MÉMOIR SUR DIVERS POINTS D'ANALYSE.

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I.
On the calculus of generating functions.

The object of this calculus is to restore to the simple development of the functions all the operations relative to the differences; and especially the integration of the equations in the ordinary or the partial differences: here is the principal idea of it. Let u be any function of t, and we suppose that by developing it with respect to the powers of t, we have

$$u = y_0 + y_1 t + y_2 t^2 + \cdots + y_x t^x + \cdots + y_\infty t^\infty;$$

u is that which we name *generating function* of y_x or of the coefficient of t^x in its development. It is clear that $y_{x+1} - y_x$, or Δy_x, will be the coefficient of t^x in the development of $u\left(\frac{1}{t} - 1\right)$; so that, in order to have the generating function of the finite difference of a variable, it suffices to multiply by $\frac{1}{t} - 1$ the generating function of this variable; $u\left(\frac{1}{t} - 1\right)^2$ will be therefore the generating function of $\Delta^2 y_x$, and, generally, the generating function of $\Delta^n y_x$ will be $u\left(\frac{1}{t} - 1\right)^n$. Now we have

$$u\left(\frac{1}{t} - 1\right)^n = u\left[\frac{1}{t^n} - \frac{n}{t^{n-1}} + \frac{n(n-1)}{1.2t^{n-2}} - \cdots\right];$$

the coefficient of t^x in $\frac{u}{t^n}$ is evidently y_{x+n}, the one of t^x in $\frac{u}{t^{n-1}}$ is y_{x+n-1} and thus in sequence; by equating therefore the coefficients of t^x in the two members of the preceding equation, that is to say by passing again from the generating functions to their coefficients, we will have

$$\Delta^n y_x = y_{x+n} - n y_{x+n-1} + \frac{n(n-1)}{1.2} y_{x+n-2} - \cdots .$$

If, instead of multiplying the function u by $\frac{1}{t} - 1$, we multiply it by every other quantity, we would have some analogous results. Let, for example, $a + \frac{b}{t} + \frac{c}{t^2} + \cdots$ be this new multiplier; the coefficient of t^x in the development of the function

$$u\left(a + \frac{b}{t} + \frac{c}{t^2} + \cdots\right)$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. August 20, 2011 will be $ay_x + by_{x+1} + cy_{x+2} + \cdots$; let ∇y_x be this coefficient, and we designate by $\nabla^2 y_x$ the quantity $a\nabla y_x + b\nabla y_{x+1} + c\nabla y_{x+2} + \cdots$, by $\nabla^3 y_x$ the quantity $a\nabla^2 y_x + b\nabla^2 y_{x+1} + \cdots$, and thus in sequence; the generating function of $\nabla^n y_x$ will be

$$u\left(a + \frac{b}{t} + \frac{c}{t^2} + \cdots\right)^n;$$

and, by developing $\left(a + \frac{b}{t} + \frac{c}{t^2} + \cdots\right)^n$ into series, we will have an equation of this form

$$u\left(a + \frac{b}{t} + \frac{c}{t^2} + \cdots\right)^n = u\left(A + \frac{B}{t} + \frac{C}{t^2} + \cdots\right).$$

This equation will give, by passing again from the generating functions to the coefficients,

$$\nabla^n y_x = Ay_x + By_{x+1} + Cy_{x+2} + \cdots.$$

I refer, for the development of this calculus of the generating functions, to the *Mémoires de l'Académie des Sciences* for the year 1779.[1] I will limit myself here to presenting some new theorems which result from it.

Let u be a function of t, and we suppose that y_x is the coefficient of t^x in its development; let likewise u' be a function of t', and we designate by y'_x the coefficient of t'^x in its development; let still u'' be a function of t'', and we designate by y''_x the coefficient of t''^x in its development, and thus in sequence. It is clear that $y_x y'_x y''_x \dots$ will be the coefficient of $t^x t'^x t''^x \dots$, in the development of $uu'u'' \dots$; $\frac{uu'u''\dots}{tt't''\dots}$ will be the generating function of $y_{x+1} y'_{x+1} y''_{x+1} \dots$; that of $\Delta(y_x y'_x y''_x \dots)$ will be therefore

$$uu'u''\dots\left(\frac{1}{tt't''\dots} - 1\right),$$

and, consequently, the generating function of $\Delta^n(y_x y'_x y''_x \dots)$ will be

$$uu'u''\dots\left(\frac{1}{tt't''\dots} - 1\right)^n;$$

by changing n into $-n$, we will have, by the principles exposed in the *Mémoires* cited of the Académie des Sciences, the generating function of $\Sigma^n(y_x y'_x y''_x \dots)$, Σ being the characteristic of the finite integrals; in such a way that we can change n into $-n$ in the generating function, provided that we change Δ^n into Σ^n in its coefficient.

We consider two functions y_x and y'_x; the generating function of $\Delta^n y_x y'_x$ will be

$$uu'\left(\frac{1}{tt'} - 1\right)^n.$$

We can put it under this form:

$$uu'\left[\frac{1}{t} - 1 + \frac{1}{t}\left(\frac{1}{t'} - 1\right)\right]^n.$$

[1] *Oeuvres de Laplace*, T. X. p. 1, "Mémoire sur les suites."

By developing it, it becomes

$$uu'\left[\left(\frac{1}{t}-1\right)^n+\frac{n}{t}\left(\frac{1}{t}-1\right)^{n-1}\left(\frac{1}{t'}-1\right)+\frac{n(n-1)}{1.2\,t^2}\left(\frac{1}{t}-1\right)^{n-2}\left(\frac{1}{t'}-1\right)^2+\cdots\right].$$

The functions

$$uu'\left(\frac{1}{t}-1\right)^n,\quad \frac{uu'}{t}\left(\frac{1}{t}-1\right)^{n-1}\left(\frac{1}{t'}-1\right),\quad \frac{uu'}{t^2}\left(\frac{1}{t}-1\right)^{n-2}\left(\frac{1}{t'}-1\right)^2,\quad \cdots$$

are respectively generators of the variables

$$y'_x\Delta^n y_x,\quad \Delta y'_x\Delta^{n-1}y_{x+1},\quad \Delta^2 y'_x\Delta^{n-2}y_{x+2},\quad \ldots;$$

the identical equation

$$uu'\left(\frac{1}{tt'}-1\right)^n=uu'\left[\left(\frac{1}{t}-1\right)^n+\frac{n}{t}\left(\frac{1}{t}-1\right)^{n-1}\left(\frac{1}{t'}-1\right)+\cdots\right].$$

will give therefore, by passing again from the generating functions to the coefficients,

$$\Delta^n y_x y'_x=y'_x\Delta^n y_x+n\Delta y'_x\Delta^{n-1}y_{x+1}+\frac{n(n-1)}{1.2}\Delta^2 y'_x\Delta^{n-2}y_{x+2}\ldots;$$

by changing n to $-n$ in it, we will have

$$\Sigma^n y_x y'_x=y'_x\Sigma^n y_x+n\Delta y'_x\Sigma^{n+1}y_{x+1}+\frac{n(n+1)}{1.2}\Delta^2 y'_x\Sigma^{n+2}y_{x+2}\ldots.$$

