Recherches sur les avantages de trois Joueurs qui font entr'eux une Poule au trictrac ou à un autre Jeu quelconque

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§ 1.

The entire calculus of probabilities is based on a very simple & very clear rule, & the difficulties which present themselves in the solution of the different Problems which we can propose on this matter, arise ordinarily only from this that the application of this rule demands the knowledge of all the cases which contribute to render more or less probable the event of which we seek the probability, that which is often very difficult to determine, & sometimes even impossible. Although the number of these cases can be infinite in some certain Problems, this does not prevent that we can sometimes come to their solution, it is necessary then to have recourse to infinite series; it is by this method there of which I have served myself in the following of this little memoir, in determining the advantages of many players who have among them a *Pool* of the following kind: There are only two who play together, the one who loses the match quits the game, & gives up his place to one of the others, the one who loses the second match, quits, & gives up also his place to the following, & thus alternately until one of them has overcome in sequence all the others, that one ends the game that way & draws the Pool which is composed of the sum of the many stakes of which each player has put each time that he is entered into the game. One plays most commonly this game among three persons, it is for that one which I have tried to calculate the advantages of the Players.

I mean by *advantage* the expectation of a Player diminished by that which he has put into the game, whether it is positive or negative.

Problem.

§ 2. *Three Players making a Pool, & putting each time that they re-enter into the game a new stake A: To determine their advantages after any number n of matches already played.*

In order to arrive more easily to the solution of this Problem, I will divide it into two parts, I will seek first in the § following the advantages of the Players, having regard

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 28, 2009 only to the pool actually existing, by supposing that one does not add any more stake, & I will determine next in the fifth §, their advantages relatively to the continuation of the game, & to the new stakes which one will put at each match, without paying any attention to the ones which are already on the game, as if each player took back those which he had put into it; & the sum of these two advantages thus determined will give for each the entire advantage, demanded in our Problem.

Problem.

§ 3. *The pool being p, & being supposed to remain the same, without receiving any increase; we demand the advantage of each player after any number of matches.*

Let the expectation of the one who has won the preceding match, & who continues the game $= r$, the expectation of the one who enters into the game $= t$, & the expectation of the one who exits $= s$; we will have by the ordinary & known rules of the calculus of probabilities the following three equations: $r = \frac{p+s}{2}$; $t = \frac{0+r}{2}$; & $s = \frac{0+r}{2}$; from which we will deduce the three values sought namely $r = \frac{4}{7}p$; $t = \frac{2}{7}p$; & $s = \frac{1}{7}p$; if the stakes of the three players are M, m, & μ, their respective advantages will be $\frac{4}{7} - M$; $\frac{2}{7}p - m$; & $\frac{1}{7}p - \mu$; which will become (in the case where the three stakes are equal, namely when each has put in $\frac{1}{3}p$) $\frac{5}{7} \cdot \frac{1}{3}p$; $-\frac{1}{7} \cdot \frac{1}{3}p$; & $-\frac{4}{7} \cdot \frac{1}{3}p$.

Corollary.

§ 4. In order to find the advantages of the Players before they have begun to play, having only decided who are those who will begin, it will suffice to consider that the one who must enter only at the second match has the same advantage before & after the first match, because it is indifferent to him which of the two others win this first match, this advantage is therefore the same as the advantage of the one who enters at the game after n matches, & what we have found in the § preceding is equal to $\frac{2}{7}p - m$, or (if $m = \frac{1}{3}p$) to $-\frac{1}{7} \cdot \frac{1}{3}p$; & consequently the advantage for each of those who begin the game is found to be equal to $+\frac{1}{14} \cdot \frac{1}{3}p$.

Problem.

§ 5. *The three Players being* Pierre*,* Paul *&* Jacques*: To determine after any number of matches, their advantages having regard only to the future increase of the pool.* Pierre *is the one who has won the preceding match & who continues;* Paul *enters into the game & puts the first stake; &* Jacques *will enter at the second match, if* Pierre *does not win the first.*

I seek for this purpose the advantage of each player, for the case where the game ends at the first match, next for the case where it ends at the second match, since for the third, the fourth &c. after which I multiply the advantage of the first case by the probability that the game will end at the first match, the advantage of the second case by the probability that it will end at the second, & thus for the others, the sum of all these products which will be an infinite series will give the advantage sought of each player.

1°. If the game ends at the first match, Pierre will win the stake of Paul, & Jacques will lose nothing, the advantage of Pierre will be $= +A$; the one of Pierre $= -A$; & the one of Jacques $= 0$.

2°. If the game ends at the second match, Paul will win the two stakes of which he has put one, his advantage will be therefore $+A$; the one of Paul $= 0$; & the one of Jacques $= -A$.

3°. If the game ends at the third match; the advantage of Pierre will be $= -A$; the one of Paul $= -A$; & the one of Jacques $= 2A$.

We will find likewise for the following cases the advantages of the three players, & we will construct easily the following small table, which will serve to show first, the law which these advantages follow:

		Advantage of Pi.	Adv. of Pa.	Adv. of Jac.
If the game ends at the	first match.	$+A$	$-A$	-0
at the	2nd	-0	$+A$	$-A$
	3rd	$-A$	$-A$	$+2A$
	4th	$+3A$	$-2A$	$-A$
	5th	$-A$	$+3A$	$-2A$
	6th	$-2A$	$-2A$	$+4A$
	7th	$+5A$	$-3A$	$-2A$
	8th	$-2A$	$+5A$	$-3A$
	9th	$-3A$	$-3A$	$+6A$
	10th	$+7A$	$-4A$	$-3A$
	&c.	&c.	&c.	&c.

