there be the same three letters, and one more A, B, C, D.

Take, according to the proposition, all the combinations of one letter in the three, B, C, D. There will be 3, namely B, C, D.

Take in the same three letters all the combinations of two; there will be 3, namely, BC, BD, CD.

Take finally in the four letters A, B, C, D, all the combinations of 2; there will be 6, namely, AB, AC, AD, BC, BD, CD.

It is necessary to demonstrate that the number of combinations of 1 in 3 and those of 2 in 3, equal those of 2 in 4.

This is easy, for the combinations of 2 in 4 are formed by the combinations of 1 in 3, and by those of 2 in 3.

In order to see the technique, it is necessary to remark that among the combinations of 2 in 4, namely, AB, AC, AD, BC, BD, CD, there are of them where the letter A is used, and the others where it is not.

Those where it is not used are BC, BD, CD, which consequently are formed of two of the three letters B, C, D; therefore these are the combinations of 2 in these three, B, C, D. Therefore the combinations of 2 in these three letters B, C, D, make a portion of the combinations of 2 in those four letters A, B, C, D, since they form those where A is not used.

Now if of the combinations of 2 in 4 where A is used, namely AB, AC, AD, one omits the A, there will remain a single letter of these three B, C, D, namely B, C, D, which are precisely the combinations of one letter in the three, B, C, D. Therefore if in the combinations of one letter in the three, B, C, D, we add to each the letter A, and that thus we have AB, AC, AD, we will form the combinations of 2 in 4, where A is used; therefore the combinations of 1 in 3 are a portion of the combinations of 2 in 4.

Whence it is seen that the combinations of 2 in 4 are formed by the combinations of 2 in 3, and of 1 in 3; and hence that the number of combinations of 2 in 4 equals that of 2 in 3, and of 1 in 3.

We will show the same thing in all the other examples, as:

The number of combinations of 29 in 40;

And the number of combinations of 30 in 40:

Equals the number of combinations of 30 in 41.

Thus the number of combinations of 15 in 55;

And the number of combinations of 16 in 55:
Equals the number of combinations of 16 in 56.
And thus to infinity. That which it was necessary to demonstrate.

PROPOSITION I

In every arithmetic Triangle, the sum of the cells of any parallel rank equals the number of combinations of the exponent of the rank in the exponent of the triangle.

Let there be any triangle, for example the fourth GDλ. I say that the sum of the cells of any parallel rank, for example the second, $\phi + \psi + \theta$, equals the sum of the combinations of this number 2, which is the exponent of this second rank, in this number 4, which is the exponent of this triangle:

Thus the sum of the cells of the 5th rank of the 8th triangle equals the sum of the combinations of 5 in 8, etc.

The demonstration of it will be short, although there are an infinity of cases, by means of these two lemmas.

The first, which is evident in itself, that in the first triangle this equality is found, because the sum of the cells of its unique rank, namely G, or unity, equals the sum of the combinations of 1, exponent of the rank, in 1, exponent of the triangle.

The second, that, if an arithmetic Triangle is found in which this proportion is encountered, that is, in which, whatever rank that one takes, it happens that the sum of the cells are equal to the number of combinations of the exponent of the rank in the exponent of the triangle: I say that the following triangle will have the same property.

Whence it follows that all the arithmetic Triangles have this equality, for it is found in the first triangle by the first lemma, and similarly it is again evident in the second; therefore by the second lemma, the following will have it likewise, and hence the next again; and also to infinity.

It is necessary therefore only to demonstrate the second lemma.

Let any triangle, for example, the third, in which we suppose that this equality is found, that is, that the sum of the cells of first rank G+$\sigma + \pi$ equals the number of combinations of 1 in 3, and that the sum of the cells of the second rank $\phi + \psi$ equals the combinations of 2 in 3; and that the sum of the cells of the third rank A equals the combinations of 3 in 3; I say that the fourth triangle will have the same equality, and that, for example, the sum of the cells of the second rank $\phi + \psi + \theta$ equals the number of combinations of 2 in 4.

Because $\phi + \psi + \theta$ equals	$\underbrace{\phi + \psi}$	$+$	$\underbrace{\theta}$
		$+$	$\underbrace{\mathrm{G} + \sigma + \pi}$
By the hypothesis	or the number of combinations of 2 in 3.	$+$	or the number of combinations of 1 in 3.
By the 4th lemma	Or the number of combinations of 2 in 4.		

One will demonstrate likewise all the others.
That which it was necessary to demonstrate.

PROPOSITION II

The number of any cell that it be equals the number of combinations of a number less by unity than the exponent of its parallel rank, in a number less by unity than the exponent of its base.

