Notes to accompany *Esame Critico* of Malfatti

Richard Pulskamp

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These notes show how to replicate Malfatti's numerical results using more modern techniques. They do not purport to produce the recurrence relations developed by him.

1 Evolution of the process

We imagine two urns A and B. Urn A originally contains n white balls and urn B contains n black. A ball is selected at random from each urn and the extracted balls are then returned respectively into the other urn. This is the *permutation* of Malfatti.

Malfatti first provides the following lemma:

Lemma 1.1 *Suppose urn A contains $n-p$ white balls and p black so that urn B contains p white balls and $n-p$ black. A ball is extracted at random from each urn and a permutation made. The number of cases for each of the events of white balls is urn A are:*

White balls in A	*Cases*
$n-p+1$	p^2
$n-p$	$2p(n-p)$
$n-p-1$	$(n-p)^2$

Proof This is straightforward. We illustrate the first case. In order to increase the number of white balls in urn A by 1, it is necessary that a black be extracted from urn A and a white from urn B. There are p ways to do the former and p ways to do the latter, in all p^2 ways.

Urn A has $n+1$ possible states (the number of white balls contained in it) which we will denote by $W=i$ for $i=0,1,\dots,n$. In addition, because the first permutation always puts urn A into the state $W=n-1$, Malfatti takes this state as the original state.

Suppose that $n=4$. In the array below, the first column represents the number of white balls in urn A before a permutation, the first row the number of white balls after 1 permutation. The remaining cells hold the number of cases as indicated by the Lemma. So, for example, urn A will transition from the state $W=3$ to the state $W=2$ in 9 ways. In general, cell (i,j) is the number of cases for the permutation $[W=i, W=j]$. Of course, $j=i-1, i, i+1$ only.

	After 1 permutation				
Before	4	3	2	1	0
$W=4$	0	16	0	0	0
3	1	6	9	0	0
2	0	4	8	4	0
1	0	0	9	6	1
0	0	0	0	16	0

Let the matrix A_4 be the matrix of cases. That is,

$$A_4 = \begin{bmatrix} 0 & 16 & 0 & 0 & 0 \\ 1 & 6 & 9 & 0 & 0 \\ 0 & 4 & 8 & 4 & 0 \\ 0 & 0 & 9 & 6 & 1 \\ 0 & 0 & 0 & 16 & 0 \end{bmatrix}$$

The successive powers of A_4 will provide the number of ways that the system of urns can evolve. That is, the cells of A_4^k will contain the events $[W=i, W=j]$ after k permutations.

Computing then

$$A_4^2 = \begin{bmatrix} 16 & 96 & 144 & 0 & 0 \\ 6 & 88 & 126 & 36 & 0 \\ 4 & 56 & 136 & 56 & 4 \\ 0 & 36 & 126 & 88 & 6 \\ 0 & 0 & 144 & 96 & 16 \end{bmatrix} \qquad A_4^3 = \begin{bmatrix} 96 & 1408 & 2016 & 576 & 0 \\ 88 & 1128 & 2124 & 720 & 36 \\ 56 & 944 & 2096 & 944 & 56 \\ 36 & 720 & 2124 & 1128 & 88 \\ 0 & 576 & 2016 & 1408 & 96 \end{bmatrix}$$

In §27, Malfatti discusses the event $[W=n-1, W=n-2]$ for various numbers of permutations. Those combinations which yield the event are called favorable. Otherwise the cases are contrary. Referring to the matrix A_4, we find (from row 2) $1+6=7$ contrary cases and 9 favorable. In matrix A_4^2, $6+88+36=130$ contrary and 126 favorable. Finally, in A_4^3, $88+1128+720+36=1972$ contrary and 2124 favorable.

2 Problems I–V

In this sequence of problems, Malfatti seeks the number of cases of the event $[W=n-1, W=n-2]$ at least once in k permutations, $k=1\ldots5$. To this end, it suffices to consider the state $W=n-2$ as an absorbing state.