If we pay attention to this table in the least, & to the manner in which it is formed, we will see that in each column, the terms taken three by three, form an Arithmetic progression, for example in the first column, the terms $1.4.7.10$ &c. forms this sequence, $+A$, $+3A$, $+5A$, $+7A$ +&c.

There remains no more than to seek what is the probability that the game will end at the first match, at the second, at the third &c.

1°. In order that it finish at the first match, it is necessary that Pierre win it, & as it is equally probable that he will lose it, the probability that the game will end at the first match is $= \frac{1}{2}$.

2°. Thence the probability that Paul will play a second match is $= \frac{1}{2}$, which being multiplied by the probability that he will win it, & which is also $\frac{1}{2}$, gives the probability that the game will end at the second match, $= \frac{1}{4}$.

Likewise we will find that the probability that the game will end at the third match; at the fourth, fifth, &c. is $= \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, &c. So that we will deduce from it

$$\begin{aligned}
&\text{The advantage of Pi.} = A\left(\tfrac{1}{2} - \tfrac{0}{4} - \tfrac{1}{8} + \tfrac{3}{16} - \tfrac{1}{32} - \tfrac{2}{64} + \tfrac{5}{128} - \text{\&c.}\right)\\
&\text{The advantage of Pa.} = A\left(-\tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} - \tfrac{2}{16} + \tfrac{3}{32} - \tfrac{2}{64} - \tfrac{3}{128} + \text{\&c.}\right)\\
&\text{The advantage of Jaq.} = A\left(-\tfrac{0}{2} - \tfrac{1}{4} + \tfrac{2}{8} - \tfrac{1}{16} - \tfrac{2}{32} + \tfrac{4}{64} - \tfrac{2}{128} - \text{\&c.}\right)
\end{aligned}$$

Each of these series can, by virtue of the remark which I have made on the formation of the small table, be decomposed into three other series of which the Numerators will go in Arithmetic progression, & the Denominators in Geometric progression, &

which can consequently be summed. We will have in this fashion

$$\text{The advantage of Pi.} = \left\{\begin{array}{ll} +A\left(\frac{1}{2}+\frac{3}{2.8}+\frac{5}{2.8^2}+\frac{7}{2.8^3}+\&\text{c.}\right) & =+\frac{36}{49}A \\ -A\left(\frac{0}{4}+\frac{1}{4.8}+\frac{2}{4.8^2}+\frac{3}{4.8^3}+\&\text{c.}\right) & =-\frac{2}{49}A \\ -A\left(\frac{1}{8}+\frac{2}{8^2}+\frac{3}{8^3}+\frac{4}{8^4}+\&\text{c.}\right) & =-\frac{8}{49}A \end{array}\right\} = +\frac{26}{49}A.$$

$$\text{The advantage of Pa.} = \left\{\begin{array}{ll} -A\left(\frac{1}{2}+\frac{2}{2.8}+\frac{3}{2.8^2}+\frac{4}{2.8^3}+\&\text{c.}\right) & =-\frac{32}{49}A \\ +A\left(\frac{1}{4}+\frac{3}{4.8}+\frac{5}{4.8^2}+\frac{7}{4.8^3}+\&\text{c.}\right) & =+\frac{18}{49}A \\ -A\left(\frac{1}{8}+\frac{2}{8^2}+\frac{3}{8^3}+\frac{4}{8^4}+\&\text{c.}\right) & -\frac{8}{49}A \end{array}\right\} = -\frac{22}{49}A.$$

$$\text{The advantage of Jaq.} = \left\{\begin{array}{ll} -A\left(\frac{0}{2}+\frac{1}{2.8}+\frac{2}{2.8^2}+\frac{3}{2.8^3}+\&\text{c.}\right) & =-\frac{4}{49}A \\ -A\left(\frac{1}{4}+\frac{2}{4.8}+\frac{3}{4.8^2}+\frac{4}{4.8^3}+\&\text{c.}\right) & =-\frac{16}{49}A \\ +A\left(\frac{2}{8}+\frac{4}{8^2}+\frac{6}{8^3}+\frac{8}{8^4}+\&\text{c.}\right) & =+\frac{16}{49}A \end{array}\right\} = -\frac{4}{49}A.$$

§ 6. We are presently in a proper condition to resolve the Problem of the ordinary pool, enunciated in § 2. because by virtue of that which we just said in this same §. if we make use of the two preceding Problems, & if we maintain the same denominations, we will have the advantage of the one who continues $= \frac{4}{7}p - M + \frac{26}{49}A$; the advantage of the one who enters into the game $= \frac{2}{7}p - m - \frac{22}{49}A$; & the advantage of the one who exits from the game $= \frac{1}{7}p - \mu - \frac{4}{49}A$. There is no more concern but to substitute for p, M, m & μ, their values: The pool p is after n matches evidently equal to $\overline{n+1}.A$; In order to determine next the quantities M, m, & μ which each player has put into the game, it is necessary to distinguish three cases.

1°. If n is a number of the sequence 1.4.7.10 &c we will find $M = \mu = \frac{n+2}{3} \cdot A$; & $m = \frac{n-1}{3} \cdot A$.