In the place of the multiplier $\frac{1}{tt'}-1$, we consider generally the multiplier

$$a+\frac{bz}{tt'}+\frac{cz^2}{t^2t'^2}+\cdots,$$

and we designate by $\nabla y_x y'_x$ the function

$$ay_xy'_x+bzy_{x+1}y'_{x+1}+cz^2y_{x+2}y'_{x+2}+\cdots;$$

$uu'\left(a+\frac{bz}{tt'}+\frac{cz^2}{t^2t'^2}+\cdots\right)^n$ will be the generating function of $\nabla^n y_x y'_x$; we designate by $\phi^n\left(\frac{z}{tt'}\right)$ the function

$$\left(a+\frac{bz}{tt'}+\frac{cz^2}{t^2t'^2}+\cdots\right)^n;$$

we will have

$$\begin{aligned}uu'\phi^n\left(\frac{z}{tt'}\right)&=uu'\phi^n\left[\frac{z}{t}+\frac{z}{t}\left(\frac{1}{t'}-1\right)\right]\\&=uu'\left[\phi^n\left(\frac{z}{t}\right)+\frac{z}{t}\left(\frac{1}{t'}-1\right)\frac{\partial\phi^n\left(\frac{z}{t}\right)}{\partial z}\right.\\&\qquad\left.+\frac{z^2}{t^2}\left(\frac{1}{t'}-1\right)^2\frac{\partial^2\phi^n\left(\frac{z}{t}\right)}{1.2\,\partial z^2}\right];\end{aligned}$$

now, $uu'\phi^n\left(\frac{z}{t}\right)$ is the generating function of $y'_x\Delta^n y_x$;

$$\frac{z}{t}\left(\frac{1}{t'}-1\right)\frac{\partial\phi^n\left(\frac{1}{t}\right)}{\partial z}$$

is the generating function of $z\Delta y'_x\frac{d\nabla^n y_{x+1}}{dz}$, and thus in sequence; we will have therefore, by passing again from the generating functions to the coefficients,

$$\nabla^n(y_x y'_x) = y'_x\nabla^n y_x + z\Delta y'_x\frac{d\nabla^n y_{x+1}}{dz} + z^2\Delta^2 y'_x\frac{d^2\nabla^n y_{x+2}}{1.2\,dz^2} + \cdots$$

We have equally

$$\begin{aligned} &uu'u''\cdots\left(\frac{1}{tt't''\cdots}-1\right)^n \\ &= uu'u''\cdots\left[\left(1+\frac{1}{t}-1\right)\left(1+\frac{1}{t'}-1\right)\left(1+\frac{1}{t''}-1\right)\cdots-1\right]^n; \end{aligned}$$

by passing again therefore from the generating functions to the coefficients, we will have

$$\Delta^n(y_x y'_x y''_x\cdots) = [(1+\Delta)(1+\Delta')(1+\Delta'')\cdots-1]^n,$$

provided that, in each term of the development of the second member of this equation, we place immediately after the power of each characteristic the corresponding variable, and that next we multiply this term by the product of the variables of which it contains the characteristic not at all: thus, in the case of three variables, we will write, instead of Δ^r, $y'_x y''_x\Delta^r y_x$; instead of $\Delta^r\Delta'^{r'}$, we will write $y''_x\Delta^r y_x\Delta^{r'}y'_x$; and instead of $\Delta^r\Delta'^{r'}\Delta''^{r''}$, we will write $\Delta^r y_x\Delta^{r'}y'_x\Delta^{r''}y''_x$; and thus of the rest.

In the case of the infinitely small differences, the characteristics Δ, Δ', Δ'', ... are changed into d, d', d'', ...; and the preceding equation gives, by neglecting the superior differences, relatively to the inferiors,

$$d^n y_x y'_x y''_x\cdots = (d+d'+d''+\cdots)^n;$$

thus, in the case of two variables, we have

$$d^n y_x y'_x = d^n + nd^{n-1}d' + \frac{n(n-1)}{1.2}d^{n-2}d'^2 + \cdots,$$

and, consequently,

$$d^n y_x y'_x = y'_x d^n y_x + ndy'_x d^{n-1}y_x + \frac{n(n-1)}{1.2}d'^2 y'_x d^{n-2}y_x + \cdots;$$

by making n negative, d^n is changed into $\int^n$, and we have

$$\begin{aligned} \int^n y_x y'_x\,dx^n = &y'_x\int^n y_x\,dx^n + n\frac{dy'_x}{dx}\int^{n+1}y_x\,dx^{n+1} \\ &+ \frac{n(n-1)}{1.2}\frac{d'^2 y'_x}{dx^2}\int^{n+2}y_x\,dx^{n+2} + \cdots. \end{aligned}$$