Let M_n be the corresponding matrix of cases for 1 permutation when there are n balls in A. Three rows suffice since the only possible states are $\{n, n-1, n-2\}$. We have

$$M_2 = \begin{bmatrix} 0 & 4 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 0 & 9 & 0 \\ 1 & 4 & 4 \\ 0 & 0 & 9 \end{bmatrix} \qquad M_4 = \begin{bmatrix} 0 & 16 & 0 \\ 1 & 6 & 9 \\ 0 & 0 & 16 \end{bmatrix}$$

Refer now to §38. When $n = 2$, we have

$$M_2^2 = \begin{bmatrix} 4 & 8 & 4 \\ 2 & 8 & 6 \\ 0 & 0 & 16 \end{bmatrix} \qquad M_2^3 = \begin{bmatrix} 8 & 32 & 24 \\ 8 & 24 & 32 \\ 0 & 0 & 64 \end{bmatrix} \qquad M_2^4 = \begin{bmatrix} 32 & 96 & 128 \\ 24 & 80 & 152 \\ 0 & 0 & 256 \end{bmatrix}$$

The combinations contrary to the event $[W = 1, W = 0]$ in 1, 2, 3 and 4 permutations are $1+2 = 3$, $2+8 = 10$, $8+24 = 32$, and $24+80 = 104$ while the favorable are 1, 6, 32, 152 respectively. Note that one can wager evenly to observe $[W = 1, W = 0]$ at least once in 3 permutations.

When $n = 3$,

$$M_3^2 = \begin{bmatrix} 9 & 36 & 36 \\ 4 & 25 & 52 \\ 0 & 0 & 81 \end{bmatrix} \qquad M_3^3 = \begin{bmatrix} 36 & 225 & 468 \\ 25 & 136 & 568 \\ 0 & 0 & 729 \end{bmatrix} \qquad M_3^4 = \begin{bmatrix} 225 & 1224 & 5112 \\ 136 & 769 & 5656 \\ 0 & 0 & 6561 \end{bmatrix}$$

The combinations contrary to the event $[W = 2, W = 1]$ in 1, 2, 3, and 4 permutations are $1+4 = 5$, $4+25 = 29$, $25+136 = 161$, and $136+769 = 905$ while the favorable are 4, 52, 568 and 5656 respectively.

3 Problems VI–IX

In this sequence of problems, Malfatti seeks the number of cases of the event $[W = n-1, W = n-3]$ at least once in k permutations, $k = 1 \ldots 4$. To this end, it suffices to consider the state $W = n-3$ as an absorbing state.

Let M'_n denote the matrix of cases for n balls and for which the element $(n-1, n-3)$ represents an absorbing state.

When $n = 3$ and $n = 4$, we have:

$$M'_3 = \begin{bmatrix} 0 & 9 & 0 & 0 \\ 1 & 4 & 4 & 0 \\ 0 & 4 & 4 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix} \qquad M'_4 = \begin{bmatrix} 0 & 16 & 0 & 0 \\ 1 & 6 & 9 & 0 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 0 & 16 \end{bmatrix}$$

Consider first the powers of M'_3

$$M'^2_3 = \begin{bmatrix} 9 & 36 & 36 & 0 \\ 4 & 41 & 32 & 4 \\ 4 & 32 & 32 & 13 \\ 0 & 0 & 0 & 18 \end{bmatrix} \qquad M'^3_3 = \begin{bmatrix} 36 & 369 & 288 & 36 \\ 41 & 328 & 292 & 68 \\ 32 & 292 & 256 & 149 \\ 0 & 0 & 0 & 729 \end{bmatrix}$$

$$M'^4_3 = \begin{bmatrix} 369 & 2952 & 2628 & 612 \\ 328 & 2849 & 2480 & 904 \\ 292 & 2480 & 2192 & 1597 \\ 0 & 0 & 0 & 6561 \end{bmatrix}$$

and the first powers of M'_4

$$M'^2_4 = \begin{bmatrix} 16 & 96 & 144 & 0 \\ 6 & 88 & 126 & 36 \\ 4 & 56 & 100 & 96 \\ 0 & 0 & 0 & 256 \end{bmatrix} \qquad M'^3_4 = \begin{bmatrix} 96 & 1408 & 2016 & 576 \\ 88 & 1128 & 1800 & 1080 \\ 56 & 800 & 1304 & 1936 \\ 0 & 0 & 0 & 4096 \end{bmatrix}$$

$$M'^4_4 = \begin{bmatrix} 1408 & 18048 & 28800 & 17280 \\ 1128 & 15376 & 24552 & 24480 \\ 800 & 10912 & 17632 & 36192 \\ 0 & 0 & 0 & 65536 \end{bmatrix}$$

Now Malfatti produces formulas to compute directly as a function of n the number of contrary combinations for the first four permutations . These are summarized in §43. It is obvious that the first entry must be n^2.