Let there be any cell, F, in the fourth parallel rank and in the sixth base: I say that it equals the number of combinations of 3 in 5, less by unity than 4 and 6, for it equals the cells A+B+C. Therefore by the preceding, etc.

PROBLEM I — *PROPOSITION* III

Two numbers being proposed, to find how many times the one is combined in the other by the arithmetic Triangle.

Let the proposed numbers be 4, 6; it is necessary to find how much 4 is combined in 6.

First way.

Let the sum of the cells of the fourth rank of the sixth triangle be taken: it will satisfy the question.

Second way.

Let the 5th cell of the 7th base be taken, because the numbers 5, 7 exceed by unity the given 4, 6: its number is that which one demands.

CONCLUSION

By the relation that there is of the cells and ranks of the arithmetic Triangle to the combinations, it is easy to see that all that which has been proved of the ones agree with the others according to their manner. It is this that I will prove in a little treatise in a small treatise that I have made on Combinations.

USAGE OF THE ARITHMETIC TRIANGLE
IN ORDER TO DETERMINE THE DIVISIONS WHICH WE MUST MAKE BETWEEN TWO PLAYER WHO PLAY IN MANY GAMES

In order to understand the rules of the divisions, the first thing that it is necessary to consider is that the money that the players have staked in the game no longer belongs to them, for they have given up the property; but they have received in exchange the right to expect that which chance is able to give to them of it, according to the conditions to which they have agreed first.

But, as this is a voluntary act, they are able to mutually interrupt it; and thus, in any term that the game is found, they are able to quit it; and, to the contrary of that which they have made on entering it, to renounce to the expectation of chance, and to return to each of them the ownership of some thing. And in this case, the settlement of that which must belong to them must be so proportioned to that which they had right to expect from fortune, that each of them finds entirely equal to take that which one assigns to him, or to continue the adventure of the game: and this just distribution is called *the division*.

The first principle which shows in what way one must make the division, is this here:

If one of the players is found in such condition that, whatever happens, a certain sum must belong to him in case of loss or of gain, without that chance is able to take it away from him, he must not make any division of it, but take the whole as guaranteed, because the division must be proportioned to chance, since there is no chance to lose, he must get all without part.

The second is this one here: if two players are found in such condition that, if the one wins, a certain sum will belong to him, and if he loses, it will belong to the other; if the game is of pure chance, and if there is as much chance for the one as for the other and consequently no more reason to win for the one than for the other, if they wish to separate without playing, and to take that which belongs to them legitimately, the division is that they separate the sum which is at risk in half, and that each takes his own.

FIRST COROLLARY

If two players play in a game of pure chance, with the condition that, if the first wins, a certain sum will be restored to him, and if he loses, a lesser will be restored to him; if they want to separate without playing, and each to take that which belongs to them, the division is that the first take that which is restored to him in the case of loss, and moreover the half of the excess by which that which would be restored to him in case of gain surpasses that which is restored to him in the case of loss.

For example, if two players play with the condition that, if the first wins, he will obtain 8 pistoles, and if he loses, he will obtain 2 of them: I say that the division is that he take these 2, plus the half of the excess of 8 over 2, that is, 3 more, because 8 surpasses 2 by 6, of which the half is 3.

For, by hypothesis, if he wins, he obtains 8, that is,, $6+2$, and if he loses, he obtains 2; therefore these 2 belong to him in case of loss and of gain: and consequently, by the first principle, he must not make any division, but take them entire. But for the 6 others they depend on chance; so that if it is favorable to him, he will win them, otherwise, they will be restored to the other; and by hypothesis, there is no more reason they be restored to the one or to the other: therefore the division is that they separate them in half, and that each take his own, which is what I have proposed.

Therefore, in order to say the same thing in other terms, the case of the loss belongs to him, plus half of the difference of the cases of loss and of gain.

And, hence, if in case of loss, A belongs to him, and in case of gain $\mathrm{A}+\mathrm{B}$, the division is that he takes $\mathrm{A}+\frac{1}{2}\mathrm{B}$.

SECOND COROLLARY

If two players are in the same condition that we just said, I say that the division is able to be made in this fashion, which returns to the same: that we collect the two sums of gain and of loss and that the first take the half of this sum; that is, that we join 2 with 8 and it will be 10, of which the half 5 will belong to the first.

Because the half of the sum of two numbers is always the same as the lesser, plus the half of their difference.