We have again

$$uu'u''\cdots\left(\frac{1}{t^i t'^i t''^i}-1\right)^n = uu'u''\cdots\left[\left(1+\frac{1}{t}-1\right)^i\left(1+\frac{1}{t'}-1\right)^i\cdots-1\right]^n:$$

by designating therefore by $'\Delta^n(y_x y'_x y''_x\cdots)$ the finite difference of the product $y_x y'_x y''_x\cdots$ when x varies with i, the preceding equation will give, by passing again from the generating functions to the coefficients,

$$\text{(a)}\qquad '\Delta^n(y_x y'_x y''_x\cdots) = [(1+\Delta)^i(1+\Delta')^i(1+\Delta'')^i\cdots-1]^n,$$

by observing the conditions prescribed above, relatively to the characteristics Δ, Δ', ... and to their powers. We suppose $x=\frac{x'}{dx'}$, $i=\frac{\alpha}{dx'}$; y_x, y'_x, ... will become some functions of x', which we will designate by $y_{x'}$, $y'_{x'}, \ldots$; x varying from unity in y_x, x' will vary only from dx' in $\Delta y_{x'}$; thus the characteristic Δ will be changed into the differential characteristic d; but in $'\Delta y_x$, x varying from i or from $\frac{\alpha}{dx'}$, x' will vary from the finite quantity α; now we have

$$(1+d)^i = (1+d)^{\frac{\alpha}{dx'}};$$

the hyperbolic logarithm of this second member is $\frac{\alpha d}{dx'}$, this which gives, by passing again from the logarithms to the numbers,

$$(1+d)^{\frac{\alpha}{dx'}} = e^{\frac{\alpha d}{dx'}},$$

e being the number of which the hyperbolic logarithm is unity; equation (a) will give therefore

$$'\Delta^n(y_{x'} y'_{x'} y''_{x'}\cdots) = (e^{\alpha dy_{x'}+\alpha dy'_{x'}+\alpha dy''_{x'}+\cdots}-1)^n,$$

provided that, in the development of the second member of this equation, we apply to the characteristic d the exponents of the powers of $dy_{x'}$, $dy'_{x'}, \ldots$.

If, in equation (a), we suppose i infinitely small and equal to dx, x will increase from dx in $'\Delta y_x$; thus $'\Delta$ will be changed into the differential characteristic d; moreover, we have $(1+\Delta)^{dx} = 1 + dx\log(1+\Delta)$; equation (a) will become therefore

$$\frac{d^n y_x y'_x y''_x\cdots}{dx^n} = [\log(1+\Delta)(1+\Delta')(1+\Delta'')\cdots]^n,$$

by observing always the conditions prescribed above, relatively to the characteristics Δ, Δ', ... We can suppose in all these equations n negative, provided that the differential characteristics corresponding to the negative exponents are changed into integral characteristics.

On the definite integrals of the equations in partial differences.

I have given, in the *Memoirs*[2] already cited in the Académie des Sciences of the year 1779, a method to integrate in a great number of cases the equations linear in the

[2] *Oeuvres de Laplace*, T. X, p. 54 ff., "Mémoire sur les suites." Section XVIII.

finite or infinitely small partial differences, by means of definite integrals, when the integration is not possible in finite terms. Many geometers have occupied themselves since on this object, but without being subject to the condition that the expression in the definite integrals becomes the integral in finite terms, when it is possible. This condition is that which renders useful this kind of integrals, and there results from it that they have often the same advantages as the finite integrals, as I have shown in the *Mémoires* cited, relatively to the propagation of sound in a plane, and as Mr. Poisson has remarked on it next in the solution of the problem of the *vibrating chain*.

Among the equations that I have considered, is the equation in the partial differences of the second order, with constant coefficients; but it offers a particular case which is found not at all in the general solution, and which, giving place to many interesting remarks on the nature of the integrals of the equations in the partial differences, has seemed to me to merit attention of the geometers.

Let

$$0 = \frac{\partial^2 z}{\partial x^2} + a\frac{\partial^2 z}{\partial x \partial y} + b\frac{\partial^2 z}{\partial y^2} + c\frac{\partial z}{\partial x} + h\frac{\partial z}{\partial y} + lz,$$

$a,\ b,\ c,\ h,$ and l being some constant coefficients; if we make

$$s = y + fx,$$
$$s' = y + f'x;$$

the proposed equation becomes

$$\text{(b)} \qquad \begin{cases} 0 =(f^2 + af + b)\dfrac{\partial^2 z}{\partial s^2} \\ \qquad + [2ff' + a(f + f') + 2b]\dfrac{\partial^2 z}{\partial s \partial s'} + (f'^2 + af' + b)\dfrac{\partial^2 z}{\partial s'^2} \\ \qquad + (cf + h)\dfrac{\partial z}{\partial s} + (cf' + h)\dfrac{\partial z}{\partial s'} + lz; \end{cases}$$

we will make the partial differences $\frac{\partial^2 z}{\partial s^2}$ and $\frac{\partial^2 z}{\partial s'^2}$ vanish, if we take for f and f' the two roots of the equation

$$0 = u^2 + au + b;$$

then we have $f + f' = -a$ and $ff' = b$; the preceding equation becomes thus

$$0 = \frac{\partial^2 z}{\partial s \partial s'} + \frac{cf + h}{4b - a^2}\,\frac{\partial z}{\partial s} + \frac{cf' + h}{4b - a^2}\,\frac{\partial z}{\partial s'} + \frac{lz}{4b - a^2}.$$

It results from the *Mémoirs* cited[3] that, if we integrate the differential of second order,

$$0 = \frac{(4b - a^2)l - bc^2 + ahc - h^2}{(4b - a^2)^2}\mu + \frac{d\mu}{d\theta} + \theta\,\frac{d^2\mu}{d\theta^2};$$

in a way that we have $\mu = 1$,

$$\frac{d\mu}{d\theta} = \frac{bc^2 - ahc + h^2 - (rb - a^2)l}{(4b - a^2)^2},$$

[3] *Oeuvres de Laplace* T. X, p. 61, "Mémoire sur les suites." Section XIX.

when θ is null, and if we designate by $r = \daleth(\theta)$ this integral, we have

$$z = e^{\frac{(ac-2h)y+(ah-2bc)x}{4b-a^2}} \Big\{ \int dt\, \daleth[(y+f'x)(y+fx-t)]\phi(t) + \int dt\, \daleth[(y+fx)(y+f'x-t)]\psi(t) \Big\},$$

e being the number of which the hyperbolic logarithm is unity. $\phi(t)$ and $\psi(t)$ are two arbitrary functions of t: the first integral must be taken from $t = 0$ to $t = y + fx$, and the second, from $t = 0$ to $t = y + f'x$.

If we have

$$(4b - a^2)l - bc^2 + ach + h^2 = 0;$$

then $\daleth(\theta)$ is reduced to unity, and we have

$$z = e^{(ac-2h)y+(ah-2bc)x}[\phi_{\prime}(y + fx) + \psi_{\prime}(y + f'x)],$$

by designating by $\phi_{\prime}(t)$ and $\psi_{\prime}(t)$ the integrals $\int dt\, \phi(t)$ and $\int dt\, \psi(t)$; we will have then also, under finite form of indefinite integrals, the expression of z; but this is the only case in which this is possible: in all other cases the integral is possible, in finite terms, only by means of definite integrals.

