# Mémoire sur les suites récurro-récurrentes et sur leurs usages dans la théorie des hasards.

P. S. de Laplace
*Professor at the École Royale militaire.*[*]

*Mém. Acad. R. Sci. Paris* (Savants étrangers) **6** (1774), pp. 353–371.
*Oeuvres de Laplace* **8**, pp. 5–24.

1. We can imagine thus the formation of a recurrent series: if $\phi$ expresses any function of $x$, and if we substitute successively in it, in place of $x$, $1$, $2$, $3$, $\ldots$, we will form a series of terms of which I designate by $y_x$ the one which corresponds to the number $x$; this put, if in this series each term is equal to any number of preceding terms, each multiplied by a function of $x$ at will, the series is then recurrent.

Such is the most general idea that is able to be formed from it, and it is under this point of view that I have considered them in a memoir previously presented to the Academy.[1]

I suppose now that $\phi$ is a function of $x$ and of $n$, and that we substitute successively in place of $x$ and of $n$ the numbers $1, 2$, $3$, $\ldots$: we will form for each value of $n$ a series in which I designate the term corresponding to the numbers $x$ and $n$ by ${}_ny_x$: now, if ${}_ny_x$ is equal to any number of preceding terms taken in any number of these series and each multiplied by a function of $x$ and of $n$, these series will be what I call *suites récurro-récurrentes*; they differ from recurrent series in that their general term has two variable indices.

As the consideration of these series has seemed very useful to me in the Theory of chances, and as they have not yet been examined by persons, that I know, I have belief that it would not be useless to develop them here to some extent.

[*]Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. August 14, 2010

[1]*See* Book IV of the *Mémoires de Turin*. *Translator's note*: "Recherches sur le calcul intégral aux différences infiniment petites, & aux différences finies." *Misc. Taurinensia* **4**, 273–345 (1766–1766).

2. PROBLEM I. — *I suppose that we have a series of equations of this form*

$$
\begin{aligned}
&{}_1y_x + A.{}_1y_{x-1} + B.{}_1y_{x-2} + \cdots + N = 0,\\
&{}_2y_x + A_2.{}_2y_{x-1} + B_2.{}_2y_{x-2} + \cdots + N_2\\
&\qquad = H_2.{}_1y_x + M_2.{}_1y_{x-1} + P_2.{}_1y_{x-2} + \cdots,\\
&{}_3y_x + A_3.{}_3y_{x-1} + B_3.{}_3y_{x-2} + \cdots + N_3\\
&\qquad = H_3.{}_2y_x + M_3.{}_2y_{x-1} + P_3.{}_2y_{x-2} + \cdots,\\
&\vdots
\end{aligned}
$$

$$
(1)\qquad \begin{cases} {}_ny_x + A_n.{}_ny_{x-1} + B_n.{}_ny_{x-2} + \cdots + N_n\\ \qquad = H_n.{}_{n-1}y_x + M_n.{}_{n-1}y_{x-1} + \cdots \end{cases}
$$

*It is necessary to determine the value of* ${}_ny_x$*;* $A_n$, $B_n$, $\ldots$, $N_n$, $H_n$, $\ldots$ *being any functions of* $n$*, and* $A_2$, $B_2$, $\ldots$, $A_3$, $B_3$, $\ldots$ *being that which these functions become when we substitute successively in them, in place of* $n$, $1$, $2$, $3$, $\ldots$*, finally* $A$, $B$, $N$, $\ldots$ *being any constants.*

We suppose first that we have

$$(a)\qquad {}_1y_x + A.{}_1y_{x-1} = 0,$$

$$(b)\qquad {}_2y_x + A_2.{}_2y_{x-1} = H_2.{}_1y_x + M_2.{}_1y_{x-1}.$$

The second of these equations will give

$${}_2y_{x-1} + A_2.{}_2y_{x-2} = H_2.{}_1y_{x-1} + M_2.{}_1y_{x-2},$$

but equation (*a*) gives

$${}_1y_{x-2} = -\frac{{}_1y_{x-1}}{A},$$

hence

$$(c)\qquad {}_2y_{x-1} + A_2.{}_2y_{x-2} = H_2.{}_1y_{x-1} - \tfrac{M_2}{A}\,{}_1y_{x-1};$$

multiplying equation (*a*) by $-\alpha$, equation (*c*) by $\beta$, and adding them with equation (*b*), we will have

$$
\begin{aligned}
&{}_2y_x + {}_2y_{x-1}(A_2+\beta) + \beta A_2.{}_2y_{x-2}\\
&\qquad = {}_1y_x(\alpha + H_2) + {}_1y_{x-1}\left[\alpha A + M_2 + \beta\left(H_2 - \frac{M_2}{A}\right)\right].
\end{aligned}
$$

We will make ${}_1y_x$ and ${}_1y_{x-1}$ vanish by means of the equations

$$
\begin{aligned}
&\alpha + H_2 = 0\\
&\alpha A + M_2 + \beta\left(H_2 - \frac{M_2}{A}\right) = 0;
\end{aligned}
$$

we see that, by following this process, it is always possible to transform equation (1) of the problem into the following

$$(2) \qquad {}_n y_x = a_n \cdot {}_n y_{x-1} + b_n \cdot {}_n y_{x-2} + c_n \cdot {}_n y_{x-3} + \cdots + u_n,$$

$a_n$, $b_n$, …, $u_n$ being some functions of $n$ and of constants that we will determine by the following method.