And this is demonstrated thus:

Let A be that which is restored in case of loss, and $\mathrm{A}+\mathrm{B}$ that which is restored in case of gain. I say that the division is made by collecting these two numbers, which are $\mathrm{A}+\mathrm{A}+\mathrm{B}$, and by giving the half to the first, which is $\frac{1}{2}\mathrm{A}+\frac{1}{2}\mathrm{A}+\frac{1}{2}\mathrm{B}$. Because this sum equals $\mathrm{A}+\frac{1}{2}\mathrm{B}$, which has been proven to make a just division.

These fundamentals being set down, we pass easily to determining the division of the two players, who play for as many games as we will wish, in any state that they find themselves, that is, what division it is necessary to make when they play in two games, and when the first has one to nothing, or when they play in three, and when the first of them has one to nothing, or when he has two to nothing or when he has two to one; and generally to any number of games that they play, and in whatever gain of games that there be, and the one, and the other.

On which the first thing that it is necessary to remark, is that two players who play to two games, of which the first of them has one to nothing, are in the same condition as two others who play to three games, of which the first has two of them, and the other one: for there is this in common that, in order to finish, the first lacks only one game and the other two: and it is in this that the difference of the advantages consists, and what must regulate the divisions; so that it is necessary properly to have regard only to the number of games which remain for the one and for the other to win, and not to the number of those which they have won, because, as we have already said, two players finding themselves in the same state, when playing to two games, one of them has one to nothing, as two who playing to twelve games, one of them has eleven to ten.

It is necessary therefore to propose the question in this way:

Being proposed two players, to each of which a certain number of games are lacking in order to end, to make the division.

I will give here the method, that I will pursue solely in two or three examples which will be so easy to continue, that it will not be necessary to give more of them.

In order to make the thing general without omitting anything, I will take for the first example, that it is perhaps not appropriate to touch, because it is too clear; I do it nevertheless in order to start at the beginning; it is this:

First case.

If to one of the players no game is lacking, and to the other some, the entire sum belongs to the first. For he has won it, since none of the games is lacking in which he must win it.

Second case.

If to one of the players a point is lacking, and one to the other, the division is that they divide the money in half, and that each take his own: this is evident by the second principle. It is likewise if two games are lacking to the one and two to the other; and likewise any number of games which are lacking to the one, if as many are lacking to the other.

Third case.

If to one of the players a game is lacking, and to the other two, here is the art to find the division.

Consider that which would belong to the first player (to whom only one point is lacking) in case of gain of the game which they are going to play, and next that which would belong to him in case of loss.

It is clear that if the one to whom only one point is lacking, wins this game which is going to be played, it will no longer be lacking to him: therefore all would belong to him by the first case. But, on the contrary, if the one to whom two games are lacking wins that which they are going to play, no more than one will be lacking to him; therefore they will be in such condition, that one will be lacking to the one, and one to the other. Therefore they must divide the money in half by the second case.

Therefore if the first wins that game which is going to be played, it all belongs to him, and if he loses, the half belongs to him; therefore, in case that they wish to separate without playing this game, $\frac{3}{4}$ belongs to him by the second corollary.

And if we wishes to propose an example of the sum that they play, the thing will be well more clear.

Let us put that this is 8 pistoles; therefore the first in case of gain, must have the whole, which is 8 pistoles, and in case of loss, he must have the half which is 4; therefore there belongs to him in case of division the half of $8 + 4$, that is, 6 pistoles of 8; for 8$+4$ makes 12, of which the half is 6.

Fourth case.

If to one of the players one game is lacking and three to the other, the division will be found likewise by examining that which belongs to the first in case of gain and of loss.

If the first wins, he will have all his games, and therefore all the money, which is, for example, 8.

If the first loses, no more than 2 games will be necessary to the other to which 3 were necessary. Therefore they will be in a state, that one game will be necessary to the first, and two to the other; and hence, by the preceding case, 6 pistoles will belong to the first.

Therefore in case of gain, 8 is necessary to him, and in case of loss 6; therefore, in case of division, the half of these two sums belong to him, namely, 7; because $6 + 8$ make 14, of which the half is 7.

Fifth case.

If to one of the players one game is lacking and to the other four, the thing is likewise.

The first, in case of gain, wins all, which is, for example, 8; and in case of loss, one game is lacking to the first and three to the other; therefore 7 pistoles of 8 belong to him; therefore in the case of division, the half of 8 belongs to him, plus the half of 7, that is, $7\frac{1}{2}$.

Sixth case.

Thus, if one game is lacking to one and five to the other; and to infinity.

Seventh case.

Likewise, if two games are lacking to the first, and three to the other; for it is necessary always to examine the case of gain and of loss.