The preceding analysis supposes that the two roots f and f' of the equation $0 = u^2 + au + b$ are unequal. If they are equals, then s is equal to s', and the preceding transformation of the variables x and y, into s and s', cannot take place. In this case, we suppose f null in equation (b), and f' the root of the equation $0 = u^2 + au + b$. The condition of equality of the roots of this equation gives $a^2 = 4b$, $f' = -\frac{1}{2}a$; equation (b) becomes thus

$$0 = b\frac{\partial^2 z}{\partial s^2} + (cf' + h)\frac{\partial z}{\partial s'} + h\frac{\partial z}{\partial s} + lz.$$

If we make next

$$z = ue^{\frac{-hs}{2b} - \frac{(4bl-h^2)x'}{4b^2(cf'+h)}}, \qquad s' = \left(\frac{cf'+h}{b}\right)x',$$

we will have this very simple equation

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial u}{\partial x'}.$$

I have shown, in the *Mémoires de l'Académie des Sciences* for the year 1773, page 360,[4] that its integral is impossible in finite terms, by means of indefinite integrals, and that the expression of u cannot be given by an ascendant series of indefinite integrals of an arbitrary function. We have observed since it can be by an ascendant series of differences of this kind of functions; and that which is worthy of remark, Mr. Poisson has shown that the expression of u depends only on a single arbitrary function, although the equation in the partial differences be of the second order.

[4] *Oeuvres de Laplace*, T. IX, p. 26, "Recherches sur le calcul intégral aux differences partielles."

In the delicate questions of the infinitesimal Analysis, it is very useful to consider the things relatively to the finite differences, and to see the modifications that they undergo in the passage from finite to infinitely small. This is thus what I have shown, in the *Mémoires* cited of the Académie des Sciences for the year 1779,[5] the necessity of introducing the discontinuous functions, in the integrals of the equations in partial differences, and the conditions to which these functions must be subject. I am going to employ the same method to determine the number of the arbitrary functions that the integral of the preceding equation must include.

Let u be a function of two quantities t and t', and we imagine that in the development into a series ordered with respect to the powers of t and of t', $y_{x,x'}$ is the coefficient of $t^x t^{x'}$ in this series; u will be the generating function of $y_{x,x'}$; $u\left[\left(\frac{1}{t}-1\right)^2-\left(\frac{1}{t'}-1\right)\right]$ will be the generating function of $\Delta^2 y_{x,x'} - \Delta' y_{x,x'}$, the characteristic Δ being relative to the variable x, and the characteristic Δ' to the variable x'. Let

$$\left(\frac{1}{t}-1\right)^2-\left(\frac{1}{t'}-1\right)=z;$$

we will have

$$\frac{1}{t'}=1+\left(\frac{1}{t}-1\right)^2-z,$$

that which gives

$$\begin{aligned}\frac{u}{t'^{x'}} &= u\left[1+\left(\frac{1}{t}-1\right)^2-z\right]^{x'}\\ &= u\Bigg\{1+x'\left(\frac{1}{t}-1\right)^2+\frac{x'(x'-1)}{1.2}\left(\frac{1}{t}-1\right)^4+\cdots\\ &\qquad -x'z\left[1+(x'-1)\left(\frac{1}{t}-1\right)^2+\cdots\right]+\cdots\Bigg\},\end{aligned}$$

if we have

$$\Delta^2 y_{x,x'}=\Delta' y_{x,x'},$$

the preceding equation will give, by passing again from the generating functions to the coefficients,

$$y_{x,x'}=y_{x,0}+x'\Delta^2 y_{x,0}+\frac{x'(x'-1)}{1.2}\Delta^4 y_{x,0}+\cdots;$$

thus the expression of $y_{x,x'}$ depends only on the sole arbitrary function $y_{x,0}$; in such a way that, if we have all the values of $y_{x,0}$ for all the positive and negative values of x, we will have those of $y_{x,x'}$ relative to all the values of x and x'. The integrations of equations in the finite differences are, properly speaking, only some eliminations of

[5] *Oeuvres de Laplace*, T. X, p. 80 ff., “Mémoire sur le suites”. Section XXII.

the variables given by a sequence of equations formed according to one same law. The preceding equation in the partial differences gives

$$y_{x,x'+1} = y_{x,x'} + \Delta^2 y_{x,x'},$$

by making $x' = 0$, we will have first

$$y_{x,1} = y_{x,0} + \Delta^2 y_{x,0},$$

by making next $x' = 1$, we will have

$$y_{x,2} = y_{x,1} + \Delta^2 y_{x,1},$$

by substituting for $y_{x,1}$ its value in $y_{x,0}$ given by the preceding equation, we will have

$$y_{x,2} = y_{x,0} + 2\Delta^2 y_{x,0} + \Delta^4 y_{x,0},$$

and by continuing thus, we will arrive to the preceding general expression of $y_{x,x'}$ in $y_{x,0}$. We see thence that the integral calculus in the finite differences is at base only a calculus of elimination, that which we can extend to the integral calculus of infinitely small differences, by observing in the successive eliminations, to reject the infinitely small of an order superior to the one that we conserve.

The equation in the finite differences,

$$\Delta^2 y_{x,x'} = \Delta' y_{x,x'},$$

is changed into an equation in the infinitely small differences, by substituting in it $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial x'}$, in the place of the characteristics Δ and Δ' (*Mémoires de l'Académie des Sciences*, 1779),[6] and by changing $y'_{x,x'}$ into y in it, we have

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial x'}.$$

In order to have then that which the preceding expression of $y_{x,x'}$ becomes, it is necessary, as we have seen in the *Mémoires* cited, to make $x', x' - 1, \ldots$ equal among them and to infinity; this which gives, by designating $y_{x,0}$ by $\phi(x)$,

$$y = \phi(x) + x' \frac{d^2\phi(x)}{dx} + \frac{x'^2}{1.2} \frac{d^4\phi(x)}{dx} + \cdots$$

It is moreover easy to be assured by the differentiation, that this value satisfies the proposed equation in the partial differences; but the preceding analysis indicates with evidence that the complete integral of this equation depends only on a single arbitrary function.

In order to have, under finite form, this expression, by means of definite integrals, we will observe that $\int dz\, e^{-z^2} = \sqrt{\pi}$, the integral being taken from $z = -\infty$ to

[6]*Oeuvres de Laplace*, T. X, p. 35, "Mémoire sur le suites." Section X.