2°. If n is a number of the sequence 2.5.8.11. &c. we will find $M = m = \mu = \frac{n+1}{3} \cdot A$.

3°. Finally if n is a number of the sequence 3.6.9.12. &c we have $M = \frac{n+3}{3} \cdot A$, & $m = \mu = \frac{n}{3} \cdot A$.

So that we will obtain the following formulas:

First Case where $\overline{n+2}$ is a multiple of 3.

The advantage of the one who continues $= \frac{4}{7} \cdot \overline{n+1} \cdot A - \frac{n+2}{3} \cdot A + \frac{26}{49}A = +\left(\frac{64+35n}{147}\right)A$
The advantage of the one who enters $= \frac{2}{7} \cdot \overline{n+1} \cdot A - \frac{n-1}{3} \cdot A - \frac{22}{49}A = -\left(\frac{7n-25}{147}\right)A$
The advantage of the one who exits $= \frac{1}{7} \cdot \overline{n+1} \cdot A - \frac{n+2}{3} \cdot A - \frac{4}{49}A = -\left(\frac{89+28n}{147}\right)A$

Second Case where $\overline{n+1}$ is a multiple of 3.

The advantage of the one who continues $= \frac{4}{7} \cdot \overline{n+1} \cdot A - \frac{n+1}{3} \cdot A + \frac{26}{49}A = +\left(\frac{113+35n}{147}\right) A$
The advantage of the one who enters $= \frac{2}{7} \cdot \overline{n+1} \cdot A - \frac{n+1}{3} \cdot A - \frac{22}{49}A = -\left(\frac{73+7n}{147}\right) A$
The advantage of the one who exits $= \frac{1}{7} \cdot \overline{n+1} \cdot A - \frac{n+1}{3} \cdot A - \frac{4}{49}A = -\left(\frac{40+28n}{147}\right) A$

Third Case where n is a multiple of 3.

The advantage of the one who continues $= \frac{4}{7} \cdot \overline{n+1} \cdot A - \frac{n+3}{3} \cdot A + \frac{26}{49}A = +\left(\frac{15+25n}{147}\right) A$
The advantage of the one who enters $= \frac{2}{7} \cdot \overline{n+1} \cdot A - \frac{n}{3} \cdot A - \frac{22}{49}A = -\left(\frac{20+7n}{147}\right) A$
The advantage of the one who exits $= \frac{1}{7} \cdot \overline{n+1} \cdot A - \frac{n}{3} \cdot A - \frac{4}{49}A = -\left(\frac{28n-9}{147}\right) A$

Coroll. 1.

§ 7. These formulas can serve for any value which we give to n except $n = 0$, because in their formation, we have supposed that after this number n of matches played, the game can end at the following match, now this is that which is not in our Problem, where it is necessary to have won two matches in sequence in order to take the pool; but the solution of this case will be deduced easily from our formulas, because each of those who must play the first match, will be found after this match, either in the case of the one who continues the game, & will have consequently the advantage $\frac{64+35n}{147}A = \frac{33}{49}A$, or in the case of the one who exits from the game & will have the advantage $-\left(\frac{89+28n}{147}\right) A = -\frac{39}{49}A$. Therefore the advantage of each of those who must begin will be $= \frac{33-39}{2.49}A = -\frac{3}{49}A$, & consequently the advantage of the one who enters into the game only at the second match must be $= +\frac{6}{49}A$ we can also find this last advantage, as we have done in §4. by making $n = 1$ in the formula $-\left(\frac{7n-25}{147}A\right)$ that which renders it equal to $\frac{18}{147}A$ or $\frac{6}{49}A$.

Coroll. 2.

§ 8. If $n = \infty$, the formulas of § 6. give the advantages of the three players, $+\frac{35nA}{147}$; $-\frac{7nA}{147}$; & $-\frac{28nA}{147}$; or $\frac{5}{21}nA$; $-\frac{1}{21}nA$; & $-\frac{4}{21}nA$. Now this is that which must effectively be so, because in this case, the pool is composed of nA, the stakes which are added again in the following matches must be counted null with respect to this infinite sum nA, it is therefore here the case where each player has put into the game $\frac{1}{3}n$, & where we add nothing more, it is this case which we have treated in § 3. where we have found the advantages of the three players $= \frac{5}{21}p$; $-\frac{1}{21}p$; & $-\frac{4}{21}p$; by understanding there by p the pool which is in this case here $= nA$.

Problem.

§ 9. *The same things being posed, excepted that instead of supposing each stake $= A$, we wish only the first three $= A$, & each of the following $= B$. To determine the advantage of each player after any number n of matches played.*

The same method as that of § 6. will give us the solution of this Problem. We will have the advantages of the players, solely as for future growth of the pool, without regard to the stakes which are already there, such as we have determined them in § 5 by putting only B in the place of A, namely $+\frac{26}{49}B$; $-\frac{22}{49}B$; & $-\frac{4}{49}B$. The pool p will be composed after n matches of $3A + \overline{n-2}.B$. And for the stakes M, m, & μ, we will have easily their values, by putting B in the place of A in those which we have found in § 6. & by adding $A - B$ to it; so that distinguishing three cases as above, we will have

1° If $\overline{n+2}$ is a multiple of 3.