If the first wins, one game will be lacking to him, and three to the other; therefore by the fourth case 7 of 8 belong to him.

If the first loses, two games will be lacking to him, and to the other two, therefore by the second case, the half belongs to each, which is four; therefore in case of gain, the first will have 7 of them and in case of loss, he will have 4 of them; therefore in case of division, he will have the half of these two together, namely, $5\frac{1}{2}$.

By this method we will make the divisions under all sorts of conditions, by taking always that which belongs in case of gain and that which belongs in case of loss, and assigning for the case of division the half of these two sums.

Here is one of the ways to make divisions.

There are two others, the one by the arithmetic triangle, and the other by combinations.

METHOD FOR MAKING THE DIVISIONS BETWEEN TWO PLAYERS WHO PLAY MANY GAMES BY MEANS OF THE ARITHMETIC TRIANGLE.

Before giving this method, it is necessary to make this lemma.

LEMMA

If two players play a game of pure chance, with condition that, if the first wins, some portion of the sum that they wager will belong to him, expressed by a fraction, and that, if he loses, a half portion of that same sum will belong to him, expressed by another fraction: if they wish to separate themselves without playing, the condition of the division will be found in this manner. Let the two fractions be reduced to the same denominator, if they are not; let a fraction be taken of which the numerator is the sum of the two numerators, and the denominator double of the preceding: this fraction expresses the portion which belongs to the first of the sum which is in the game.

For example, let belong in case of gain $\frac{3}{5}$ of the sum which is in play, and let in the case of loss, $\frac{1}{5}$ of it belong to him. I say that that which belongs to him in case of division, will be found by taking the sum of the numerators, which is 4, and the double of the denominator, which is 10, from which one makes the fraction $\frac{4}{10.}$

For, by that which has been demonstrated in the second corollary, it was necessary to collect the case of gain and of loss, and to take the half; now the sum of the two fractions $\frac{3}{5} + \frac{1}{5}$ is $\frac{4}{5}$, which is made by the addition of the numerators, and its half is found by doubling the denominator, and thus one has $\frac{4}{10}$. That which it was necessary to demonstrate.

Now, the rules are general and without exception, whatever is restored in case of loss or of gain; because if, for example, in case of gain, $\frac{1}{2}$ belongs, and in case of loss nothing, in reducing the two fractions to the same denominator, we will have $\frac{1}{2}$ for the case of gain, and $\frac{0}{2}$ for the case of loss; therefore, in case of division, it is necessary this fraction $\frac{1}{4}$, of which the numerator equals the sum of the others, and the denominator is the double of the preceding.

Thus if in case of gain, all belongs, and in case of loss $\frac{1}{3}$, by reducing the fractions to like denomination, we will have $\frac{3}{3}$ for the case of gain, and $\frac{1}{3}$ for the one of the loss; therefore in case of division, $\frac{4}{6}$ belongs.

Thus, if in case of gain all belongs and in case of loss nothing, the division will be clearly $\frac{1}{2}$; for the case of gain is $\frac{1}{1}$, and the case of loss$\frac{0}{1}$; therefore the division is $\frac{1}{2}$.

And thus of all the possible cases.

PROBLEM I — PROPOSITION I

Two players being proposed, to each of whom a certain number of games are lacking to end, to find by the arithmetic Triangle the division that it is necessary to make (if they wish to separate themselves without playing), having regard to the games which are lacking to each.

Let the base in the triangle be taken in which there are as many cells as games lacking to the two together: next let be taken in this base as many contiguous cells starting with the first, as games lacking to the first player, and let us take the sum of the numbers. Therefore there remain as many cells as games lacking to the other. Let us take further the sum of the numbers. These sums are the one to the other as the advantages of the players reciprocally; so that if the sum that they play is equal to the sum of the numbers of all the cells from the base, there will belong to each that which is contained in as many cells as games lacking to the other; and if they play for another sum, it will belong to each of them in proportion.

For example, let there be two players, to the first of which two games are lacking, and to the other 4: it is necessary to find the division.

Let these two numbers 2 and 4 be added, and let their sum be 6; let the sixth base of the arithmetic Triangle Pδ be taken, in which there are consequently six cells P, M, F, ω, S, δ. Let as many cells be taken, starting at the first P, as games lacking to the first player, that is, the first two P, M; therefore there remains as many games to the other, that is, 4, F, ω, S, δ.