Equation (2) will give the following:

$$\begin{aligned} H_n \cdot {}_{n-1}y_x &= H_n(a_{n-1} \cdot {}_{n-1}y_{x-1} + b_{n-1} \cdot {}_{n-1}y_{x-2} + \cdots + u_{n-1}), \\ M_n \cdot {}_{n-1}y_{x-1} &= M_n(a_{n-1} \cdot {}_{n-1}y_{x-2} + b_{n-1} \cdot {}_{n-1}y_{x-3} + \cdots + u_{n-1}), \\ P_n \cdot {}_{n-1}y_{x-2} &= P_n(a_{n-1} \cdot {}_{n-1}y_{x-3} + b_{n-1} \cdot {}_{n-1}y_{x-4} + \cdots + u_{n-1}), \\ &\vdots \end{aligned}$$

By comparing these equations with equation (1), we will have

$$\begin{aligned} {}_n y_x + A_n \cdot {}_n y_{x-1} + B_n \cdot {}_n y_{x-2} + \cdots + N_n &= a_{n-1}({}_n y_{x-1} + A_n \cdot {}_n y_{x-2} + \cdots + N_n) \\ &\quad + b_{n-1}({}_n y_{x-2} + A_n \cdot {}_n y_{x-3} + \cdots + N_n) \\ &\quad + \cdots \\ &\quad + u_{n-1}(H_n + M_n + P_n + \cdots). \end{aligned}$$

If we compare this equation with equation (2), we will have

$$\begin{aligned} a_n &= a_{n-1} - A_n, \\ b_n &= b_{n-1} + a_{n-1}A_n - B_n, \\ c_n &= c_{n-1} + b_{n-1}A_n + a_{n-1}B_n - c_n, \\ &\vdots \\ u_n &= u_{n-1}(H_n + M_n + P_n + \cdots) - N_n(1 - a_{n-1} - b_{n-1} - c_{n-1} - \cdots). \end{aligned}$$

Equation (1) of the problem will be therefore in this way transformed into equation (2) which is ordinary in the recurrent series, and that we will integrate easily by the method explained in the memoir cited above.

3. Problem II. — *We propose to integrate the differentio-differential* equation

$$(3) \qquad \left\{ \begin{aligned} &{}_n y_x + A_n \cdot {}_n y_{x-1} + B_n \cdot {}_n y_{x-2} + \cdots + N_n \\ &\qquad = H_n \cdot {}_{n-1}y_x + M_n \cdot {}_{n-1}y_{x-1} + P_n \cdot {}_{n-1}y_{x-2} + \cdots \\ &\qquad + \alpha_n \cdot {}_{n-2}y_x + \beta_n \cdot {}_{n-2}y_{x-1} + \gamma_n \cdot {}_{n-2}y_{x-2} + \cdots, \end{aligned} \right.$$

*by supposing that we have*

$${}_2 y_x + A \cdot {}_2 y_{x-1} + B \cdot {}_2 y_{x-2} + \cdots + N = H \cdot {}_1 y_x + M \cdot {}_1 y_{x-1} + P \cdot {}_1 y_{x-2} + \cdots$$

We will show easily, as in the preceding problem, that it is always possible to transform equation (3) into another, such that

$$(4)\qquad \begin{cases} {}_ny_x = a_n\cdot{}_ny_{x-1} + b_n\cdot{}_ny_{x-2} + c_n\cdot{}_ny_{x-3} + \cdots + u_n \\ \qquad + h_n\cdot{}_{n-1}y_x + l_n\cdot{}_{n-1}y_{x-1} + p_n\cdot{}_{n-1}y_{x-2} + \cdots; \end{cases}$$

we will have therefore

$$\begin{aligned}
\alpha_n\cdot{}_{n-1}y_x &= \alpha_n(a_{n-1}\cdot{}_{n-1}y_{x-1} + b_{n-1}\cdot{}_{n-1}y_{x-2} + c_{n-1}\cdot{}_{n-1}y_{x-3} + \cdots + u_{n-1} \\
&\quad + h_{n-1}\cdot{}_{n-2}y_x + l_{n-1}\cdot{}_{n-2}y_{x-1} + p_{n-1}\cdot{}_{n-2}y_{x-2} + \cdots); \\
\beta_n\cdot{}_{n-1}y_{x-1} &= \beta_n(a_{n-1}\cdot{}_{n-1}y_{x-2} + b_{n-1}\cdot{}_{n-1}y_{x-3} + c_{n-1}\cdot{}_{n-1}y_{x-4} + \cdots + u_{n-1} \\
&\quad + h_{n-1}\cdot{}_{n-2}y_{x-1} + l_{n-1}\cdot{}_{n-2}y_{x-2} + p_{n-1}\cdot{}_{n-2}y_{x-3} + \cdots); \\
\gamma_n\cdot{}_{n-1}y_{x-2} &= \gamma_n(a_{n-1}\cdot{}_{n-1}y_{x-3} + \cdots + u_{n-1} \\
&\quad + h_{n-1}\cdot{}_{n-2}y_{x-2} + \cdots); \\
&\vdots,
\end{aligned}$$

