SUR LE JEU DE PAIR OU NON

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If there is anything which seems commonly clear & incontestable, it is that in the game of *pair ou non*, when one presents to us a fistful of tokens, & when one asks of us if the number in it is even or odd, it is worth as much to respond the one as the other, because certainly there are as many even numbers as odd numbers; this so simple reason everyone will determine. However to consider it more closely, it is no longer found thus, as much as these kinds of questions on the probabilities are delicate. Mr. de Mairan has found that there was advantage to say odd rather than even, & he has said since, that some sophisticated players had noticed it themselves also.

The tokens, cached in the hand of the player who proposes the wager, have been taken at random from a certain pile, which the player may have even taken completely whole. We suppose that this pile may be odd only. If it is 3, the player can take only 1 or 2, or 3 tokens, here are therefore two cases where he takes some odd numbers, & one alone where he takes an even number. There are therefore odds 2 against 1 for the odds, which makes an advantage of $\frac{1}{2}$. If the pile is 5, the player can take 3 odds, & only 2 evens, there are odds 3 against 2 for the odd, & the advantage is $\frac{1}{3}$. Likewise if the pile is 7, one will find that the advantage of odd is $\frac{1}{4}$, in such a way that for all the odd piles the advantages of the odds corresponding to each pile will be the sequence $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, whence one sees that the pile 1 would give an infinite advantage, having odds 1 against 0, because the denominators of all these fractions diminished by unity, express the lot of the even against the odd.

If one assumes on the contrary that the pile can be only even, there will be advantage neither for the even nor the odd, it is clear that in all even piles there are no more even numbers to take than odds, nor of odds than of evens.

When one plays, one knows not if the tokens have been taken from an even or an odd pile, if this pile had been 2 or 3, 4 or 5, &c. And as there may have been equally the one or the other, the advantage of the odd is diminished by half because of the possibility that the pile had been even. Thus the sequence $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. becomes $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, &c.

One can get a more sensible idea of this little theory. If one imagines a teetotum with four faces, marked 1, 2, 3, 4, it is evident that when it will rotate there are as great odds that it will fall on an even face as on an odd. If it had five faces, it would have then an odd more, & consequently there would be advantage to wager that it would fall on an odd face: but if it is permitted to a player to rotate one of these two teetotums as he would wish, certainly the advantage of the odd is at least half that it was in the case where only the odd teetotum would be rotated. This which makes precisely the case of the game of *pair ou non*.

One sees by the sequence $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. or by the other $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, &c. that the advantage of the odd always diminishes, accordingly as the pile or the number of tokens

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which one is able to take is greater. The essential reason is that 1 being always the difference by which the number of odds exceeds those of the evens in any odd, this 1 is always less in relation to a greater number. Those players if sophisticated, who have suspected some advantage for the odd, had apparently not suspected this diminution.

If one wished to play in a fair game, it would be necessary that the player who presents the wager say if the pile where he has taken the tokens is even or odd, & in the 2nd case which odd it is. If he says that it is even, it is not necessary further to know that the wager is equal, whatever even that this be. If he says that the pile is odd, it is necessary that he determine it, for example 7, so that one knows that there is $\frac{1}{4}$ more to wager for odd, & that the one who takes this part sets this a quarter more than the other, that he sets 4 against 3. Then the game is perfectly equal. We take here $\frac{1}{4}$, advantage of the odd, in the 1st sequence, & not in the 2nd, where it would be $\frac{1}{8}$, because this 2nd assumes that the pile is able to be equally even or odd, which it is not here.