I say that the advantage of the first is to the advantage of the second, as $\mathrm{F}+\omega+\mathrm{S}+\delta$ to $\mathrm{P}+\mathrm{M}$, that is that, if the sum which is played is equal to $\mathrm{P}+\mathrm{M}+\mathrm{F}+\omega+\mathrm{S}+\delta$, to the one to whom two games are lacking belong the sum of the four cells $\delta+\mathrm{S}+\omega+\mathrm{F}$ and to the one to whom 4 games are lacking, the sum of the two cells $\mathrm{P}+\mathrm{M}$. And if they play for another sum, it belongs to them in proportion.

And in order to say it generally, any sum that they wager, there belongs to the first a portion expressed by this fraction $\frac{\mathrm{F}+\omega+\mathrm{S}+\delta}{\mathrm{P}+\mathrm{M}+\mathrm{F}+\omega+\mathrm{S}+\delta}$ of which the numerator is the sum of the 4 cells of the other, and the denominator is the sum of all the cells; and to the other a portion expressed by this fraction, $\frac{\mathrm{P}+\mathrm{M}}{\mathrm{P}+\mathrm{M}+\mathrm{F}+\omega+\mathrm{S}+\delta}$ of which the numerator is the sum of the two cells of the other, and the denominator the same sum of all the cells.

And, if one game is lacking to the one, and five to the other, to the first belongs the sum of the first five cells $\mathrm{P}+\mathrm{M}+\mathrm{F}+\omega+\mathrm{S}+\delta$, and to the other the sum of the cell δ.

And if six games are lacking to the one, and two to the other, the division will be found in the eighth base, in which the first six cells contain that which belongs to the one to whom two games are lacking, and the two others, that which belongs to the one to whom six of them are lacking; and thus to infinity.

Although this proposition has an infinity of cases, I will demonstrate it nevertheless in a few words by means of two lemmas.

The first, that the second base contains the divisions of the players to whom two games are lacking in all.

The second, that if any base contains the divisions of those to whom as many games are lacking as it has cells, the following base will be the same, that is it will contain also the divisions of the players to whom as many games are lacking as it has cells.

Whence I conclude, in a word, that all the bases of the arithmetic Triangle have this property: for the second has it by the first lemma; therefore, by the second lemma, the third has it also, and consequently the fourth; and to infinity. That which it was necessary to demonstrate.

It is necessary therefore only to demonstrate these 2 lemmas.

The first is evident of itself; because if one game is lacking to the one and one to the other, it is evident that their conditions are as ϕ to σ, that is as 1 to 1, and that this fraction belongs to each,

$$\frac{\sigma}{\phi+\sigma} \text{ which is } \frac{1}{2}.$$

The second will be demonstrated in this way.

If any base, as the fourth Dλ, contains the divisions of those to whom four games are lacking, that is that, if one game is lacking to the first, and three to the second, the portion which belongs to the first of the sum which is played, is that which is expressed by the fraction $\frac{\mathrm{D}+\mathrm{B}+\theta}{\mathrm{D}+\mathrm{B}+\theta+\lambda}$, which has for denominator the sum of the cells of this base, and for numerator its first three; and that, if two games are lacking to the one, and two to the other, the fraction which belongs to the first is $\frac{\mathrm{D+B}}{\mathrm{D+B}+\theta+\lambda}$; and that, if three games are lacking to the first, and one to the other, the fraction of the first is $\frac{\mathrm{D}}{\mathrm{D+B}+\theta+\lambda}$ etc.

I say that the fifth base contains also the divisions of those to whom five games are lacking; and that if two games, for example, are lacking to the first, and three to the other, the portion which belongs to the first of the sum which is played, is expressed by this fraction:

$$\frac{\mathrm{H+E+C}}{\mathrm{H+E+C+R}+\mu}$$

For in order to know that which belongs to two players to each of whom some games are lacking, it is necessary to take the fraction which would belong to the first in case of gain, and that which would belong to him in the case of loss, putting them with same denominator, if they are not, and forming a fraction, of which the numerator is the sum of the two others, and the denominator double of the other, by the preceding lemma.

Examine therefore the fractions which would belong to our first player in case of gain or loss.

If the first, to whom two games are lacking, wins that which they are going to play, no more than one game will be lacking to him, and to the other, always three; therefore four games are lacking to them in all: therefore, by hypothesis, their division is found in the fourth base, and to the first will belong this fraction $\frac{\mathrm{D+B}+\theta}{\mathrm{D+B}+\theta+\lambda}$.

