ESSAI D'ANALYSE SUR LES JEUX DE HAZARD

PIERRE DE MONTMORT

Fourth Part

Where we give the solution of diverse Problems on chance, & in particular the five Problems proposed in the year 1657 by Mr. Huygens.

PROBLEM I.
PROPOSITION XXXII.

Pierre & Paul play together with two dice: Here are the conditions of the game. Pierre will win by bringing forth six, & Paul by bringing forth seven. Each of the two will play two coups consecutively when he will have the dice: however Pierre who will begin will play only once for the first time. The concern is to determine the lot of each of these two Players, or the expectation that each will have to win the game.

SOLUTION.

169. Since each face of one of the dice is able to be found successively with all the faces of the other, it is clear that the two dice are able to give thirty-six coups, & that of these thirty-six coups there are five which give the number of six, namely ace & five, five & ace, two & four, four & two & tern, & six which give the number of seven, namely ace & six, six & ace, two & five, five & two, three & four, four & three.

Presently let A be named the money of the game, x the lot of Pierre when he is going to play his coup, y his lot when Paul is going to play his first coup, z his lot when Paul is going to play his second coup, & finally u his lot when the turn of Pierre returning he is going to play the first of his two coups.

We will have these four equalities, $S = \frac{5}{36}A + \frac{31}{36}y$, $y = \frac{30}{36}z$, $z = \frac{30}{36}u$, $u = \frac{5}{36}A + \frac{31}{36}x$; this which gives $S = \frac{5}{36}A + \frac{31}{36} \times \overline{\frac{5}{36}A + \frac{31}{36}x}$; whence we draw by reduction & transposition $S = \frac{10355}{22631}A$, this which expresses the lot of Pierre, & $A - S = \frac{12276}{22631}A$ which expresses the lot of Paul.

REMARK

170. The analytic method which we have employed here is always the best & shortest, when it happens that at the end of a certain number of coups the Players are found again in the same state where they were before; but when this happens not at all, we fall into some infinite series; & in order to find them, all the skill consists in observing well the conditions of the game, & to draw from it the Law of the progression. For example, if we supposed that Pierre played first one coup, & Paul two coups, next Pierre two coups, & Paul three coups; next Pierre three coups, & Paul four coups, & thus consecutively, Paul playing always one more coup than Pierre, the lot of Pierre would be expressed by a series of which it would be quite difficult to have the sum, this series would be $= \frac{b}{f}A +$

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$\frac{d\times c^2\times b}{f^4}A++\frac{d^2\times c^2\times b}{f^5}A+\frac{d^3\times c^5\times b}{f^9}A+\frac{d^4\times c^5\times b}{f^{10}}A+\frac{d^5\times c^5\times b}{f^{12}}A+\frac{d^6\times c^9\times b}{f^{16}}A+\frac{d^7\times c^9\times b}{f^{17}}A+\frac{d^8\times c^9\times b}{f^{18}}A+\frac{d^9\times c^9\times b}{f^{19}}A+\frac{d^{10}\times c^{14}\times b}{f^{25}}A+$&c. by supposing $b=5$, $c=30$, $d=31$, $f=36$.

It is easy to note the order of the series, & to continue it to infinity, the expression of the lot of Paul would be the quantity which is lacking to the series which expresses the lot of Pierre for the value of A.

If Pierre & Paul play with one die, according to the order that we just showed, to which the first will bring forth a six, we will have for the expression of the lot of Pierre a quite simple series, namely $1-p+p^3-p^5+p^8-p^{11}+p^{15}-p^{19}+p^{24}-p^29+$ &c. by supposing that p is $=\frac{5}{6}$.

GENERALLY

171. According as some order that two Players are able to play we will have always their lot expressed by the rule which follows.

Let p be the fraction which expresses how much are the odds that that which we intend will not happen on the first coup. If Pierre & Paul play alternately, Pierre b coups, Paul c coups, Pierre d coups, Paul e coups, Pierre f coups, Paul g coups, Pierre, &c. & thus consecutively according as any law, the lot of Pierre will be $1-p^b+p^{b+c}-p^{b+c+d}+p^{b+c+d+e}-p^{b+c+d+e+f}+$ &c. & the one of Paul will be the complement of unity. We are able to seek parallel formulas for many players.