The advantage of the one who continues$=\frac{4}{7}(3A + \overline{n-2} \cdot B) - \left(A + \frac{n-1}{3} \cdot B\right) + \frac{26}{49}B = \frac{35}{49}A + \left(\frac{35n-41}{147}B\right)$
The advantage of the one who enters $=\frac{2}{7}(3A + \overline{n-2} \cdot B) - \left(A + \frac{n-4}{3} \cdot B\right) - \frac{22}{49}B = \frac{-7}{49}A - \left(\frac{7n-46}{147}B\right)$
The advantage of the one who exits $=\frac{1}{7}(3A + \overline{n-2} \cdot B) - \left(A + \frac{n-1}{3} \cdot B\right) - \frac{4}{49}B = \frac{-28}{49}A - \left(\frac{28n+5}{147}B\right)$

2°. If $\overline{n+1}$ is a multiple of 3.

The advantage of the one who continues $=\frac{4}{7}(3A + \overline{n-2} \cdot B) - \left(A + \frac{n-2}{3}B\right) + \frac{26}{49}B = \frac{35}{49}A + \left(\frac{35n+8}{147}\right)B$
The advantage of the one who enters $=\frac{2}{7}(3A + \overline{n-2} \cdot B) - \left(A + \frac{n-2}{3} \cdot B\right) - \frac{22}{49}B = \frac{-7}{49}A - \left(\frac{7n+52}{147}\right)B$
The advantage of the one who exits $=\frac{1}{7}(3A + \overline{n-2} \cdot B) - \left(A + \frac{n-2}{3} \cdot B\right) - \frac{4}{49}B = \frac{-28}{49}A - \left(\frac{28n-44}{147}B\right)$

3°. Third Case where n is a multiple of 3.

The advantage of the one who continues $=\frac{4}{7}(3A + \overline{n-2} \cdot B) - \left(A + \frac{n}{3}B\right) + \frac{26}{49}B = \frac{35}{49}A + \left(\frac{35n-90}{147}\right)B$
The advantage of the one who enters $=\frac{2}{7}(3A + \overline{n-2} \cdot B) - \left(A + \frac{n-3}{3} \cdot B\right) - \frac{22}{49}B = \frac{-7}{49}A - \left(\frac{7n+3}{147}\right)B$
The advantage of the one who exits $=\frac{1}{7}(3A + \overline{n-2} \cdot B) - \left(A + \frac{n-3}{3} \cdot B\right) - \frac{4}{49}B = \frac{-28}{49}A - \left(\frac{28n-93}{147}B\right)$

Cor. 1.

§ 10. If $B = A$, these formulas will be changed into those of § 6.

Cor. 2.

§ 11. If $B = 0$, that is to say if we add nothing more to the first three stakes A, the advantages of the three players will become $\frac{35}{49}A$; $-\frac{7}{49}A$; & $-\frac{28}{49}A$; or $\frac{5}{7}A$; $-\frac{1}{7}A$; & $-\frac{4}{7}A$; as we have found in § 3. by putting $\frac{1}{3}p$ in place of A.

Cor. 3.

§ 12. These formulas are seen as there is a case where the one who comes to win the preceding match can however have some disadvantage, that is the case where $n = 1$, his advantage is $\frac{35}{49}A - \frac{2}{49}B$ which will become negative, everytime that we take B greater than $\frac{35}{2}A$, or than $17\frac{1}{2}A$, & which will be null if $B = 17\frac{1}{2}A$. In this last case the one who enters into the game, will have the advantage $\frac{9}{2}A$ or $4\frac{1}{2}A$, & the one who exits the same disadvantage.

Cor. 4

§ 13. We see also that there is a case where the one who exits from the game, can however have a positive advantage, but which will always be smaller than the advantage of the one who continues, that is the case where $n = 3$; the advantage of the one who exits is $-\frac{28}{49}A + \frac{3}{49}B$ which will become null if $B = \frac{28}{3}A = 9\frac{1}{3}A$, & which will be positive everytime that B will be greater than $9\frac{1}{3}A$.

Cor. 5.

§ 14. The case where $n = 0$, will be determined as § 7. we will find the advantage of those who must begin $= \frac{7}{2.49}A - \frac{13}{2.49}B$ & consequently the one of the 3rd will be $= \frac{13}{49}B - \frac{7}{49}A$. Whence it follows that

If $B = 0$, these advantages become $\frac{1}{14}A$ & $-\frac{2}{14}A$; as we have found in § 4.

If $B = \frac{7}{13}A$, the quantity $\frac{13}{49}B - \frac{7}{49}A$ vanishes; in this case it is indifferent to the players to begin, or to not begin the game, & the one who enters only at the second match, has no advantage over the others, he can even have a disadvantage, namely if we make B smaller than $\frac{7}{13}A$, if for example $B = \frac{6}{13}A$, this disadvantage will be $\frac{1}{49}A$.

In general if $B = fA$, the advantage of the one who does not begin the game is $= \frac{13f-7}{49}A$ that is to say as much greater as B is greater; if we wish that this advantage be $= mA$, it will be necessary to take $B = \frac{49m+7}{13}A$.

Problem.

§ 15. *The first three stakes being $= A$, & the following forming an Arithmetic Progression, namely the $4^{th} = A + B$; the $5^{th} = A + 2B$; &c. To determine the advantage of each player after any number n of matches.*

The pool after n matches will be composed of $\overline{n+1} \cdot A + B$ multiplied by the n^{th} term of the sequence $0, 0, 1, 3, 6, 10$, &c. that is to say $= \overline{n+1} \cdot A + \frac{\overline{n-1} \cdot \overline{n-2}}{2}B$.