One sees therefore that if in place of the alternative of an even or odd pile, one assumed more probability to the one than to the other, or, what reverts to the same, three piles in place of two, the advantage of the player who says not-even, could diminish in one case, & increase in the other. It would diminish in the case where he could have a single of the three odd piles against two even, & it would increase on the contrary, if he had possibility of two odd piles against one even. For example, if the player who presents the wager said to you, that the pile from which he takes some tokens, & where you have to say *pair ou non*, is 6, 7, or 8, it is evident that the only possibility of a pile which would be 7, or the advantage $\frac{1}{4}$ which would follow to say odd, must be divided by 3, because of the three possible cases: this which would give $\frac{1}{12}$, smaller than $\frac{1}{8}$. As to the contrary if the three possible cases were 5, 6 & 7, the advantage being then $\frac{1}{3}$ in the first case, 0 in the second, & $\frac{1}{4}$ in the third, one would have $\frac{4}{12}$ plus 0, plus $\frac{3}{12}$, which makes $\frac{7}{12}$, to divide by 3; this which would give $\frac{7}{36}$, an advantage greater than $\frac{1}{6}$, & consequently than $\frac{1}{8}$.

In such manner the advantage which there is to say non-even in a number of any possible piles, either evens, with non-evens, or only odds, will always be expressed by the sum of the advantages of each of the possible cases, divided by the number of piles, by including the evens with them, if there are them, which always give 0 advantage. This is here the formula or the general rule.

On which Mr. de Mairan made yet this question. If the player who presents the wager said, the pile in which I have to take will not exceed a certain number of tokens, for example 7, or 12, &c. but it may be smaller at my choice; what is the advantage that there is then to say non-even? It is evident that it will be composed of the risk or the advantage of all the possible piles, from 7, or 12, to 1 inclusively. Thus under the condition that it cannot exceed 7, the rule will give $\frac{1}{1}$, plus 0, plus $\frac{1}{2}$, &c. divided by 7, this which makes in total $\frac{25}{84}$, nearly a third of the wager of one who says odd. If the greatest pile possible had been 12, the advantage had been less, not only because the number of possible piles, or the divisor, had been greater, but again because it would be able to have as many evens as odds. There would be therefore $\frac{147}{720}$ or approximately $\frac{1}{5}$ advantage to say odd under this assumption.

Among all the objections which have been made to Mr. de Mairan against the inequality of the game of *pair ou non*, & his manner to evaluate it, one of the most specious is this. Let the pile have three tokens. By that which has been said above, there are two odds against one even, or odds 2 against 1 for odd, & leaving advantage $\frac{1}{2}$. This is true, one says, in regard to a teetotum with three faces marked 1, 2, 3. But it is not the same of a pile with three tokens; because I can take each of these tokens alone, this which makes three cases, or all three together, this which makes a fourth case, & always for the odd; &

because three things are able to be taken two by two, in three different ways, there will be at the same time three cases favorable to the even: this which gives odds 4 against 3, or advantage $\frac{1}{4}$, & not $\frac{1}{2}$, as had been found.

But one must take care, of this that the player has his hand on the first, the second, or the third of the tokens in the pile, there does not result three different events in favor of the odd, as of this that he will take the second & the third, or the first & the second, does not make two in favor of even, but a one & the same event, & one same expectation for the players. Because as soon as chance, or whim, or some prudent reason, has determined the one who has his hand on the pile of three tokens, in order to take one or two of them, it matters not which of the three he takes. It changes nothing in the game. And in order to render this more sensible, there is only to remark that in the case where the player would take from one pile of two tokens, & where one agrees the game is perfectly equal, there would be inequality, & 2 against 1 for the odd, if the objection held; because by the same reasoning he could take alone one or the other of the two tokens for the odd, & only both together for the even. The pile of three tokens does not give therefore four possibilities for the odd with respect to the risk & to the expectation of the players, but only two. The combinations, the changes in order, & the configurations of the numbers are of applicable speculations, in all or in part, to the questions of chance & of the game, according to the hypothesis & the law which makes the foundation of it; & it is clear that here the right or the left, & the first or the second token, does not commit me more to the one than the other to take them alone, or accompanied. These are therefore some foreign circumstances in the risk of the players in the present question.

There would be many ways to introduce equality in the game of *pair ou non*; those which one practices sometimes, reducing all to the case of two tokens, the one white, & the other black, as if the player who presents the wager would demand white or black: but this here is not what is asked, & we pass over them in silence, likewise some other objections that were made to Mr. de Mairan, & of which it is easy to find the solution in his theory.