PROBLEM II.

PROPOSITION XXXIII.

Three Players, Pierre, Paul & Jacques play together, & agree that drawing the one after the other a token at random among twelve, of which eight will be black & four will be white, the one who first will have drawn a white token will win. Here is the order according to which they play: Pierre draws first, Paul second, & Jacques third; next Pierre begins again, & the others following him according to their rank, until one of the Players has won. The concern is to find that which each Player must set into the game, in order that the division is equal.

SOLUTION

172. It is clear that each of the Players in order to wager equally & without disadvantage, must set into the game in proportion more or less by right that he has for the division, or the expectation that he has to win. We see well, for example, that because of the primacy, Pierre has more advantage in this game than Paul, & Paul more advantage than Jacques, since it is able to be that Pierre wins without Paul & Jacques having played, & also that Paul wins without the turn of Jacques be come. But how much Pierre has more advantage than Paul, & Paul has more advantage than Jacques, & what is, proportionally with these different advantages of the Players, the difference of the advances that each must make in order to compose the fund of the game? It is that which it is necessary to seek.

It is necessary to note first that the lot of a person who wagers to take a white token among twelve, of which eight are black & four are white, is to have one against two.

This supposed, if we name A the money of the game, S the lot of Jacques when Pierre is going to draw his token, y his lot when Paul is going to draw his, z his lot when it is for him to draw, we will have these three equalities $S=\frac{2}{3}y$, $y=\frac{2}{3}z$, $z=\frac{1}{3}A+\frac{2}{3}S$; whence we will draw $S=\frac{4}{19}A$, this which expresses the lot of Jacques.

Similarly in order to find the lot of Pierre, I name u his lot when he draws his token, t his lot when Paul draws his, q his lot when Jacques draws his token. This supposed, I have these three inequalities $u=\frac{1}{3}A+\frac{2}{3}t$, $t=\frac{2}{3}q$, $q=\frac{2}{3}u$; whence we draw $u=\frac{9}{19}A$, this which expresses the lot of Pierre. Now the lot of Paul being to have the money of the game, less the sum of the just claims of Pierre & of Jacques, we will have the lot of Paul $= A - \frac{4}{19}A - \frac{9}{19}A = \frac{6}{19}A$. Consequently if we wish that the game be of nineteen écus, it will be necessary that Pierre set nine into it, Paul six, & Jacques four.

REMARK

173. If the sense of the Problem is that each Player after having drawn a token no longer replaces it, we will find in the same manner that the lot of the three Players, as these three numbers 77, 53, 35; & if we wish to consider the Problem generally for any number of Players, of black tokens & of white tokens, we will have some infinite series, of which we will find the sums by the methods that I give in *art. 54 & 73*.

PROBLEM III.
PROPOSITION XXXIV.

Pierre wagers against Paul taking, the eyes shut, seven tokens among twelve, of which eight are black & four are white, he will take from them three white & four black. We demand how much Pierre & Paul must wager in order that the stake of each is in the same proportion as their lot.

SOLUTION.

174. This Problem & all the similar kinds are only some particular examples of Proposition 7, *art. 20*. One finds by the Table, *art. I*, that eight tokens are able to be taken in 70 different ways 4 by 4; that four tokens are able to be taken differently four times three by three; & finally that twelve tokens are able to be taken seven by seven in 792 different ways. We will have therefore for expression of the lot of Pierre $\frac{35}{99}$, & consequently $\frac{64}{99}$ for the one of Paul. If we wish that Pierre has also won when he will take four white & three black, we will have likewise, by *art. 20*, $\frac{70\times4+1\times56}{792} = \frac{14}{33}$ for the lot of Pierre, & in this case it would be necessary that Paul set into the game 19 against Pierre 14.

PROBLEM IV.
PROPOSITION XXXV.

Pierre wagers against Paul that drawing, eyes shut, four cards among forty, namely ten diamonds, ten hearts, ten spades & ten clubs, he will draw from it one of each kind. We demand what is the lot of these two Players, or that which they must set into the game in order to wager equally.

SOLUTION.