If on the contrary the first loses, two games will always be lacking to him, and two alone to the other; therefore by hypothesis, the fraction of the first will be $\frac{\mathrm{D+B}}{\mathrm{D+B}+\theta+\lambda}$. Therefore, in the case of division, to the first will belong this fraction

$$\frac{\mathrm{D+B}+\theta+\mathrm{D+B},}{2\mathrm{D}+2\mathrm{B}+2\theta+2\lambda,}\quad \frac{\text{that is,}}{\text{that is,}}\quad \frac{\mathrm{H+E+C}}{\mathrm{H+E+C+R}+\mu}.$$

That which it was necessary to demonstrate.

Thus this is demonstrated among all the other bases without any difference, because the foundation of this proof is that a base is always double of its preceding by the seventh consequence, and that, by the tenth consequence, as many cells as one will wish of one same base are equal to as many of the base preceding (which is always the denominator of the fraction in case of gain) plus again in the same cells, one excepted (which is the numerator of the fraction in case of loss); that which being true generally everywhere, the demonstration will be always without obstacle and universal.

PROBLEM II — PROPOSITION II

Having proposed two players who stake each one same sum with a certain number of games proposed, to find in the arithmetic Triangle the value of the last game out of the money of the loser.

For example, let two players each wager three pistoles in four games: one demands the value of the last game out of the three pistoles of the loser.

Let the fraction be taken, which has unity for numerator, and for denominator the sum of the cells of the fourth base, since we play to four games: I say that this fraction is the value of the last game out of the stake of the loser.

For if two players playing to four games, one of them has three to nothing, and that thus one is lacking to the first, and four to the other, it has been demonstrated that that which belongs to the first for the gain that he has made for his first three games, is expressed by this fraction $\frac{\mathrm{H+E+C+R}}{\mathrm{H+E+C+R}+\mu}$ which has for denominator the sum of the cells of the fifth base, and for numerator its first four cells; therefore, there remains out of the total sum of the two stakes only this fraction $\frac{\mu}{\text{H+E+C+R+}\mu}$, which will be acquired by the one who has already the first three games in case that he won the last; of which the value of this last out of the sum of the two stakes is

$$\frac{\mu}{\mathrm{H+E+C+R}+\mu} \quad \frac{\textit{that is,}}{\textit{that is,}} \quad \frac{\textit{unity}}{2\mathrm{D}+2\mathrm{B}+2\theta+2\lambda}.$$

Now, since the total sum of the stakes is $2\text{D}+2\text{B}+2\theta + 2\lambda$, the sum of each stake is $\text{D+B}+\theta + \lambda$; therefore the value of the last game out of the sole stake of the loser is this fraction $\frac{1}{\text{D+B}+\theta+\lambda}$, double of the preceding, and which has for numerator unity, and for denominator the sum of the cells of the fourth base. That which was necessary to demonstrate.

PROBLEM III — PROPOSITION III

Two players being proposed who each stake one same sum in a certain number of given games, to find in the arithmetic Triangle the value of the first game out of the stake of the loser.

For example, let two players each stake 3 pistoles on four games, one demands the value of the first out of the stake of the loser.

Let be added to the number 4 the number 3, less by unity, and let the sum be 7; let the fraction be taken which has for denominator all the cells of the seventh base, and for numerator the cell of this base which is encountered in the divide, namely, this fraction

$$\frac{\rho}{V+\mathrm{Q}+\mathrm{K}+\rho+\xi+\mathrm{N}+\zeta},$$

I say that it satisfies the problem.

For if two players playing to four games, the first has one to nothing, there will remain three to win to the first, and four to the other; therefore there belongs to the first out of the sum of the two stakes this fraction $\frac{V+\text{Q+K}+\rho}{V+\text{Q+K}+\rho+\xi+\text{N}+\zeta}$, which has for denominator all the cells of the seventh base, and for numerator its first four cells.

Therefore $V+\text{Q+K}+\rho$ out of the sum total of the two stakes belongs to him, expressed by $V+\text{Q+K}+\rho + \xi+\text{N}+\zeta$; but this last sum being the collection of two stakes, the half was set into the game, namely $\text{V+Q+K}+\frac{1}{2}\rho$ (because V+Q+K is equal to $\zeta+\text{N}+\xi$).

Therefore this is $\frac{1}{2}\rho$, that is, ω, no more that he had in entering the game; therefore he has won out of the total sum of the two stakes a portion expressed by this fraction $\frac{\omega}{V+\text{Q+K}+\rho+\xi+\text{N}+\zeta}$, therefore he has won out of the stake of the loser a portion which will be double of this here, namely, that which is expressed by this fraction:

$$\frac{\rho}{V+\mathrm{Q}+\mathrm{K}+\rho+\xi+\mathrm{N}+\zeta}.$$

Therefore the gain of the first game has acquired this fraction to him; therefore its value is such.