The values of M, m, & μ are not difficult to detemine; we seek for example the value of M in the case where n is a number of the sequence $1,\ 4,\ 7,\ 10$, &c.

It is clear that the one who will enter into the game there, & will put a stake into the 1st match, into the 4^{th}; 7^{th}; 10^{th} &c. putting successively A, then $A + 5B$, next $A + 8B$, &c. So that the sum of the stakes will be:

After 1 match	$= A + 0.B$	Whence we see that after a number of matches
4 matches	$= 2A + 2.B$	expressed by the x^{th} term of the sequence 1,
7	$= 3A + 7.B$	$4, 7,\ 10$, &c. the sum of the stakes will be
10	$= 4A + 15.B$	$= xA + B$ multiplied by the x^{th} term of the
15	$= 5A + 26.B$	sequence $0,\ 2,\ 7,\ 15,\ 26$, &c. which term is

found easily by the ordinary methods, $= \frac{3xx-5x+2}{2}$; M or the sum of the stakes will be therefore $= xA + \frac{3xx-5x+2}{2}B$.

Now we know that the x^{th} term of the sequence $1,\ 4,\ 7,\ 10$, &c. is $= 3x-2$, which we suppose $= n$, that which gives $x = \frac{n+2}{3}$, & substituting this value of x, we will have the sum of the stakes or $M = \frac{n+2}{3}A + \frac{n.\overline{n-1}}{2.3}B$.

We will find by the same method this sum of stakes for all the other cases, & we will have

If n is a number of the sequence $1,\ 4,\ 7,\ 10$ &c: $M = \frac{n+2}{3}A + \frac{n.\overline{n-1}}{6}B$; $m = \frac{n-1}{3}A + \frac{\overline{n-1}.\overline{n-4}}{6}B$; & $\mu = \frac{n+2}{3}A + \frac{\overline{n-1}.\overline{n-2}}{6}B$.

If n is a number of the sequence $2,\ 5,\ 8$ &c: $M = \frac{n+1}{3}A + \frac{\overline{n+1}.\overline{n-2}}{6}B$; $m = \frac{n+1}{3}A + \frac{\overline{n-2}.\overline{n-3}}{6}B$; & $\mu = \frac{n+1}{3}A + \frac{\overline{n-1}.\overline{n-2}}{6}B$.

If n is a number of the sequence $2,\ 6,\ 9,\ 12$ &c: $M = \frac{n+3}{3}A + \frac{n.\overline{n-1}}{6}B$; $m = \frac{n}{3}A + \frac{\overline{n-2}.\overline{n-3}}{6}B$; & $\mu = \frac{n}{3}A + \frac{n.\overline{n-3}}{6}B$.

Having thus the value of the pool, & the sum of the stakes of each player, we will know by § 3. their advantages relatively to this value of the pool, & there remains no more but to seek their advantages for the future pool, we will construct for this purpose, as in § 5. the following Table, where Pierre is the one who continues the game, Paul is the one who enters, & who puts a stake $A+\overline{n-1}.B$, & Jacques will enter only next, in putting the stake $A+nB$.

If the game ends	Advant. of Pierre	Advant. of Paul	Advant. of Jacques	Probabilities that the game will end
at the 1st match	$+(A+\overline{n-1}.B)$	$-(A+\overline{n-1}.B)$	0	$\frac{1}{2}$
$2nd$	-0	$+(A+nB)$	$-(A+nB)$	$\frac{1}{4}$
$3rd$	$-(A+\overline{n+1}.B)$	$-(A+\overline{n-1}.B)$	$+(2A+2n.B)$	$\frac{1}{8}$
$4th$	$+(3A+\overline{3n-1}.B)$	$+(2A+\overline{2n+1}.B)$	$-(A+nB)$	$\frac{1}{16}$
$5th$	$-(A+\overline{n+1}.B)$	$+(3A+\overline{3n+4}.B)$	$-(2A+\overline{2n+3}.B)$	$\frac{1}{32}$
$6th$	$-(2A+\overline{2n+5}.B)$	$-(2A+\overline{2n+1}.B)$	$-(4A+\overline{4n+6}.B)$	$\frac{1}{64}$
$7th$	$+(5A+\overline{5n+9}.B)$	$-(3A+\overline{3n+6}.B)$	$-(2A+\overline{2n+3}.B)$	$\frac{1}{128}$
$8th$	$-(2A+\overline{2n+5}.B)$	$+(5A+\overline{5n+14}.B)$	$-(3A+\overline{3n+9}.B)$	$\frac{1}{256}$
$9th$	$-(2A+\overline{3n+12}.B)$	$-(3A+\overline{3n+6}.B)$	$+(6A+\overline{6n+18}.B)$	$\frac{1}{512}$
$10th$	$+(7A+\overline{7n+23}.B)$	$-(4A+\overline{4n+14}.B)$	$-(3A+\overline{3n+9}.B)$	$\frac{1}{1024}$
&c.	&c.	&c.	&c.	&c.