175. This Problem, in the same way as the previous, is only a particular case of Proposition 7, *art. 20*, the lot of Pierre is here expressed by a fraction of which the numerator is the fourth power of 10, & the denominator the number which expresses in how many ways four things are able to be taken in forty; & consequently the lot of Pierre is to the lot of Paul :: 10000 : 91390 − 10000 :: 1000 : 8139.

If we would demand how much are the odds that Paul drawing thirteen cards at random in fifty-two, will not draw all one color, we would find that the odds are 158753389899 against 1.

If we wish to know how much are the odds that Pierre drawing ten cards at random among forty, namely an ace, a deuce, a three, a four, a five, a six, a seven, an eight, a nine, & a ten of diamonds, as many hearts, spades & clubs, he will draw a complete "dizaine," we will find that the odds are 1048576 against 846611952, nearly 1 against 808.

PROBLEM V.
PROPOSITION XXXVI.

Pierre and Paul take each twelve tokens & play with three dice with the conditions which follow. If the dice bring forth eleven, Paul will give a token to Pierre. If the dice bring forth fourteen, Pierre will give a token to Paul. The one of the two who first will have all the tokens, will win. We demand what is the lot of the two Players.

SOLUTION.

176. It is necessary to note first that three dice give two hundred sixteen coups, since two dice give thirty-six coups, with each of which each face of the third die is able to be found successively. We will observe next that among these two hundred sixteen coups there are twenty-seven which give a token to Pierre, namely, six four & ace which happens in six ways; six three & deuce which happens in six ways; five four & deuce which happens in six ways; five five & ace which happens in three ways; four four & three which happens in three ways; three three five which happens in three ways; & fifteen which give a token to Paul, namely six five three which happens in six ways; five five four which happens in three ways; six six deuce which happens in in three ways; four four six which happens in three ways. This supposed, let A name the money in the game, x the lot of Pierre, when he has twelve tokens & Paul twelve; y, z, u, t, r, s, q, p, n, m, o, his lot, when he has won from Paul either one token, or two tokens, or three tokens,..., or eleven tokens. k, i, l, h, g, f, e, d, c, b, w, his lot, when he as lost against Paul either one token, or two tokens, or three tokens, ... or eleven tokens. We have $x = \frac{27y+15k+174x}{216}$, or that which is the same thing $x = \frac{9y+5k}{14}$.

And likewise all the other equalities.

$$\begin{array}{lll}
14y = 9z + 5x & 14p = 9n + 5q & 14h = 9l + 5g \\
14z = 9u + 5y & 14n = 9m + 5p & 14g = 9h + 5f \\
14u = 9t + 5z & 14m = 9o + 5n & 14f = 9g + 5e \\
14t = 9r + 5u & 14o = 9A + 5m & 14e = 9f + 5d \\
14r = 9s + 5t & 14k = 9x + 5i & 14d = 9e + 5c \\
14s = 9q + 5r & 14i = 9k + 5l & 14c = 9d + 5b \\
14q = 9p + 5s & 14l = 9i + 5h & 14b = 9c + 5w \\
 & & 14w = 9b + 5 \times \text{zero}
\end{array}$$

From all these equalities one will draw

$$y = \frac{31381059609A + 39165289355x}{70546348964},$$

&

$$K = \frac{70497520839}{70546348964}x.$$

And consequently we will find

$$x = \frac{9y + 5K}{14} = \frac{282492953648A + 352487604195x}{987648885496},$$

by substituting for y & k their values in x, & finally we will have by reducing

$$x = \frac{282429536481}{282673677106}A,$$

this which expresses the lot of Pierre, &

$$A - x = \frac{244140625}{282673677106},$$

this which expresses the lot of Paul.

Remark

177. It is proper to observe that the analytic way is perhaps not here the best, since we are able to discover otherwise that the lots of Pierre & Paul are as the twelfth powers of the numbers 9 & 5, thus as it has been observed by Messers. Bernoulli who have warned me of it in their letters of 17 March 1710, & 26 February 1711, & since by Mr. Moivre in his Treatise *de Mensura Sortis*, which appeared the past year.