which will give, by combining these equations with equation (3) of the problem,

$$\begin{aligned}
\alpha_n\cdot{}_{n-1}y_x + \beta_n\cdot{}_{n-1}y_{x-1} + \gamma_n\cdot{}_{n-1}y_{x-2} + \cdots \\
= a_{n-1}(\alpha_n\cdot{}_{n-1}y_{x-1} + \beta_n\cdot{}_{n-1}y_{x-2} + \cdots) \\
+ b_{n-1}(\alpha_n\cdot{}_{n-1}y_{x-2} + \cdots) \\
+ \cdots \\
+ h_{n-1}({}_ny_x + A_n\cdot{}_ny_{x-1} + B_n\cdot{}_ny_{x-2} + \cdots \\
- H_n\cdot{}_{n-1}y_x - M_n\cdot{}_{n-1}y_{x-1} - \cdots \\
+ l_{n-1}({}_ny_{x-1} + A_n\cdot{}_ny_{x-2} + B_n\cdot{}_ny_{x-3} + \cdots \\
- H_n\cdot{}_{n-1}y_{x-1} - M_n\cdot{}_{n-1}y_{x-2} - \cdots \\
+ \cdots \\
+ u_{n-1}(\alpha_n + \beta_n + \gamma_n + \cdots) \\
+ N_n(h_{n-1} + l_{n-1} + \cdots).
\end{aligned}$$

Therefore

$$\begin{aligned}
{}_ny_x = {}_ny_{x-1}&\left(-A_n - \frac{l_{n-1}}{h_{n-1}}\right) + {}_ny_{x-2}\left(-B_n - \frac{l_{n-1}A_n}{h_{n-1}} - \frac{p_{n-1}}{h_{n-1}}\right) + \cdots \\
&+ {}_{n-1}y_x\left(\frac{\alpha_n}{h_{n-1}} + H_n\right) + {}_{n-1}y_{x-1}\left(\frac{\beta_n}{h_{n-1}} + M_n + l_{n-1}\frac{H_n}{h_{n-1}} - \frac{\alpha_n a_{n-1}}{h_{n-1}}\right) \\
&+ {}_{n-1}y_{x-2}\left(\frac{\gamma_n}{h_{n-1}} + P_n + \frac{l_{n-1}M_n}{h_{n-1}} + \frac{p_{n-1}H_n}{h_{n-1}} - \frac{\alpha_n b_{n-1}}{h_{n-1}} - \frac{a_{n-1}\beta_n}{h_{n-1}}\right) \\
&+ \cdots \\
&- u_{n-1}\frac{\alpha_n + \beta_n + \cdots}{h_{n-1}} - N_n\frac{h_{n-1} + l_{n-1} + \cdots}{h_{n-1}};
\end{aligned}$$

whence we will deduce, by comparing with equation (4),

(I) $\frac{\alpha_n}{h_{n-1}} + H_n = h_n,$

(II) $a_n = -A_n - \frac{l_{n-1}}{h_{n-1}},$

(III) $l_n = \frac{\beta_n}{h_{n-1}} + M_n + \frac{l_{n-1}H_n}{h_{n-1}} - \frac{\alpha_n a_{n-1}}{h_{n-1}},$

(IV) $b_n = -B_n - \frac{l_{n-1}A_n}{h_{n-1}} - \frac{p_{n-1}}{h_{n-1}},$

(V) $p_n = \frac{\gamma_n}{h_{n-1}} + P_n + \frac{l_{n-1}M_n}{h_{n-1}} + \frac{p_{n-1}H_n}{h_{n-1}} - \frac{\alpha_n b_{n-1}}{h_{n-1}} - \frac{a_{n-1}\beta_n}{h_{n-1}},$

$\ldots$

From the first of these equations, we will conclude $h_n$, the second will give

$$l_n = -a_{n+1}h_n - A_{n+1}h_n;$$

this value of $l_n$, substituted into the third, will give

$$-a_{n+1}h_n - A_{n+1}h_n = \frac{\beta_n}{h_{n-1}} + M_n - H_n a_n - H_n A_n - \frac{\alpha_n a_{n-1}}{h_{n-1}},$$

whence we will conclude $a_n$, and hence $l_n$; by combining in the same manner the fourth and the fifth, ... equation, we will have the expression of $b_n$, $c_n$, $p_n$, and thus the rest, and we will determine $u_n$ by the equation

$$u_n = -u_{n-1}\frac{\alpha_n + \beta_n + \cdots}{h_{n-1}} - N_n\frac{h_{n-1} + l_{n-1} + \cdots}{h_{n-1}}.$$