COROLLARY

Therefore the value of the first game of two out of the stake of the loser, is expressed by this fraction $\frac{1}{2}$.

For in taking this value according to the rule which comes from being given it, it is necessary to take the fraction that has for denominator the cells of the third base (because the number of the games on which we play is two, and the number less by unity is 1, which with 2 makes 3), and for numerator the cell of this base which is in the divide; therefore one will have this fraction $\frac{\psi}{\mathrm{A}+\psi+\pi}$.

Now the number of the cell ψ is 2, and the numbers of the cells $\mathrm{A}+\psi+\pi$, are $1+2+1$.

Therefore we have this fraction $\frac{2}{1+2+1}$, that is, $\frac{2}{4}$ that is, $\frac{1}{2}$.

Therefore the gain of the first game has acquired to him this fraction; therefore its value is such. That which it was necessary to demonstrate.

PROBLEM IV — PROPOSITION IV

Two players being proposed who each stake one same sum on a certain number of given games, to find by the arithmetic Triangle the value of the second game on the stake of the loser.

Let the given number of games on which we play be given, 4; it is necessary to find the value of the second game on the stake of the loser.

Let the value be taken of the first game by the preceding problem. I say that it is the value of the second.

For two players playing in 4 games, if one of them has two to none, the fraction which belongs to him is this,

$$\frac{\mathrm{P}+\mathrm{M}+\mathrm{F}+\omega}{\mathrm{P}+\mathrm{M}+\mathrm{F}+\omega+\mathrm{S}+\delta},$$

which has for denominator the sum of the cells of the sixth base, and for numerator the sum of the first four; but there was set into the game this fraction:

$$\frac{\mathrm{P}+\mathrm{M}+\mathrm{F}+\omega}{\mathrm{P}+\mathrm{M}+\mathrm{F}+\omega+\mathrm{S}+\delta}$$

namely, the half of all. Therefore there remains to him of gain this fraction: $\frac{\mathrm{P}+\mathrm{M}+\mathrm{F}+\omega}{\mathrm{P}+\mathrm{M}+\mathrm{F}+\omega+\mathrm{S}+\delta}$, which is the same thing as this:

$$\frac{\rho}{V+\mathrm{Q}+\mathrm{K}+\rho+\xi+\mathrm{N}+\zeta};$$

therefore he has won out of the half of the entire sum, that is, out of the stake of the loser, this fraction:

$$\frac{2\rho}{V+\mathrm{Q}+\mathrm{K}+\rho+\xi+\mathrm{N}+\zeta},$$

double the preceding.

Therefore the gain of the first two games had acquired to him this fraction out of the money of the loser, which is the double of that which the first game had acquired to him by the previous; therefore the second game has as much acquired to him as the first.

CONCLUSION

One is able to conclude easily, by the relationship that there is of the arithmetic Triangle to the divisions which must be made between two players, that the proportions of the cells which have been given in the *Treatise of the Triangle*, has some consequences which are extended to the value of the divisions, which are very easy to draw, and of which I have made a small discourse, in treating some divisions, which give the intelligence and the means to extend them further.

USAGE OF THE ARITHMETIC TRIANGLE
TO FIND THE POWERS OF BINOMIALS AND OF APOTOMES

If it is proposed to find any power, as the fourth degree of a binomial, of which the first term is A, the other unity, that is that it is necessary to find the square-square of $\mathrm{A}+1$, it is necessary to take in the arithmetic Triangle the fifth base, namely that of which the exponent 5 is greater by unity than 4, exponent of the order proposed. The cells of this fifth base are 1, 4, 6, 4, 1, of which it is necessary to take the first number 1 for coefficient of A in the degree proposed, that is, of A^4; next it is necessary to take the second number of the base, which is 4, for coefficient of A at the degree next inferior, that is of A^3, and take the next number of the base, namely 6, as coefficient of A at the degree inferior, namely A^2, and the next number of the base, namely 4, as coefficient of A at the degree inferior, namely root A, and to take the last number of the base 1 for absolute number: and thus we will have $1\mathrm{A}^4+4\mathrm{A}^3+6\mathrm{A}^2+4\mathrm{A}+1$ which will be the square-square power of the binomial $\mathrm{A}+1$. So that if A (which represents all numbers) is unity, and that thus the binomial $\mathrm{A}+1$ is the binary, this power

$$1\mathrm{A}^4 + 4\mathrm{A}^3 + 6\mathrm{A}^2 + 4\mathrm{A} + 1$$

will be now

$$1.1^4 + 4.1^3 + 6.1^2 + 4.1 + 1;$$

that is	*one time the square-square of the unit A, that is,*	*1*
	Four times the cube of 1, that is,	*4*
	Six times the square of 1, that is,	*6*
	Four times unity, that is,	*4*
	Plus unity	*1*
	Which added make	*16*

And indeed the square-square of 2 is 16.