Extract from the letter of 17 March 1710
From Johann Bernoulli to Montmort
pp. 291–294

You claim, Sir, to have resolved the second Problem of Mr. Huygens, that which you have effected indeed in the sense that you give to this Problem, which is that one must suppose that each Player having removed a black token returns it immediately into the pot, in order to leave to his successor the dozen tokens always complete, this which renders the Problem quite easy, & find the lot of the three Players in the ratio of 9, 6 & 4, as you have found. But it seems that Mr. Huygens has proposed this problem in another sense which appears more natural, which is that all the time that one draws a black token, it is no longer returned to the pot; so that the first drawer having failed by drawing a black token, the second when he comes to draw, finds no more than eleven tokens; & the second having also failed, the third find no more than ten tokens; & the one having similarly drawn a black, leaves only nine tokens to the first who must recommence to draw, & thus consecutively: the Problem being concluded in this sense, becomes a little more difficult, & renders the calculation longer. Test it to see if you yourself agree with me; I have set the solution after the proposition in the Treatise *De Ratioc. Lud. aleae*, twelve years ago; by consulting it I find that I have written these three numbers 77, 53, 35 for the ratio of the lots of the three Players.

⋮

The two Problems[1] that you set here as the third & the fourth, are in the Treatise of Mr. Huygens the fourth & third; for that which is former, that is of the fourth according to the order of Mr. Huygens, or the third in your Book, it is well resolved, as Pierre wagers among the 7 tokens that he is going to take, he will find justly 3 of them white neither more nor less; for I find also that the lot of Pierre will be the one of Paul under this assumption as 35 to 64, or as 280 to 512; but if we wish that Pierre has won also when he draws four white, this which appears to be the true sense of the words of Mr. Huygens *inter quos 3 albi erunt*, where it is necessary to supply *minimum*, as if Pierre was engaged to draw three whites at least among the seven tokens that he takes among the twelve: This sense was given to the Problem, we find that the lots of Pierre & of Paul will be as 14 & 19.

[1] *Translator's note*: Bernoulli refers to the first edition of 1708.

The proposition that you make here of the 5^{th} Problem of Mr. Huygens gives to it a sense entirely different, & it is no longer the same Problem. In order for you to see the greatest difference, I am going to give you here the simple solutions of the one & the other proposed in general. According to the conditions of Mr. Huygens, let n be named the number of tokens that each Player takes at the beginning, a the number of coups which make Pierre win a token from Paul, & b the number of coups which make Paul win a token from Pierre; I say that their lots will be as a^n & b^n; & thus for the particular case which is in question of the 3 dice, where $n = 12$, $a = 27$ & $b = 15$; so that $a : b :: 27 : 15 :: 9 : 5$, I say that the lot of Pierre will be to the one of Paul :: $9^{12} : 5^{12} :: 282429536481 : 244140625$. You have not observed, I believe, that these two large numbers are nothing other than the 12^{th} power of 9 & 5. But according to the conditions of your proposition, we name again n the number of tokens that one of the two Players must win first in order to win the game, that is the half of the tokens that distributor Jacques holds at the beginning between his hands; let also a be the number of coups favorable to Pierre, & b the one of the coups favorable for Paul. I find that the lots of the two Players will be expressed by the two sums of the two halves of the terms of the binomial $a + b$ raised to power $2n - 1$: for example if $n = 3$, the lots of Pierre & of Paul will be as the sum of the first three terms & the sum of the last three terms of $\overline{a+b}^5$, that is as $a^5 + 5ab^4 + 10a^3bb$, & $b^5 + 5b^4a + 10b^3aa$; & thus in general supposing $2n - 1 = p$, these two series $a^p + \frac{p}{1}a^{p-1}b^1 + \frac{p.p-1}{1.2}a^{p-2}b^2 + \frac{p.p-1.p-2}{1.2.3}a^{p-3}b^3 +$ &c. & $b^p + \frac{p}{1}b^{p-1}a^1 + \frac{p.p-1}{1.2}b^{p-2}a^2 + \frac{p.p-1.p-2}{1.2.3}b^{p-3}a^3 +$ &c. continued each to the number of terms expressed by n, will give the ratio of the lots of Pierre & of Paul. In our particular case where $n = 12$, & $a : b :: 9 : 5$, it would be necessary to take twelve terms of each of these two series $9^{23} + \frac{23}{1}9^{22}5^1 + \frac{23.22}{1.2}9^{21}5^2 + \frac{23.22.21}{1.2.3}9^{20}5^3 + \frac{23.22.21.20}{1.2.3.4}9^{19}5^4 +$ &c. & $5^{23} + \frac{23}{1}5^{22}9^1 + \frac{23.22}{1.2}5^{21}9^2 + \frac{23.22.21}{1.2.3}5^{20}9^3 + \frac{23.22.21.20}{1.2.3.4}5^{19}9^4 +$ &c., this which would produce two numbers so great, that the first would constitute (according to my conjecture) for at least 25 figures, if you have desire to make the calculation, here is the work to exercise your patience; however you see the extreme difference that there is between the two ways to propose the fifth Problem of Mr. Huygens; so that these are effectively two entirely different Problems, of which I have resolved each generally for you. You will be perhaps amazed that your 23 equalities furnish you nonetheless the same solution as Mr. Huygens, notwithstanding that you proposed the Problem in a sense which makes it, as I just showed you, so different from the one of Mr. Huygens; but the reason for it is, since you follow effectively in forming the equalities, the conditions of Mr. Huygens, & not those of your proposition; for I see that you suppose that the lots of the two Players are the same when there is one same difference between the number of tokens that the one has already won, & the number of tokens that the other has won; so that you suppose, for example, that their lots are the same, be it that Pierre having won five tokens, Paul has three of them; or that Pierre having two tokens, Paus has none of them at all: or it is this proposition which is not just, being able to subsist with the Problem taken in the sense that you give to it.