4. If we call *equation of the first order* an equation in recurrent series, *equation of the second order* an equation such as that of problem I, *equation of the third order* an equation such as that of problem II, and thus in sequence, we see that it is always possible to reduce by the preceding method an equation of any order $r$ to another of an inferior order, provided that, in one particular assumption for $n$, the equation of order $r$ becomes of order $r - 1$, and the same method would take place again if the constant difference, instead of being unity, was any number $q$; it would be useless for us to pause on this further: we are going presently to give some applications of this theory.

5. The most complicated problems in all the theory of chances have for object the duration of events, and we will see with what facility they are able to be resolved by the method of the récurro-récurrente series.

PRINCIPLE

*The probability of an event is equal to the sum of the products of each favorable case by its probability divided by the sum of the products of each possible case by its* *probability, and if each case is equally probable, the probability of the event is equal to the number of favorable cases divided by the number of all possible cases.*

PROBLEM III. — *Two players A and B play on this condition, that* on each trial the one who will lose will give an écu to the other; *I suppose that the skill of A be to that of B as* $a : b$*, and that A has a number* $m$ *of écus and B a number* $n$*; we ask what is the probability that the game will not end before or at the number* $x$ *of trials.*

I suppose first $a = b$, $m = n$, and that $n$ is an even number; it is clear that $x$ must be then even; let ${}_0y_x$ be the number of possible cases according to which, on trial $x$, the gain of the two players is zero; ${}_2y_x$ the number of cases according to which it is equal to 2, and thus in sequence; it is evident that the number of all the possible cases is $2^x$; if therefore we call ${}_nz_x$ the probability that the game will not end at trial $x$, we will have

$$ {}_nz_x = \frac{{}_0y_x + {}_2y_x + {}_4y_x + \cdots + {}_{n-2}y_x}{2^x}; $$

but it is easy to form the following equations, according to the conditions of the problem,

$$ \begin{aligned} {}_0y_x &= 2.{}_0y_{x-2} + {}_2y_{x-2}, \\ {}_2y_x &= 2.{}_2y_{x-2} + 2.{}_0y_{x-2} + {}_4y_{x-2}, \\ {}_4y_x &= 2.{}_4y_{x-2} + {}_2y_{x-2} + {}_6y_{x-2}, \\ {}_6y_x &= 2.{}_6y_{x-2} + {}_4y_{x-2} + {}_8y_{x-2}, \\ &\vdots \\ {}_{n-2}y_x &= 2.{}_{n-2}y_{x-2} + {}_{n-4}y_{x-2}, \end{aligned} $$

whence we deduce

$$ {}_0y_x + {}_2y_x + {}_4y_x + \cdots + {}_{n-2}y_x = 4.{}_0y_{x-2} + 4.{}_2y_{x-2} + \cdots + 4.{}_{n-2}y_{x-2} - {}_{n-2}y_{x-2}, $$

which gives

$$ {}_nz_x = {}_nz_{x-2} - \frac{{}_{n-2}y_{x-2}}{2^x} $$

or

$$ 2^x\Delta.{}_nz_{x-2} = -{}_{n-2}y_{x-2}, $$

$\Delta.{}_nz_{x-2}$ designating the finite difference of ${}_nz_{x-2}$, by regarding $x$ alone as variable, the constant difference being 2.

We take now the two equations

$$ \begin{aligned} {}_0y_x &= 2.{}_0y_{x-2} + {}_2y_{x-2}, \\ {}_2y_x &= 2.{}_2y_{x-2} + 2.{}_0y_{x-2} + {}_4y_{x-2}. \end{aligned} \tag{1} $$

The first gives

$$ {}_0y_{x-2} = 2.{}_0y_{x-4} + {}_2y_{x-4}, \tag{2} $$

and the second gives

(3) $$ {}_2y_{x-2} = 2.{}_2y_{x-4} + 2.{}_0y_{x-4} + {}_4y_{x-4}. $$

If we multiply equation (2) by $\alpha$, and equation (3) by $\beta$, and if next we add them with equation (1), we will have

$$ \begin{aligned} {}_2y_x = (2-\beta).{}_2y_{x-2} + (\alpha+2\beta).{}_2y_{x-4} + (2-\alpha).{}_0y_{x-2} \\ + (2\alpha+2\beta).{}_0y_{x-4} + {}_4y_{x-2} + \beta.{}_4y_{x-4}. \end{aligned} $$

Let

$$ 2-\alpha = 0 \quad \text{or} \quad \alpha = 2, \quad \text{and} \quad 2\alpha + 2\beta = 0 \quad \text{or} \quad \beta = -2; $$

we will have thus the following equations:

(*h*) $$ \begin{cases} {}_2y_x = 4.{}_2y_{x-2} - 2.{}_2y_{x-4} + {}_4y_{x-2} - 2.{}_4y_{x-4}, \\ {}_4y_x = 2.{}_4y_{x-2} + {}_2y_{x-2} + {}_6y_{x-2}, \\ \quad\vdots \\ {}_qy_x = 2.{}_qy_{x-2} + {}_{q-2}y_{x-2} + {}_{q+2}y_{x-2}, \\ \quad\vdots \\ {}_{n-2}y_x = 2.{}_{n-2}y_{x-2} + {}_{n-4}y_{x-2}. \end{cases} $$

These equations evidently correspond to problem II; I suppose therefore that we have in general

(*k*) $$ \begin{cases} {}_qy_x = a_q.{}_qy_{x-2} + b_q.{}_qy_{x-4} + c_q.{}_qy_{x-6} + \cdots + u_q \\ \qquad + h_q.{}_{q+2}y_{x-2} + l_q.{}_{q+2}y_{x-4} + p_q.{}_{q+2}y_{x-6} + \cdots \end{cases} $$

We will have therefore

$$ \begin{aligned} {}_{q-2}y_{x-2} = a_{q-2}.{}_{q-2}y_{x-4} + b_{q-2}.{}_{q-2}y_{x-6} + c_{q-2}.{}_{q-2}y_{x-8} + \cdots + u_{q-2} \\ + h_{q-2}.{}_qy_{x-4} + l_{q-2}.{}_qy_{x-6} + p_{q-2}.{}_qy_{x-8} + \cdots \end{aligned} $$

Substituting into this equation, in place of ${}_{q-2}y_{x-2}$, ${}_{q-2}y_{x-4}$, ..., their values which equation (*h*) furnishes, we will have

$$ \begin{aligned} {}_qy_x = (2+a_{q-2}).{}_qy_{x-2} + (b_{q-2} - 2a_{q-2} + h_{q-2}).{}_qy_{x-4} \\ + (c_{q-2} - 2b_{q-2} + l_{q-2}).{}_qy_{x-6} + \cdots \\ + {}_{q+2}y_{x-2} - a_{q-2}.{}_{q+2}y_{x-4} - b_{q-2}.{}_{q+2}y_{x-6} - \cdots + u_{q-2}. \end{aligned} $$

Therefore, by comparing this equation with equation (*k*), we will have:

1° $$ 2 + a_{q-2} = a_q; $$

now, as here the constant difference is 2, we will have by integrating $a_q = q+c$, $c$ being a constant, and, putting $q = 2$, we have $a_q = 4$; therefore $c = 2$, hence $a_q = q+2 = m$, by making $q + 2 = m$.

2 °
$$h_q = 1.$$

3 °
$$b_{q-2} - 2a_{q-2} + h_{q-2} = b_q;$$

whence we will conclude by integrating and adding the appropriate constant $b_q = -\frac{m(m-3)}{1.2}$.

4 °
$$l_q = -a_{q-2} = -(m-2).$$

5 °
$$c_q = c_{q-2} - 2b_{q-2} + l_{q-2};$$

therefore $c_q = \frac{m(m-4)(m-5)}{1.2.3}\cdots$, finally $u_q = u_{q-2}$; hence $u_q = C$; now $q$ being 2, we have $C = 0$, therefore $u_q = 0$; thus we will have

$$ {}_q y_x = m.{}_q y_{x-2} - \frac{m(m-3)}{1.2}.{}_q y_{x-4} + \frac{m(m-4)(m-5)}{1.2.3}.{}_q y_{x-6} - \cdots + {}_{q+2} y_{x-2} - \frac{m-2}{1}.{}_{q+2} y_{x-4} + \cdots $$

If we suppose now $q = n-2$, then it will not be necessary to take account of the terms ${}_{q+2} y_{x-2}$, ${}_{q+2} y_{x-4}, \cdots$ and we will have under this assumption $m = n$; therefore

$$ {}_{n-2} y_x = n.{}_{n-2} y_{x-2} - \frac{n(n-3)}{1.2}.{}_{n-2} y_{x-4} + \cdots $$

Substituting into this equation, in place of ${}_{n-2} y_x$, ${}_{n-2} y_{x-2}, \ldots$, their values $-2^{x+2}\Delta.{}_n z_x$, $-2^x\Delta.{}_n z_{x-2}, \ldots$, we will have, after having integrated,

$$ {}_n z_x = \frac{n}{4}\cdot {}_n z_{x-2} - \frac{n(n-3)}{1.2}\frac{1}{4^2}\cdot {}_n z_{x-4} + \frac{n(n-4)(n-5)}{1.2.3}\frac{1}{4^3}\cdot {}_n z_{x-6} - \cdots + H; $$

in order to determine this constant $H$, we must observe that by supposing $x = n$, we will have

$$ {}_n z_x = 1 - \frac{1}{2^{n-1}}, \quad {}_n z_{x-2} = 1, \quad {}_n z_{x-4} = 1, \quad \cdots, $$

therefore

$$ 1 - \frac{1}{2^{n-1}} = \frac{1}{4}n - \frac{n(n-3)}{1.2}\frac{1}{4^2} + \frac{n(n-4)(n-5)}{1.2.3}\frac{1}{4^3} - \cdots + H. $$