We will deduce from there

$$\text{The advant. of Pi.}\left\{\begin{aligned} &+(A+nB)\left(\frac{1}{2}+\frac{3}{2.8}+\frac{5}{2.8^2}+\frac{7}{2.8^3}+\text{\&c.}\right)=\frac{36}{49}(A+nB)\\ &\quad+B\left(-\frac{1}{2}+\frac{1}{2.8}+\frac{9}{2.8^2}+\frac{23}{2.8^3}+\text{\&c.}\right)=\frac{116}{7.49}B\\ &\quad-(A+nB)\left(\frac{0}{4}+\frac{1}{4.8}+\frac{2}{4.8^2}+\frac{3}{4.8^3}+\text{\&c.}\right)=\frac{2}{49}(A+nB)\\ &\quad-B\left(\frac{0}{4}+\frac{1}{4.8}+\frac{5}{4.8^2}+\frac{12}{4.8^3}+\text{\&c.}\right)=\frac{20}{7.49}B\\ &\quad-(A+nB)\left(\frac{1}{8}+\frac{2}{8^2}+\frac{3}{8^3}+\frac{4}{8^4}+\text{\&c.}\right)=\frac{8}{89}(A+nB)\\ &\quad-B\left(\frac{1}{8}+\frac{5}{8^2}+\frac{12}{8^3}+\frac{22}{8^4}+\text{\&c.}\right)=\frac{80}{7.49}B \end{aligned}\right\}=\begin{aligned}&\frac{26}{49}A\\ +&\frac{26}{49}nB\\ -&\frac{216}{7.49}B\end{aligned}$$

$$\text{The advant. of Pa.}\left\{\begin{aligned}
&+(A+nB)\left(\frac{1}{4}+\frac{3}{4.8}+\frac{5}{4.8^2}+\frac{7}{4.8^3}+\text{\&c.}\right)=\frac{18}{49}(A+nB)\\
&\quad+B\left(\frac{0}{4}+\frac{4}{4.8}+\frac{14}{4.8^2}+\frac{30}{4.8^3}+\text{\&c.}\right)=\frac{68}{7.49}B\\
&\quad-(A+nB)\left(\frac{1}{8}+\frac{2}{8^2}+\frac{3}{8^3}+\frac{4}{8^4}+\text{\&c.}\right)=-\frac{8}{49}(A+nB)\\
&\quad-B\left(\frac{1}{8}+\frac{1}{8^2}+\frac{6}{8^3}+\frac{14}{8^4}+\text{\&c.}\right)=+\frac{32}{7.49}B\\
&\quad-(A+nB)\left(\frac{1}{2}+\frac{2}{2.8}+\frac{3}{2.8^2}+\frac{4}{2.8^3}+\text{\&c.}\right)=-\frac{32}{49}(A+nB)\\
&\quad-B\left(-\frac{1}{2}+\frac{1}{2.8}+\frac{6}{2.8^2}+\frac{14}{2.8^3}+\text{\&c.}\right)=+\frac{128}{7.49}B
\end{aligned}\right\}=\begin{aligned}&-\frac{22}{49}A\\&-\frac{22}{49}nB\\&+\frac{228}{7.49}B\end{aligned}$$

$$\text{The advant. of Jaq.}\left\{\begin{aligned}
&+(A+nB)\left(\frac{2}{8}+\frac{4}{8^2}+\frac{6}{8^3}+\frac{8}{8^4}+\text{\&c.}\right)=\frac{16}{49}(A+nB)\\
&\quad+B\left(\frac{0}{8}+\frac{6}{8^2}+\frac{18}{8^3}+\frac{36}{8^4}+\text{\&c.}\right)=\frac{48}{7.49}B\\
&\quad-(A+nB)\left(\frac{1}{2.8}+\frac{2}{2.8^2}+\frac{3}{2.8^3}+\frac{4}{2.8^4}+\text{\&c.}\right)=-\frac{4}{49}(A+nB)\\
&\quad-B\left(\frac{0}{2.8}+\frac{3}{2.8^2}+\frac{9}{2.8^2}+\frac{18}{2.8^3}+\text{\&c.}\right)=\frac{12}{7.49}B\\
&\quad-(A+nB)\left(\frac{1}{4}+\frac{2}{4.8}+\frac{3}{4.8^2}+\frac{4}{4.8^3}+\text{\&c.}\right)=-\frac{16}{49}(A+nB)\\
&\quad-B\left(\frac{0}{4}+\frac{3}{4.8}+\frac{9}{4.8^2}+\frac{18}{4.8^3}+\text{\&c.}\right)=-\frac{48}{7.49}B
\end{aligned}\right\}=\begin{aligned}&\frac{4}{49}A\\&-\frac{4}{49}nB\\&-\frac{12}{7.49}B.\end{aligned}$$

By making now usage of all that which we just found, we will determine as in §§ 6 & 9. the advantages demanded in this Problem, & we will have

11°. If $n+2$ is a multiple of 3.

The advantage of the one who continues $=\frac{64+35n}{3.7^2}A+\left(\frac{245nn-329n-120}{6.7^3}\right)B$
The advantage of the one who enters $=\frac{25-7n}{3.7^2}A+\left(\frac{-49nn-91n+584}{6.7^3}\right)B$
The advantage of the one who exits $=\frac{-89-28n}{3.7^2}A+\left(\frac{-196nn+420n-464}{6.7^3}\right)B$

2°. If $n+1$ is a multiple of 3.

The advantage of the one who continues $=\frac{113+35n}{3.7^2}A+\left(\frac{245nn-329n+566}{6.7^3}\right)B$
The advantage of the one who enters $=\frac{-73-7n}{3.7^2}A+\left(\frac{-49nn-91n-102}{6.7^3}\right)B$
The advantage of the one who exits $=\frac{-40-28n}{3.7^2}A+\left(\frac{-196nn+420n-464}{6.7^3}\right)B$

3°. If n is a multiple of 3.