Extract from the letter of 15 November 1710
From Montmort to Johann Bernoulli
pp. 305

I was rather carried to believe as you that the sense of the Problem of Mr. Huygens is rather the one that you give to it than the one of my solution: I have found likewise as you these three numbers 77, 53, 35.

I do not believe that Mr. Huygens has wished *minimum* understood, as you say, after these words: *Inter quos tres albi erunt*; in each case I have been able to conjecture it, it has been also easy in one way than in the other.

This observation is very important, & I am very obliged to you to have communicated it to me. You have perfectly reasoned, & I have erred: Here is the manner by which I myself remember that which is arrived. I have resolved the five Problems of Mr. Huygens five or six years ago; it was only on the occasion of the extract of the Book of Mister your brother which is found in the Memoirs of the Academy, that I conclude the project of my Book. This Book achieved, I will give in the third part these five Problems without reviewing them. Now resolving them for the first time, I have resolved them with respect to the enunciated Latin; but being on the point of printing them, I made nothing other than to translate the text of Mr. Huygens; & finding it obscure, & myself no longer recollecting the solution, I will give to him a sense which squares not at all with the solution. It is true that these two great numbers 282429536481, 244140625 are the 12$^{\text{th}}$ power of 9 & of 5: the remark is excellent, your two solution are exquisite. I am no able more than you to take the pain to resolve the Problem on the foot of the enunciation of the proposition, I am content with your solution, the calculation of it would be too long.

Extract from the letter of 26 February 1711
From Nicolas Bernoulli to Montmort
pp. 309–311