Now we know that, if we call $\cos\phi = y$, we will have

$$ \cos n\phi = 2^{n-1}y^n - n2^{n-3}y^{n-2} + \frac{n(n-3)}{1.2}2^{n-5}y^{n-4} + \cdots $$

Putting therefore $\phi = 0$, we will have

$$1 - \frac{1}{2^{n-1}} = \frac{n}{4} - \frac{n(n-3)}{1.2.4^2} + \frac{n(n-4)(n-5)}{1.2.3.4^3} - \cdots,$$

whence we conclude $H = 0$.

We suppose actually that $n$ is an odd number, $x$ will be then odd, and we will have

$${}_{n}z_{x} = \frac{{}_{1}y_{x} + {}_{3}y_{x} + {}_{5}y_{x} + \cdots + {}_{n-2}y_{x}}{2^x};$$

next we will form the following equations:

$$\begin{aligned}
{}_{1}y_{x} &= 3.{}_{1}y_{x-2} + {}_{3}y_{x-2}, \\
{}_{3}y_{x} &= 2.{}_{3}y_{x-2} + {}_{1}y_{x-2} + {}_{5}y_{x-2}, \\
{}_{5}y_{x} &= 2.{}_{5}y_{x-2} + {}_{3}y_{x-2} + {}_{7}y_{x-2}, \\
&\vdots \\
{}_{n-2}y_{x} &= 2.{}_{n-2}y_{x-2} + {}_{n-4}y_{x-2};
\end{aligned}$$

by operating next as previously, we will have the equation

$${}_{n}z_{x} = \frac{n}{4} \cdot {}_{n}z_{x-2} - \frac{n(n-3)}{1.2}\frac{1}{4^2} \cdot {}_{n}z_{x-4} + \cdots,$$

the same as for the even numbers.

We suppose now that the numbers of écus of the two players are equal and even, and that the skills of these players are unequal and in the ratio of $a$ to $b$; by naming ${}_{0}y_{x}$ the number of cases according to which at the trial $x$ the gain of the two players is zero, ${}_{2}y_{x}$ the number of cases according to which the gain of A can be 2, and ${}_{2}'y_{x}$ the number of cases according to which the gain of B can be 2, and thus in sequence, we will form easily the following equations:

(*t*) $${}_{0}y_{x} = 2ab.{}_{0}y_{x-2} + b^2.{}_{2}y_{x-2} + a^2.{}_{2}'y_{x-2},$$

(*v*) $${}_{2}y_{x} = 2ab.{}_{2}y_{x-2} + a^2.{}_{0}y_{x-2} + b^2.{}_{4}y_{x-2},$$

(*w*) $${}_{4}y_{x} = 2ab.{}_{4}y_{x-2} + a^2.{}_{2}y_{x-2} + b^2.{}_{6}y_{x-2},$$

$$\begin{aligned}
&\vdots \\
{}_{n-2}y_{x} &= 2ab.{}_{n-2}y_{x-2} + a^2.{}_{n-4}y_{x-2}, \\
&\vdots \\
{}_{2}'y_{x} &= 2ab.{}_{2}'y_{x-2} + b^2.{}_{0}y_{x-2} + a^2.{}_{4}'y_{x-2}, \\
{}_{4}'y_{x} &= 2ab.{}_{4}'y_{x-2} + b^2.{}_{2}'y_{x-2} + a^2.{}_{6}'y_{x-2}, \\
&\vdots \\
{}_{n-2}'y_{x} &= 2ab.{}_{n-2}'y_{x-2} + b^2.{}_{n-4}'y_{x-2};
\end{aligned}$$

now here the number of all the cases multiplied by their particular probability is $(a + b)^x$: we will have therefore

$$\Delta . {}_n z_{x-2} = -\frac{a^2 . {}_{n-2} y_x}{(a+b)^x} - \frac{b^2 . {}_{n-2}' y_x}{(a+b)^x};$$

now it is easy to see that we have ${}_2' y_x = \frac{b^2}{a^2} \cdot {}_2 y_x$; substituting this value of ${}_2' y_x$ into equation (*t*), we will have

$${}_0 y_x = 2ab . {}_0 y_{x-2} + 2b^2 . {}_2 y_{x-2} .$$

Eliminating from this equation[2] ${}_0 y_x$ by means of equation (*v*), we will have

$${}_2 y_x = 4ab . {}_2 y_{x-2} - 2a^2b^2 . {}_2 y_{x-4} + b^2 . {}_4 y_{x-4} - ab^3 \cdot {}_4 y_{x-4} .$$