$z = \infty$; π being the ratio of the semi-circumference to the radius. We will observe next that in these limits we have

$$\int z^{2r-1} dz\, e^{-z^2} = 0,$$
$$\int z^{2r}\, dz\, e^{-z^2} = \frac{1.3.5\ldots(2r-1)}{2^r}\sqrt{\pi};$$

the preceding expression of y can be put therefore under this finite form,

$$y = \frac{1}{\sqrt{\pi}} \int dz\, e^{-z^2} \phi(x + 2z\sqrt{x'}),$$

because it is clear that by developing into series, with respect to the powers of z, the function $\phi(x + 2z\sqrt{x'})$, and by integrating it, we will have the preceding expression of y; this integral satisfies thus the condition to represent exactly the series of the differences, as those that I have given in the *Mémoires* cited represent the series of indefinite integrals. It is easy moreover to assure ourselves by differentiation, that the equation

$$y = \int dz\, e^{-z^2} \phi(x + 2z\sqrt{x'})$$

satisfies the equation in the partial differences

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial x},$$

because we have

$$\frac{\partial^2 y}{\partial x^2} = \int dz\, e^{-z^2} \phi''(x + 2z\sqrt{x'}),$$

$\phi'(x)$ being equal to $\frac{d\phi(x)}{dx}$ and $\phi''(x)$ to $\frac{d\phi'(x)}{dx}$, we have next

$$\frac{\partial y}{\partial x'} = \int \frac{zdz}{\sqrt{x'}}\, e^{-z^2} \phi'(x + 2z\sqrt{x'});$$

now, by integrating by parts, we have

$$\frac{\partial y}{\partial x'} = -\frac{1}{2\sqrt{x'}} e^{-z^2} \phi'(x + 2z\sqrt{x'}) + \int dz\, e^{-z^2} \phi''(x + 2z\sqrt{x'}),$$

the integral being taken from $z = -\infty$ to $z = \infty$, $e^{-z^2}\phi'(x + 2z\sqrt{x'})$ is null at these limits; because we suppose the function $\phi'(x + 2z\sqrt{x'})$ such that its product by e^{-z^2} remains null when z is infinity; we have therefore then

$$\frac{\partial y}{\partial x'} = \int dz\, e^{-z^2} \phi''(x + 2z\sqrt{x'}) = \frac{\partial^2 y}{\partial x^2}.$$

The preceding expression of y, by means of a definite integral, is complete, although it contains only one arbitrary function; however, by developing y with respect to the powers of x, we find that we satisfy the proposed equation in the partial differences, by making

$$y = \phi(x') + \frac{x^2}{1.2}\,\frac{d\phi(x')}{dx'} + \frac{x^4}{1.2.3.4}\,\frac{d^2\phi(x')}{dx'^2} + \cdots$$
$$+ x\psi(x') + \frac{x^3}{1.2.3}\,\frac{d\psi(x')}{dx'} + \frac{x^5}{1.2.3.4.5}\,\frac{d^3\psi(x')}{dx'^3} + \cdots ;$$

$\phi(x')$ and $\psi(x')$ being two arbitrary functions of x'. This expression appears therefore, at first glance, more general than the preceding, which contains only one arbitrary function; but we are going to show that it derives from it.

We suppose that $\Gamma(x + 2z\sqrt{x'})$ is an arbitrary function which contains only some even powers of $x + 2z\sqrt{x'}$, we will satisfy by that which precedes, the proposed equation in the partial differences, by making

$$y = \int dz\, e^{-z^2}\Gamma(x + 2z\sqrt{x'}).$$

By developing this expression of y with respect to the powers of x, we will have

$$y = \int dz\, e^{-z^2}\left[\Gamma(2z\sqrt{x'}) + x\Gamma'(2z\sqrt{x'}) + \frac{x^2}{1.2}\Gamma''(2z\sqrt{x'}) + \cdots\right],$$

$\Gamma(2z\sqrt{x'})$ containing only some even powers of $2z\sqrt{x'}$, $\Gamma'(2z\sqrt{x'})$ will contain only some odd powers of the same quantity; in such a way that we will have

$$\Gamma'(-2z\sqrt{x'}) = -\Gamma'(2z\sqrt{x'}),$$

and, consequently, $\int dz\, e^{-z^2}\Gamma'(2z\sqrt{x'})$ is null in the limits $z = -\infty$ and $z = \infty$. Moreover, we have

$$\int dz\, e^{-z^2}\Gamma^{(2r)}(2z\sqrt{x'}) = \frac{e^{-z^2}}{2\sqrt{x'}}\Gamma^{(2r-1)}(2z\sqrt{x'}) + \int \frac{e^{-z^2}zdz}{\sqrt{x'}}\Gamma^{(2r-1)}(2z\sqrt{x'}).$$

The first of these two terms is null in the limits $z = -\infty$ and $z = \infty$, because we suppose generally $\Gamma^{(2r-1)}(2z\sqrt{x'})$ such that its product by e^{-z^2} vanishes when z is infinity. The term $\int \frac{e^{-z^2}zdx}{\sqrt{x'}}\Gamma^{(2r-1)}(2z\sqrt{x'})$ is equal to

$$\frac{\partial}{\partial x'} = \int e^{-z^2}dz\Gamma^{(2r-2)}(2z\sqrt{x'}),$$

we will have generally

$$\int dz\, e^{-z^2}\Gamma^{(2r)}(2z\sqrt{x'}) = \frac{d^r}{dx'^r}\int e^{-z^2}dz\Gamma(2z\sqrt{x'});$$

by designating therefore by $\phi(x')$ the integral $\int dz\, e^{-z^2}\Gamma(2z\sqrt{x'})$, we will have

$$y = \phi(x') + \frac{x^2}{1.2}\,\frac{d\phi(x')}{dx'} + \frac{x^4}{1.2.3.4}\,\frac{d^2\phi(x')}{dx'^2} + \cdots = \int dz\, e^{-z^2}\Gamma(x + 2z\sqrt{x'}).$$

If we designate now by $\Pi(x + 2z\sqrt{x'})$ a function which contains only the odd powers of $x + 2z\sqrt{x'}$, we will have

$$y = \int dz\, e^{-z^2} \left[x\Pi'(2z\sqrt{x'}) + \frac{x^3}{1.2.3}\Gamma'''(2z\sqrt{x'}) + \cdots \right],$$

a function which we will reduce, as above, to the following, by making $\int dz\, e^{-z^2} x\Pi'(2z\sqrt{x'}) = \psi(x')$,

$$y = x\psi(x') + \frac{x^3}{1.2.3}\frac{d\psi(x')}{dx'} + \cdots = \int dz\, e^{-z^2}\Pi(x + 2z\sqrt{x'}).$$