…The Problem that you propose on the games which are played in many parts by reducing is quite difficult; nonetheless seeing that you wished that I find a solution of it, I have applied myself, & I have found a general rule in order to express the lot of the one who would wager that one of the Players will have won in such number of trials as we will wish, be it that they play in an equal or unequal game, or that one has already won some games or not: here it is in few words. Let the two Players be Pierre & Paul, the number of parts which are lacking to Pierre$= m$, the number of parts which are lacking to Paul$= n$, their sum$=$ $m+n=s$, the number of cases favorable to Pierre$= p$, the number of cases favorable to Paul$= q$, their sum $= p+q=r$, the number of trials $= h = m+2k$, the number of times that s is contained in $k=t$; this assumed, I say that the difference between the sum of all the possible values (that is by setting for t all the values which it is able to have from 0 to the greatest) of this series $1 \times \overline{p^{2k-2ts} + q^{2k-2ts}} + h \times \overline{p^{2k-2ts-1}q + q^{2k-2ts-1}p} + \frac{h.h-1}{1.2} \times \overline{p^{2k-2ts-2}qq + q^{2k-2ts-2}pp} + \frac{h.h-1.h-2}{1.2.3} \times \overline{p^{2k-2ts-3}p^3 + q^{2k-2ts-3}q^3} +$ &c. to $\frac{h.h-1.h-2\cdots h-k+ts+1}{1.2.3.4\cdots k-ts} \times pq^{k-ts}$ the whole multiplied by $\frac{p^{ts+m}q^{ts}}{r^h}$, & the sum of all the values of this here $1 \times \overline{p^{2k-2ts-2n} + q^{2k-2ts-2n}} + h \times \overline{p^{2k-2ts-2n-1}q + q^{2k-2ts-2n-1}p} + \frac{h.h-2}{1.2} \times \overline{p^{2k-2ts-2n-2}qq + q^{2k-2ts-2n-2}pp} +$ &c. to $\frac{h.h-1.h-2\cdots h-k+ts+n+1}{1.2.3.4\cdots k-ts-n} pq^{k-ts-n}$, the whole multiplied by $\frac{p^{ts+s}q^{ts+n}}{r^h}$, will express the lot of the one who would wager that Pierre will win the game in at least h trials. If k is smaller than $ts+n$; that is, if after having divided k by s, the remainder of the division is smaller than n, it is not necessary in the last series to set for t all the values from 0 to t, but only to $t-1$. In order to have the lot of the one who would wager that Paul will win it in h coups, it will be necessary only to substitute into this formula the letters q, p, n, m, instead of p, q, m, n. The sum of these two lots together will be the lot of the one who would wager that the game will be decided in h trials. The application of this formula to some particular cases, when $p=q=1$, is quite easy; I have found as well as you, nearly without calculation, that for six games the lot of the one who would wager that the game will be ended in 26 trials will be $\frac{16607955}{33554432}$, & in 28 trials $\frac{35485125}{67108864}$, or $\frac{7090250}{134217738}$, & not $\frac{70970250}{133432831}$, as you have written in error; but for twelve games I have found that we can already wager with advantage that the game will be ended in 110, & it would be disadvantageous to wager that it will end in 108 trials; because the lot for these two numbers of trials will be $\frac{329\cdots}{649\cdots}$ & $\frac{810\cdots}{1622\cdots}$,[2] it is necessary therefore that you yourself are mistaken, since you say that we can wager with advantage when the game will be decided only in 124 trials. It is ncessary however to confess that it is necessary by groping in order to find when the strength will be $\frac{1}{2}$; this is why if you have a better method than that here, I pray you to communicate it to me, & I will be much obliged to you. It is clear that this formula, which I just gave, will serve also to find the lot of the same Players; because for this end it will be necessary only to suppose that the number of trials is infinite, by setting therefore $h,\ k,$ & $l =$ *inf.* we will find that the lot of Pierre will be $= \frac{\overline{p+q}^h \times p^s - p^{s-n} q^n}{r^h \times p^s - q^s} = \frac{p^s - p^m q^n}{p^s - q^s}$; & consequently that of Paul $\frac{p^m q^n - q^s}{p^s - q^s}$, this which I have found formerly by a different way from that which I have followed in the research on this Problem. If $m = n$, & $s = 2m$, their lots will be as $p^{2m} - p^m q^m$ & $p^m q^m - q^{2m}$, or as p^m & q^m; & by supposing $m = 12$, $p = 9$, $q = 5$, we will have 9^{12} & 5^{12} for the lots of Pierre & Paul, which is the case of the fifth Problem of Huygens. If $p = q$, the lots of the two Players are as n & m, which is found easily by dividing $p^s - p^m q^n$, & $p^m q^n - q^s$ by $p - q$; because we will have by this division two geometric progressions, of which the number of terms of the 1st will be $= n$, & that of the 2nd $= m$, & of which the terms, by supposing $p = q$, will become all equal. If $p = q$, & $s = m + n = 12$, we have the case of page 178[3] of your Book.

[2] Translator's note. These values are correct. Indeed, the exact probability that the game terminate in 110 trials is $\frac{32975629612261143154616804262 6736}{64903710731685345356631204115 2512}$ and the exact probability that the game terminate in 108 trials is $\frac{810572622764486688482230467324 61}{162259276829213363391578010288128}$. The first quotient is approximately 0.5080700200 and the second is 0.4995539476.

[3] *Translator's note*. This refers to the first edition of 1708.