The advantage of the one who continues $=\frac{15+35n}{3.7^2}A+\left(\frac{245nn-329n-120}{6.7^3}\right)B$
The advantage of the one who enters $=\frac{-24-7n}{3.7^2}A+\left(\frac{-49nn-91n-102}{6.7^3}\right)B$
The advantage of the one who exits $=\frac{9-28n}{3.7^2}A+\left(\frac{-196nn+420n+222}{6.7^3}\right)B$

Coroll. 1.

§ 16. If $B = 0$, this is then the case of § 6 & our formulas are changed into those which we have found in this said §.

Cor. 2.

§ 17. If $B = A$, these advantages become

For the 1th case

$$\frac{245nn+161n+776}{6.7^3}A; \quad \frac{-49nn-189n+934}{6.7^3}A; \quad \frac{-196nn+28n-1710}{6.7^3}A;$$

For the 2nd case

$$\frac{245nn+161n+2148}{6.7^3}A; \quad \frac{-49nn-189n-1124}{6.7^3}A; \quad \frac{-196nn+28n-1024}{6.7^3}A;$$

For the 3rd case

$$\frac{245nn+161n+90}{6.7^3}A; \quad \frac{-49nn-189n-438}{6.7^3}A; \quad \frac{-196nn+28n+348}{6.7^3}A;$$

Cor. 3.

§ 18. If $n = \infty$, we obtain the three formulas $\frac{245nn}{6.7^3}B$; $\frac{-49nn}{6.7^3}B$; & $\frac{-196nn}{6.7^3}B$ or $\frac{5}{42}nnB$; $-\frac{1}{42}nnB$; & $\frac{-4}{42}nnB$, as in § 3. by putting $p = \frac{1}{2}nnB$.

Cor. 4.

§ 19. The advantage from the beginning for the one who enters into the game only at the second match, will always be positive, whatever positive value which we give to B, namely $= \frac{6}{49}A + \frac{74}{7^3}B$, that which will become $= \frac{116}{7^3}A$, in the case of $B = A$.

Problem.

§ 20. *The first three stakes being A, & the following forming any Geometric Progression, bA, b^2A, b^3A, &c. To determine the advantage to each player after any number n of matches.*

This Problem will be resolved by the same method as above, & consequently having no difficulty but the length of the calculation, I will content myself to be neither annoying nor too long, to give here the results of the calculation, & I will add the formulas for the case $b = 2$.

1°. If $n + 2$ is multiple of 3.

The advantage of the one		If $b = 2$.
who continues	$=\left(\frac{5}{7} + \frac{4}{7}\cdot\frac{b-b^{n-1}}{1-b} + \frac{b^{n+1}-bb}{1-b^3} + \frac{32b^{n-1}+4b^n-10b^{n+1}}{7.(8-b^3)}\right)A=$	$\left(\frac{2+3.2^n}{14}\right)A$
who enters	$=\left(-\frac{1}{7} + \frac{2}{7}\cdot\frac{b-b^{n-1}}{1-b} + \frac{b^{n-1}-b^3}{1-b^3} + \frac{16b^n+2b^{n+1}-40b^{n-1}}{7.(8-b^3)}\right)A=$	$\left(\frac{6-2^n}{14}\right)A$
who exits	$=\left(-\frac{4}{7} + \frac{1}{7}\cdot\frac{b-b^{n-1}}{1-b} + \frac{b^n-b}{1-b^3} + \frac{8b^{n-1}+8b^{n+1}-20b^n}{7.(8-b^3)}\right)A=$	$\left(\frac{-2.2^n-8}{14}\right)A$

2°. If $n+1$ is multiple of 3.

The advantage of the one

$$\begin{array}{ll|l}
\text{who continues} = & \left(\frac{5}{7} + \frac{4}{7}\cdot\frac{b-b^{n-1}}{1-b} + \frac{b^{n+1}-b^3}{1-b^3} + \frac{32b^{n-1}+4b^n-10b^{n+1}}{7.(8-b^3)}\right)A = & \left(\frac{10+3.2^n}{14}\right)A \\
\text{who enters} = & \left(-\frac{1}{7} + \frac{2}{7}\cdot\frac{b-b^{n-1}}{1-b} + \frac{b^{n-1}-b}{1-b^3} + \frac{16b^n+2b^{n+1}-40b^{n-1}}{7.(8-b^3)}\right)A = & \left(\frac{-2^n-6}{14}\right)A \\
\text{who exits} = & \left(-\frac{4}{7} + \frac{1}{7}\cdot\frac{b-b^{n-1}}{1-b} + \frac{b^n-bb}{1-b^3} + \frac{8b^{n-1}+8b^{n+1}-20b^n}{7.(8-b^3)}\right)A = & \left(\frac{-2.2^n-4}{14}\right)A
\end{array}$$

3°. If n is multiple of 3.