By reuniting these two expressions of y, as we may, the proposed equation in the partial differences being linear, we will have

$$\begin{aligned} y = \phi(x') &+ \frac{x^2}{1.2}\frac{d\phi(x')}{dx'} + \frac{x^4}{1.2.3.4}\frac{d^2\phi(x')}{dx'^2} + \cdots \\ &+ x\psi(x') + \frac{x^3}{1.2.3}\frac{d\psi(x')}{dx'} + \cdots \\ &= \int dz\, e^{-z^2}\left[\Gamma(x + 2z\sqrt{x'}) + \Pi(x + 2z\sqrt{x'})\right] = \int dz\, e^{-z^2}\phi(x + 2z\sqrt{x'}), \end{aligned}$$

by making

$$\phi(x + 2z\sqrt{x'}) = \Gamma(x + 2z\sqrt{x'}) + \Pi(x + 2z\sqrt{x'}).$$

We see therefore with evidence how the expression of y, which seems to contain two arbitrary functions $\phi(x')$ and $\psi(x')$, depends however only on one arbitrary function.

On the reciprocal passage of real results to imaginary results.

When the results are expressed in indeterminate quantities, the generality of the notation embraces all the cases, either reals, or imaginaries. Analysis has deduced a great part from this extension, chiefly in the calculus of the sines and cosines, which can, as we know, be represented by some imaginary exponentials. I have shown, in my *Theorie des approximations des formules qui sont fonctions de très grands nombres*, inserted into the *Mémoires de l'Académie des Sciences* for the year 1782,[7] that this passage from the real to the imaginary can yet take place, even when the results are expressed by determined quantities; and I have concluded from it the values of some definite integrals, which it would be difficult to obtain by other means. I am going to give here some new applications of this remarkable artifice.

I consider generally the integral $\int \frac{dx\, e^{x\sqrt{-1}}}{x^\alpha}$, α being positive and less than unity. Let $x = t^{\frac{1}{1-\alpha}}\sqrt{-1}$; this integral will become

$$\frac{1}{1-\alpha}(-1)^{\frac{1-\alpha}{2}} \int dt\, e^{-t^{\frac{1}{1-\alpha}}}.$$

[7] *Oeuvres de Laplace* T. X., p. 209, "Mémoire sur les approximations des formules qui sont fonctions de très-grands nombres."

By taking the first integral from $x = 0$ to x infinity; the second integral should be taken from $t = 0$ to t infinity.

We name k the integral $\int dt\, e^{-t^{\frac{1}{1-\alpha}}}$, taken within this interval; we will have

$$\int \frac{dx\, e^{x\sqrt{-1}}}{x^\alpha} = \frac{1}{1-\alpha}(-1)^{\frac{1-\alpha}{2}} k;$$

$(-1)^{\frac{1-\alpha}{2}}$ can be represented by $\cos\phi + \sqrt{-1}\sin\phi$, and then we have

$$-1 = (\cos\phi + \sqrt{-1}\sin\phi)^{\frac{2}{1-\alpha}} = \cos\frac{2}{1-\alpha}\phi + \sqrt{-1}\sin\frac{2}{1-\alpha}\phi;$$

this equation gives $\frac{2}{1-\alpha}\phi = (2r+1)\pi$, r being a positive or negative whole number, and π being the semi-circumference; we have therefore

$$\phi = (2r+1)(1-\alpha)\frac{\pi}{2},$$

and, consequently,

$$(-1)^{\frac{1-\alpha}{2}} = \cos(2r+1)(1-\alpha)\frac{\pi}{2} + \sqrt{-1}\sin(2r+1)(1-\alpha)\frac{\pi}{2};$$

we have therefore

$$\begin{aligned}\int \frac{dx\, e^{x\sqrt{-1}}}{x^\alpha} &= \int \frac{dx\cos x}{x^\alpha} + \sqrt{-1}\int \frac{dx\sin x}{x^\alpha} \\ &= \left[\cos(2r+1)(1-\alpha)\frac{\pi}{2} + \sqrt{-1}\sin(2r+1)(1-\alpha)\frac{\pi}{2}\right]\frac{k}{1-\alpha};\end{aligned}$$

by comparing the real quantities to the reals and the imaginaries to the imaginaries, we will have

$$\int \frac{dx\cos x}{x^\alpha} = \frac{k}{1-\alpha}\cos(2r+1)(1-\alpha)\frac{\pi}{2}, \tag{1}$$

$$\int \frac{dx\sin x}{x^\alpha} = \frac{k}{1-\alpha}\sin(2r+1)(1-\alpha)\frac{\pi}{2}, \tag{2}$$

the integrals being taken from x null to x infinity. Within this interval, $\int \frac{dx\sin x}{x^\alpha}$ is a positive and finite quantity, when α is less than 2. In fact, in the first semi-circumference, all the elements of the integral being positives, the entire integral is positive. In the second semi-circumference, all the elements are negatives; but the element which corresponds to $\sin x$, in the first, is $\frac{dx\sin x}{x^\alpha}$, and the element which corresponds to the same sine, in the second, is $-\frac{dx\sin x}{(\pi+x)^\alpha}$; the sum of these two elements is evidently positive; thus the sum of their integrals, taken from $x = 0$ to $x = \pi$, is positive: now, this sum is the integral $\int \frac{dx\sin x}{x^\alpha}$ taken from $x = 0$ to $x = 2\pi$; this integral, taken in the extent of the circumference, is therefore positive. We will prove in the same manner that it is positive in the extent of the second, in the third, etc. circumference; and it is the sum of all these positive quantities which form the entire integral $\int \frac{dx \sin x}{x^\alpha}$, taken from $x = 0$ to x infinity.

This integral, taken to infinity, is smaller than its value taken in the extent of the first semi-circumference. In fact, if we suppose $x = \pi + x'$, it becomes $-\int \frac{dx' \sin x'}{(\pi + x')^\alpha}$, and we will prove, as above, that this last integral taken from x' null to x' infinity is a negative quantity and, as it must be added to the integral $\int \frac{dx \sin x}{x^\alpha}$ taken within the extent of the first semi-circumference, there results from it that this last integral surpasses the entire integral taken to x infinity.