Permutations	Formula	$n = 3$	$n = 4$
1	n^2	9	16
2	$6n^3 - 13n^2 + 12n - 4$	$77 = 3^4 - 4$	$220 = 4^4 - 36$
3	$33n^4 - 126n^3 + 198n^2 - 144n + 40$	$661 = 3^6 - 68$	$3016 = 4^6 - 1080$
4	$176n^5 - 943n^4 + 2132n^3 - 2484n^2 + 1472n - 352$	$5657 = 3^8 - 904$	$41056 = 4^8 - 24480$

4 Problems X–XVII

These will be dispensed with rather quickly. Problems X, XI and XII consider the events $[W = n-1, W = n-4]$, $[W = n-1, W = n-5]$ and $[W = n-1, W = n-z-1]$ occuring at least once for any number of permutations. Of course, these may all be handled by making $n-z-1$ an absorbing state as we have previously. Problems XIII–XVIII refine formulas.

5 Problems XIX and the Last

Problem XIX considers the event $[W = n-1, W = n-1]$ occurring at least once in m permutations. Naturally, one would make the state $W = n-1$ an absorbing state but only after the first permutation.

Consider the case of $n = 3$. Let

$$M''^k_3 = \begin{bmatrix} 0 & 9 & 0 & 0 \\ 1 & 4 & 4 & 0 \\ 0 & 4 & 4 & 1 \\ 0 & 0 & 9 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 9 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 4 & 4 & 1 \\ 0 & 0 & 9 & 0 \end{bmatrix}^{k-1}$$

Consequently,

$$M''^2_3 = \begin{bmatrix} 0 & 81 & 0 & 0 \\ 0 & 61 & 16 & 4 \\ 0 & 52 & 25 & 4 \\ 0 & 36 & 36 & 9 \end{bmatrix} \qquad M''^3_3 = \begin{bmatrix} 0 & 729 & 0 & 0 \\ 0 & 613 & 100 & 16 \\ 0 & 568 & 136 & 25 \\ 0 & 468 & 225 & 36 \end{bmatrix}$$

It is easy to confirm the formulas of Malfatti for small values of n and k:

Permutations	Formula	$n=3$	$n=4$
1	$1+(n-1)^2$	5	10
2	$(n-1)^2(n-2)(n+2)$	20	108
3	$(n-1)^2(n-2)^2(n^2+4n+8)$	116	1440
4	$(n-1)^2(n-2)^2(n^4+4n^2+8n^2-100)$	644	19440

The Last Problem investigates the return to the original state given that there are $n-1$ white balls in A, that is, the event $[W=n-1, W=n]$ in any number of permutations. To this end, the state $W=n$ is an absorbing state.

If $n=2$, then the matrix of events is

$$M''_2 = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \quad M''^2_2 = \begin{bmatrix} 16 & 0 & 0 \\ 6 & 8 & 2 \\ 4 & 8 & 4 \end{bmatrix} \quad M''^3_2 = \begin{bmatrix} 64 & 0 & 0 \\ 32 & 24 & 8 \\ 24 & 32 & 8 \end{bmatrix}$$

$$M''^4_2 = \begin{bmatrix} 256 & 0 & 0 \\ 152 & 80 & 24 \\ 128 & 96 & 32 \end{bmatrix}$$

Thus it is clear that there are 1, 6, 32 and 152 favorable cases as Malfatti claims.

If $n=3$, the matrix of cases becomes

$$M''_3 = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 1 & 4 & 4 & 0 \\ 0 & 4 & 4 & 1 \\ 0 & 0 & 9 & 0 \end{bmatrix}$$

It is easy to confirm the values given by Malfatti and to verify that with fewer than 13 permutations, the number of favorable cases is less than the number of contrary, but with 13 permutations the number of favorable exceeds those of the contrary.

6 Observations

Malfatti has two concerns in his paper:

- What is the probability that the number of white balls in Urn A assume a certain value at least once within a fixed number of permutations?
- How long will it take for the number of white balls to be half of the original number?

From the previous it is clear that three variables are of interest. These are the total number of balls in an urn n, the number of white balls W in Urn A, and the number of permutations k. Malfatti defines certain events for which he determines not the probability but rather the number of cases contrary or favorable to it.

The general problem is this: *Given $W=n-1$ as the original state, to find the number of contrary combinations to have at least once $W=n-z-1$ white balls in Urn A in k permutations.* We have denoted this event as $[W=n-1, W=n-z-1]$ in k permutations for given n. The second item listed above is the special case where $W=n/2$ for n even or $(n-1)/2$ for n odd. Malfatti sought *for each value of* k a series or polynomial in n for which z is fixed.