The advantage of the one

$$\begin{array}{ll|l}
\text{who continues} = & \left(\frac{5}{7} + \frac{4}{7}\cdot\frac{b-b^{n-1}}{1-b} + \frac{b^{n+1}-b}{1-b^3} + \frac{32b^{n-1}+4b^n-10b^{n+1}}{7.(8-b^3)}\right)A = & \left(\frac{3.2^n-2}{14}\right)A \\
\text{who enters} = & \left(-\frac{1}{7} + \frac{2}{7}\cdot\frac{b-b^{n-1}}{1-b} + \frac{b^{n-1}-bb}{1-b^3} + \frac{16b^n+2b^{n+1}-40b^{n-1}}{7.(8-b^3)}\right)A = & \left(\frac{-2^n-2}{14}\right)A \\
\text{who exits} = & \left(-\frac{4}{7} + \frac{1}{7}\cdot\frac{b-b^{n-1}}{1-b} + \frac{b^n-b^3}{1-b^3} + \frac{8b^{n-1}+8b^{n+1}-20b^n}{7.(8-b^3)}\right)A = & \left(\frac{4-2.2^n}{14}\right)A
\end{array}$$

Corollary.

§ 21. The advantage of the one who enters into the game only at the second match, will be found by putting $n = 1$ into the 2nd formula, that which will give this advantage $= \frac{16b+2bb-4b^3-8}{7.(8-b^3)}A$ which vanishes in one case alone, namely when $b = \frac{1}{2}$, it will always be negative if $b < \frac{1}{2}$, & always positive if $b > \frac{1}{4}$. If the stakes follow the law of the double geometric progression 2, 4, 8 &c. that is to say if we make $b = 2$, the quantity $\frac{16b+2bb-4b^3-8}{7.(8-b^3)}A$ becomes $\frac{0}{0}$, but by applying here the rule of Mr. Bernoulli for such quantities, we will obtain this advantage $= \frac{2}{7}A$.

§ 22. Let in general the first stake be $= a$, the 2nd $= b$, the 3rd $= c$, &c. so that the sequence which form the stakes is:

	1st stake,	2nd,	3rd,	4th,	5th,	6th,	7th,	8th,	9th,	&c.	$\ldots$	
	a,	b,	c,	d,	e,	f,	g,	h,	i,	&c.	$\ldots$	
$\overline{n-1}^{\text{st}}$,	n^{th},	$\overline{n+1}^{\text{st}}$,	$\overline{n+2}^{\text{nd}}$,	$\overline{n+3}^{\text{rd}}$,	$\overline{n+4}^{\text{th}}$,	$\overline{n+5}^{\text{th}}$,	$\overline{n+6}^{\text{th}}$,	$\overline{n+7}^{\text{th}}$,	$\overline{n+8}^{\text{th}}$,	$\overline{n+9}^{\text{th}}$,	$\overline{n+10}^{\text{th}}$,	&c.
k,	l,	m,	p,	q,	r,	s,	t,	u,	z, y,	x,	&c.	

Let P be the pool which after n matches will be equal to the sum of the first $n+1$ terms of this sequence, & M, m & μ, the sums of the stakes of each player, who will have for the three cases distinguished above the following values:

1°.If $n+2$ is multiple of 3.	2°.If $n+1$ is multiple of 3.	3°.If n is multiple of 3.
$M = (a+e+h+\text{\&c}\ldots+m)$	$= (c+f+i+\text{\&c}\ldots+m)$	$= (b+d+g+\text{\&c}\ldots+m)$
$m = (c+f+i+\text{\&c}\ldots+k)$	$= (b+d+g+\text{\&c}\ldots+k)$	$= (a+e+h+\text{\&c}\ldots+k)$
$\mu = (b+d+g+\text{\&c}\ldots+l)$	$= (a+e+h+\text{\&c}\ldots+l)$	$= (c+f+i+\text{\&c}\ldots+l)$

We will have by our method

$$\text{The advantage of the one who continues} = \tfrac{4}{7}P - M + \tfrac{4}{7}\left\{\begin{matrix} p + \dfrac{1}{8}s + \dfrac{1}{8^2}z + \&\text{c.} \\ \dfrac{1}{8}q + \dfrac{1}{8^2}t + \dfrac{1}{8^3}y + \&\text{c.} \end{matrix}\right\} - \tfrac{5}{4.7}\left(r + \tfrac{1}{8}u + \tfrac{1}{8^2}x + \&\text{c.}\right)$$

$$\text{The advantage of the one who enters} = \tfrac{2}{7}P - m + \tfrac{2}{7}\left\{\begin{matrix} q + \dfrac{1}{8}t + \dfrac{1}{8^2}y + \&\text{c.} \\ \dfrac{1}{8}r + \dfrac{1}{8^2}u + \dfrac{1}{8^3}x + \&\text{c.} \end{matrix}\right\} - \tfrac{5}{7}\left(p + \tfrac{1}{8}s + \tfrac{1}{8^2}z + \&\text{c.}\right)$$

$$\text{The advantage of the one who exits} = \tfrac{1}{7}P - \mu + \tfrac{1}{7}\left\{\begin{matrix} p + \dfrac{1}{8}s + \dfrac{1}{8^2}z + \&\text{c.} \\ r + \dfrac{1}{8}u + \dfrac{1}{8^2}x + \&\text{c.} \end{matrix}\right\} - \tfrac{5}{2.7}\left(q + \tfrac{1}{8^2}y + \&\text{c.}\right)$$

§ 23. The advantage since the beginning for the one who must enter only at the second match will be generally

$$\frac{2}{7}(a+b) + \frac{2}{7}\left\{\begin{matrix} d + \dfrac{1}{8}g + \&\text{c.} \\ \dfrac{1}{8}e + \dfrac{1}{8^2}h + \&\text{c.} \end{matrix}\right\} - \frac{5}{7}(c + \frac{1}{8}f + \frac{1}{8^2}i + \&\text{c.})$$