The integral $\int \frac{dx \cos x}{x^\alpha}$ is equal to $\frac{\sin x}{x^\alpha} + \alpha \int \frac{dx \sin x}{x^{\alpha+1}}$, and this last quantity is reduced to its second term, when the integrals are taken from $x = 0$ to x infinity: now, we just showed that the second integral is always positive and finite, when α is less than unity. The integral $\int \frac{dx \cos x}{x^\alpha}$ is therefore also positive and finite. All the elements of this integral are positive from $x = 0$ to $x = \frac{\pi}{2}$. By making next $x = \frac{\pi}{2} + x'$, the integral is reduced to $-\int \frac{dx' \sin x'}{(\frac{\pi}{2} + x')^\alpha}$, and we see by that which precedes, that this last integral, taken from x' null to x' infinity, is a negative quantity; the partial integral $\int \frac{dx \cos x}{x^\alpha}$, taken from x null to $x = \frac{\pi}{2}$, surpasses therefore the entire integral taken to infinity.

We resume now equations (1) and (2) and we suppose first $1 - \alpha$ infinitely small, equation (2) will give

$$\int \frac{dx \sin x}{x^\alpha} = (2r+) \frac{\pi}{2} k,$$

k is equal to the integral $\int dt\, e^{-t^{\frac{1}{1-\alpha}}}$, and this integral becomes here $\int dt\ e^{-t^\infty}$. So much as t is less than unity, e^{-t^∞} is equal to unity; and it becomes null, when t surpasses unity; k is therefore equal to unity. Now, the integral $\int \frac{dx \sin x}{x^\alpha}$ is less than this same integral, taken from $x = 0$ to $x = \pi$; and this last integral is smaller than the integral $\int \frac{x\, dx}{x}$, taken within the same interval, and, consequently, smaller than π; it is necessary therefore here to make $r = 0$ and $k = 1$, that which gives

$$\int \frac{dx \sin x}{x} = \frac{\pi}{2},$$

equation (1) gives then $\int \frac{dx \cos x}{x}$ infinity, as this must be.

If we suppose $\alpha = \frac{1}{2}$, we will have $k = \int dt\, e^{-t^2}$, and this last quantity is $\frac{1}{2}\sqrt{\pi}$, as I have shown in the *Mémoires de l'Académie des Sciences* for the year 1782;[8] equations (1) and (2) become therefore

$$\int \frac{dx \cos x}{\sqrt{x}} = \sqrt{\pi} \cos \frac{2r+1}{4} \pi,$$
$$\int \frac{dx \sin x}{\sqrt{x}} = \sqrt{\pi} \sin \frac{2r+1}{4} \pi,$$

the sine and the cosine of $\frac{(2r+1)\pi}{4}$ must therefore be positives, that which supposes r

[8] *Oeuvres de Laplace,* T. X., p. 223. "Mémoire sur les approximations des formules qui sont fonctions de très-grands nombres." Section IV.

null or a multiple of 4; then, we have

$$\sin\frac{(2r+1)\pi}{4} = \cos\frac{(2r+1)\pi}{4} = \frac{1}{\sqrt{2}},$$

hence

$$\int\frac{dx\sin x}{\sqrt{x}} = \int\frac{dx\cos x}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}.$$

Mascheroni, in a work entitled *Annotations in Calculum integralem Euleri*, has found $\int\frac{dx\cos x}{\sqrt{x}} = \sqrt{2\pi}$; but this value is evidently too great, because we have seen that $\int\frac{dx\cos x}{\sqrt{x}}$ is less than the partial integral, taken from $x = 0$ to $x = \frac{\pi}{2}$, and this partial integral is smaller itself than the integral $\int\frac{dx}{\sqrt{x}}$, taken in the same interval: now, this last integral is $\sqrt{2\pi}$; therefore $\int\frac{dx\cos x}{\sqrt{x}}$ is less than $\sqrt{2\pi}$.

If $\alpha = \frac{3}{4}$, we will have

$$k = \int dt\, e^{-t^4}.$$

By naming π' the integral $\int\frac{du}{(1-u^4)^{\frac{1}{2}}}$, taken from $u = 0$ to $u = 1$, we have

$$\pi' = 1.311\,028\,777\,246\,059\,87$$

and

$$k = \frac{1}{2}\sqrt{\pi'\sqrt{2\pi}} = 0.906\,402$$

(*Mémoires de l'Académie des Sciences*, 1782, p. 21)[9]; we have next

$$\int\frac{dx\cos x}{x^{\frac{3}{4}}} = 4k\cos\frac{(2r+1)}{4}\,\frac{\pi}{2},$$
$$\int\frac{dx\sin x}{x^{\frac{3}{4}}} = 4k\sin\frac{(2r+1)}{4}\,\frac{\pi}{2}.$$

Here, we can suppose again r null; because the integral $\int\frac{dx\sin x}{x^{\frac{3}{4}}}$ must be contained between the integrals $\int\frac{dx\sin x}{x^{\frac{1}{2}}}$ and $\int\frac{dx\sin x}{x}$, and this is that which holds by supposing r null, because then these three integrals are 1.2533; 1.3875; 1.5708: the value of the integral $\int\frac{dx\cos x}{x^{\frac{3}{4}}}$ is 3.34963.

If α is infinitely small, then $k = \int dt\, e^{-t^{\frac{1}{1-\alpha}}} = \int dt\, e^{-t} = 1$, next we have

$$\int\frac{dx\sin x}{x^\alpha} = \sin(2r+1)\,\frac{\pi}{2} = 1,$$
$$\int\frac{dx\cos x}{x^\alpha} = \sin(2r+1)\,\alpha\,\frac{\pi}{2} = (2r+1)\frac{\alpha\pi}{2};$$

[9] *Oeuvres de Laplace*, T. X, p. 226. "Mémoire sur les approximations des formules qui sont fonctions de très-grands nombres." Section V.

now we have, for that which precedes,

$$\int \frac{dx \cos x}{x^\alpha} = \alpha \int \frac{dx \sin x}{x^{\alpha+1}},$$

and, in the case of α infinitely small,

$$\int \frac{dx \sin x}{x^{\alpha+1}} = \int \frac{dx \sin x}{x} = \frac{\pi}{2},$$

therefore

$$\int \frac{dx \cos x}{x^\alpha} = \frac{\alpha\pi}{2}.$$

By comparing this value to the preceding, we see that r must be supposed null.