If A is another number, as 4, and therefore that the binomial $\mathrm{A}+1$ is 5, then its square-square will be always, according to this method, $1\mathrm{A}^4+4\mathrm{A}^3+6\mathrm{A}^2+4\mathrm{A}+1$, which signifies now:

$$1.4^4 + 4.4^3 + 6.4^2 + 4.4 + 1;$$

that is	*one time the square-square of 4, namely*	*256*
	Four times the cube of 4, namely	*256*
	Six times the square of 4	*96*
	Four times the root 4,	*4*
	Plus unity	*1*
	of which the sum	*625*

makes the square-square of 5: and indeed the square-square of 5 is 625. And thus some other examples.

If one wishes to find the same degree of the binomial $\mathrm{A}+2$, it is necessary to take of the same

$$1\mathrm{A}^4 + 4\mathrm{A}^3 + 6\mathrm{A}^2 + 4\mathrm{A} + 1$$

and next write these four numbers 2, 4, 8, 16, which are the first four degrees of 2, under the numbers 4, 6, 4, 1; that is, under each of the numbers of the base, in leaving the first in this manner:

$$1\mathrm{A}^4 + \underset{2}{4}\mathrm{A}^3 + \underset{4}{6}\mathrm{A}^2 + \underset{8}{4}\mathrm{A} + \underset{16}{1}$$

and multiply the numbers which correspond by one another so that:

$$\begin{array}{ccccc} 1A^4+ & 4A^3+ & 6A^2+ & 4A^1+ & 1 \\ & 2 & 4 & 8 & 16 \\ \hline 1A^4+ & 8A^3+ & 24A^2+ & 32A^1+ & 16 \end{array}$$

And thus we will have the square-square of the binomial A$+2$; so that if A is unity, this square-square will be such:

One times the square-square of the unit A	*1*
Eight times the cube of unity	*8*
24, 1^2	*24*
32, 1	32
Plus the square-square of 2	<u>*16*</u>
of which the sum	*81*

will be the square-square of 3: and indeed 81 is the square-square of 3.

And if A is 2, then A$+2$ will be 4, and its square-square will be:

One times the square-square of A or of 2, namely	*16*
8, 2^3	*64*
24, 2^2	*96*
32, 2	*64*
Plus the square-square of 2	<u>*16*</u>
of which the sum	*256*

will be the square-square of 4.

In the same manner we will find the square-square of A$+3$, in setting out in the same way:

$$\begin{array}{ccccc} A^4+ & 4A^3+ & 6A^2+ & 4A+ & 1 \end{array}$$

and below the numbers

$$\begin{array}{ccccc} & 3 & 9 & 27 & 81 \\ \hline 1\mathrm{A}^4+ & 12\mathrm{A}^3+ & 54\mathrm{A}^2+ & 108\mathrm{A}+ & 81 \end{array}$$

which are the first four degrees of 3; and multiplying the corresponding numbers, we will find that the square-square of $A+3$.

And thus to infinity. If instead of square-square we wish to find the square-cube, or the fifth degree, it is necessary to take the sixth base, and to use as I have said of the fifth; and thus of all the other degrees.

We will find likewise the powers of the apotomes A-1, A-2, etc. The method is entirely similar, and differs only in the signs, because the signs of $+$ and of $-$ always follow themselves alternately, and the sign of $+$ is always the first.

Thus the square-square of A-1 will be found in this way. The square-square of A$+1$ is by the preceding rule $1\mathrm{A}^4 + 4\mathrm{A}^3 + 6\mathrm{A}^2 + 4\mathrm{A}{+}1$. Therefore in changing the signs as I have said, one will have $1\mathrm{A}^4 - 4\mathrm{A}^3 + 6\mathrm{A}^2 - 4\mathrm{A}{+}1$. Thus the cube of A$-2$ will be found similarly. For the cube of A$+2$, by the preceding rule, is $\mathrm{A}^3 + 6\mathrm{A}^2 + 12\mathrm{A}{+}8$. Therefore the cube of A-2 will be found by changing the signs

$$\mathrm{A}^3 - 6\mathrm{A}A^2 + 12\mathrm{A} - 8.$$

And thus to infinity.

I have not given the demonstration of all this, because of others who have treated it already, as Hérigone, besides that the thing is evident in itself.

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