This equation, with equation (*w*) and the following, will give, by a process similar to the preceding,

$${}_{n-2} y_x = nab . {}_{n-2} y_{x-2} - \frac{n(n-3)}{1.2} a^2 b^2 . {}_{n-2} y_{x-4} + \cdots ,$$

whence we will conclude

$${}_n z_x = \frac{nab}{(a+b)^2} \cdot {}_n z_{x-2} - \frac{n(n-3)}{1.2} \frac{a^2b^2}{(a+b)^4} \cdot {}_n z_{x-4} + \cdots + H.$$

If $n$ was odd, the problem would be resolved exactly in the same manner; thus it would be useless for us to delay further.

But here is another way to treat the same problem, always following the method of the recurro-recurrente series; we will make

$${}_{n-2}' y_x = {}_0 \nu_x , \quad {}_{n-4}' y_x = {}_2 \nu_x , \quad \ldots ,$$

and we will have the equations

$$\begin{aligned} {}_0 \nu_x &= 2ab . {}_0 \nu_{x-2} + b_2 . {}_2 \nu_{x-2} , \\ {}_2 \nu_x &= 2ab . {}_2 \nu_{x-2} + b_2 . {}_4 \nu_{x-2} , \\ &\vdots \end{aligned}$$

Whence we will deduce easily, by Problem II, an equation between ${}_{n-4} \nu_x , {}_{n-4} \nu_{x-2}$, ...and ${}_{n-2} \nu_{x-2} , {}_{n-2} \nu_{x-4} , \ldots$, or, what is the same thing, between ${}_2' y_x , {}_2' y_{x-2} , \ldots$ and ${}_0 y_{x-2} , {}_0 y_{x-4} , \ldots$; by aid of this equation and the two equations (*t*) and (*v*), we will eliminate easily ${}_2' y_x , {}_2' y_{x-2}$ and ${}_0 y_{x-2} , {}_0 y_{x-4}$, and we will have an equation between ${}_2 y_x , {}_2 y_{x-2} , \ldots$ and ${}_4 y_{x-2} , {}_4 y_{x-4} , \ldots$, whence next it will be easy, by

[2] It is necessary to change in this equation $x$ into $x - 2$, and to eliminate ${}_0 y_{x-2}$ and ${}_0 y_{x-1}$ enters the equation thus obtained, equation (*v*) and that which one deduces from it by replacing $x$ by $x - 2$. (*Note of the editor.*)

Problem II, to find an equation between ${}_{n-2}y_{x}$, ${}_{n-2}y_{x-2}, \ldots$ and, changing in this equation $a$ to $b$ and $b$ to $a$, we will have a second equation between ${}_{n-2}y'_{x}$, ${}_{n-2}y_{x-2}, \ldots$, and from these two equations we will have easily ${}_{n}z_{x}$.

It would be the same process if the number of écus were different for the two players, and the problem have no other difficulty than the length of the calculation.

6. I pass now to the following Problem, which had been proposed to me on the occasion of a wager made on the lottery of the military school.

PROBLEM IV. — *A lottery being composed of a number $n$ of tickets $1, 2, 3, \ldots, n$, of which there is extracted a number $p$ at each drawing, we ask the probability that after $x$ drawings all the tickets will be extracted.*

We suppose that $S$ wagers that all the tickets will not be extracted after this number of drawings, and we seek all the cases favorable to $S$; it is clear that their number is equal:

1° To the number of cases according to which the ticket 1 is not able to be extracted after the drawing $x$;

2° To the number of cases according to which the ticket 2 is not able to be extracted, the ticket 1 being extracted;

3° To the number of cases according to which the ticket 3 is not able to be extracted, the tickets 1 and 2 being extracted, and thus in sequence; if therefore we name ${}_{q}y_{x}$ the sum of all these cases to the ticket $q$, we will have

$${}_{q}y_{n} = {}_{q-1}y_{n} - {}_{q-1}y_{n-1} + \left[\frac{(n-1)\cdots(n-p)}{1.2\ldots p}\right]^{x},$$

an equation which corresponds to Problem I, $q$ and $n$ being supposed variables and $x$ constant; here is how we can integrate in this particular case; putting $q$ successively equal to $1, 2, 3, \ldots,$ we will have

$${}_{1}y_{n} = \left[\frac{(n-1)\cdots(n-p)}{1.2\ldots p}\right]^{x},$$
$${}_{2}y_{n} = 2\left[\frac{(n-1)\cdots(n-p)}{1.2\ldots p}\right]^{x} - \left[\frac{(n-2)\cdots(n-p-1)}{1.2\ldots p}\right]^{x},$$
$${}_{3}y_{n} = 3\left[\frac{(n-1)\cdots(n-p)}{1.2\ldots p}\right]^{x} - 3\left[\frac{(n-2)\cdots(n-p-1)}{1.2\ldots p}\right]^{x} + \left[\frac{(n-3)\cdots(n-p-2)}{1.2\ldots p}\right]^{x},$$

whence we will conclude easily

$$\begin{aligned}{}_{n}y_{n} =& n\left[\frac{(n-1)\cdots(n-p)}{1.2\ldots p}\right]^{x} - \frac{n(n-1)}{1.2}\left[\frac{(n-2)\cdots(n-p-1)}{1.2\ldots p}\right]^{x} \\ &+ \frac{n(n-1)(n-2)}{1.2.3}\left[\frac{(n-3)\cdots(n-p-2)}{1.2\ldots p}\right]^{x} + \cdots\end{aligned}$$

Now here the sum of all the possible cases is $\left[\frac{n(n-1)\cdots(n-p+1)}{1.2\ldots p}\right]^x$; naming therefore $z_x$ the probability of $S$, we will have

$$z_x = n\left[\frac{(n-1)(n-2)\cdots(n-p)}{n(n-1)\ldots(n-p+1)}\right]^x - \frac{n(n-1)}{1.2}\left[\frac{(n-2)\cdots(n-p-1)}{n\ldots(n-p+1)}\right]^x + \cdots$$

If we wish to apply this formula to the lottery of the military school, it is necessary, according to the nature of this lottery, to suppose $n = 90$ and $p = 5$.

7. The notation that we have employed and the manner in which we consider the calculus in the finite differences in two variables are, as we see, of an extended use in the theory of chances. In order to give yet a very simple example, let us propose the following problem:

PROBLEM V. — *If in a pile of $x$ pieces, we take a number at random, we ask the probability that this number will be even or odd.*

Let ${}_p y_x$ be the number of cases according to which this number can be even, and ${}_{p-1} y_x$ the number of cases according to which it can be odd; we will have

$$(1) \qquad {}_p y_{x+1} = {}_p y_x + {}_{p-1} y_x,$$

$$(2) \qquad {}_{p-1} y_{x+1} = {}_{p-1} y_x + {}_p y_x + 1.$$

This second equation will give

$${}_{p-1} y_x = {}_{p-1} y_{x-1} + {}_p y_{x-1} + 1.$$

The first gives

$${}_p y_x = {}_p y_{x-1} + {}_{p-1} y_{x-1};$$

therefore we will have

$${}_p y_{x+1} = 2^p y_x + 1;$$

whence we deduce, by integrating,

$${}_p y_x = A2^x - 1;$$

now, putting $x = 1$, we have

$${}_p y_x = 0;$$

therefore $2A - 1 = 0$ and $A = \frac{1}{2}$, hence ${}_p y_x = 2^{x-1} - 1$, and since equation (1) gives

$${}_{p-1} y_x = {}_p y_{x+1} - {}_p y_x,$$

we will have

$${}_{p-1} y_x = 2^{x-1}.$$

The sum of all the possible cases is clearly

$$ {}_{p}y_{x} + {}_{p-1}y_{x} = 2^x - 1. $$

If therefore we call ${}_{p}z_{x}$ the probability that the number will be even, and ${}_{p-1}z_{x}$ the probability that it will be odd, we will have

$$ {}_{p}z_{x} = \frac{2^{x-1} - 1}{2^x - 1}, $$
$$ {}_{p-1}z_{x} = \frac{2^{x-1}}{2^x - 1}; $$

whence it is easy to see that there is always greater advantage to wager on the odd numbers than on the even.

I suppose that we are assured that the number $x$ cannot exceed $n$, but that this number and all the inferior numbers are equally possible; we will have, for the sum of all the favorable cases on the odds,

$$ S2^{x-1} = 2^x + C; $$

now, $x$ being 1, we have

$$ 2^x + C = 1; $$

therefore $C = -1$ and $2^x + C = 2^x - 1$; we will have similarly

$$ S(2^{x-1} - 1) = 2^x - x + C; $$

now, $x$ being 1, we have

$$ 2^x - x + C = 0, $$

therefore $C = -1$; hence, the sum of all the cases favorable to the odds is $2^n - 1$, and the sum of all the cases favorable to the evens is $2^n - n - 1$; thus the probability for the odds is

$$ \frac{2^n - 1}{2^{n-1} - n - 2}, $$

and the probability for the evens is

$$ \frac{2^n - n - 1}{2^{n-1} - n - 2}. $$

In the *Histoire de l'Académie des Sciences*, for the year 1728, we see that Mr. de Mairan has likewise observed that there is always a greater advantage to wager for the odds than for the evens; but it seems to me that the manner in which this ingenious author considers the problem is not correct, and that, in order to appreciate this advantage, it is necessary to consider it under the point of view under which we have assessed it.

We can imagine in the same manner some récurro-récurrente series, of which the general term would have three or even a greater number of variable indices, and if they are met in the resolution of some problems, we can treat them by a method analogous to the preceding.

[The remainder of this paper, section 8, concerns the solution of a differential equation and is omitted.]