We consider again the case of $\alpha = \frac{1}{4}$. In this case, we have

$$k = \int dt\, e^{-t^{\frac{4}{3}}};$$

by making $t = t'^3$, we will have

$$k = 3 \int dt'\, t'^2\, e^{-t'^4};$$

now we have (page cited in the *Mémoires de l'Académie des Sciences*)

$$16 \int dt\, e^{-t^4} \int dt' t'^2\, e^{-t'^4} = \pi\sqrt{2},$$

we will have therefore

$$k = \frac{3\pi\sqrt{2}}{16 \int dt\, e^{-t^4}} = 0.919\,062.$$

We can again here suppose $r = 0$, because $\int \frac{dx \sin x}{x^{\frac{1}{4}}}$ must be contained between $\int \frac{dx \sin x}{x^\alpha}$, α being infinitely small, and $\int \frac{dx \sin x}{x^{\frac{1}{2}}}$; we have thus[10]

$$\int \frac{dx \sin x}{x^{\frac{1}{4}}} = 1.1321, \qquad \int \frac{dx \cos x}{x^{\frac{1}{4}}} = 0.4689.$$

[10] *Translator's note*: It may be observed that the value of these integrals are respectively $\sqrt{\pi}\, 2^{\frac{3}{4}}\, \Gamma\left(\frac{7}{8}\right) / 2\Gamma\left(\frac{5}{8}\right) \approx 1.132\,137$ and $\sqrt{\pi}\, 2^{\frac{3}{4}} \csc\left(\frac{3\pi}{8}\right) \sin\left(\frac{\pi}{8}\right) \Gamma\left(\frac{7}{8}\right) / 2\Gamma\left(\frac{5}{8}\right) \approx 0.468\,947$. Likewise, the other values appearing in the table have closed-form representations.

If we assemble these diverse results, we will form the following Table from them:

α	$\int \frac{dx \sin x}{x^\alpha}$	$\int \frac{dx \cos x}{x^\alpha}$
0	1.0000	0.0000
$\frac{1}{4}$	1.1321	0.4689
$\frac{2}{4}$	1.2533	1.2533
$\frac{3}{4}$	1.3875	3.34963
$\frac{4}{4}$	1.5708	∞
$\frac{5}{4}$	1.8756	∞
$\frac{6}{4}$	2.2507	∞
$\frac{7}{4}$	4.4662	∞
$\frac{8}{4}$	∞	∞

Thence we can generally conclude that, in equations (1) and (2), r can be supposed null, and then they become

$$\int \frac{dx \cos x}{x^\alpha} = \frac{k}{1-\alpha} \sin \frac{\alpha\pi}{2}, \tag{3}$$

$$\int \frac{dx \sin x}{x^\alpha} = \frac{k}{1-\alpha} \cos \frac{\alpha\pi}{2}, \tag{4}$$

we must join the equation

$$\int \frac{dx \sin x}{x^{\alpha+1}} = \frac{k}{\alpha(1-\alpha)} \sin \frac{\alpha\pi}{2}. \tag{5}$$

In order to give an application of this analysis, we consider an elastic blade folded naturally onto itself in form of a spiral. We imagine that its interior extremity is fixed, and that the blade can be developed into a horizontal line, by a weight p suspended at its other extremity. In this state, the action of the weight on an element of the blade, placed at the distance s from the extremity, will be ps; and the elasticity of the element must make equilibrium to it. This elasticity is reciprocal to the osculating radius of the blade in its natural state. By naming therefore r this radius relative to the part s of the blade, taken from its exterior extremity, we will have

$$ps = \frac{g}{r},$$

g being a constant depending on the elasticity proper to the blade. We will make $\frac{g}{p} = a^2$, a being a right, in order to conserve the homogeneity of the dimensions; we will have therefore, in the natural state of the blade,

$$s = \frac{a^2}{r}.$$

Now we imagine in this state, and by the exterior extremity of the blade, two orthogonal coordinates x and y of which the first is, at this origin, tangent to the blade; we will have

$$\frac{ds}{r} = \frac{d\frac{dy}{ds}}{\sqrt{1 - \frac{dy^2}{ds^2}}},$$

this which gives

$$\frac{dy}{ds} = \sin\left(\int \frac{ds}{r}\right),$$

and, consequently,

$$\frac{dx}{ds} = \cos\left(\int \frac{ds}{r}\right);$$

substituting for $\frac{1}{r}$ its value $\frac{s}{2a^2}$, we will have

$$x = \int ds \cos\frac{s^2}{2a^2}, \qquad y = \int ds \sin\frac{s^2}{2a^2}.$$

Euler arrived to the same equations, in this beautiful work *Sur les isopérimètres*,[11] page 276; but he adds: *"Curva ergo erit ex spiralium genere, ita ut infinitis peractis spiris, in certo quodam puncto tanquam centro convolvatur, quod punctum ex hâc constructione invenire difficillimum videtur."* The determination of this point is deduced easily from the preceding analysis; because, by making $\frac{s^2}{2a^2} = \phi$, we will have

$$ds = a\frac{d\phi}{\sqrt{2\phi}} \quad \text{and} \quad x = a\int \frac{d\phi}{\sqrt{2\phi}}\cos\phi, \quad y = a\int \frac{d\phi\sin\phi}{\sqrt{2\phi}},$$

the integrals being taken from ϕ null to ϕ infinity; then, we have, by that which precedes,

$$x = y = \frac{1}{2}a\sqrt{\pi}.$$

We can generalize the preceding analysis, by applying to the integral

$$\int \frac{dx}{x^\alpha} e^{-fx+gx\sqrt{-1}}.$$

If we make

$$fx - gx\sqrt{-1} = t^{\frac{1}{1-\alpha}},$$

the integral becomes

$$\int \frac{dt\, e^{-t^{\frac{1}{1-\alpha}}}}{(1-\alpha)(f-g\sqrt{-1})^{1-\alpha}};$$

[11] *Translator's note*: This is E65: *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti, Additamentum* I, 1744. Euler derives the integrals but is unable to evaluate them.

by naming therefore, as above, k the integral $\int dt\, e^{-t^{\frac{1}{1-\alpha}}}$ taken from t null to t infinity, and substituting, in the place of $e^{gx\sqrt{-1}}$, $\cos gx + \sqrt{-1}\sin gx$, we will have

$$\int \frac{dx\, e^{-fx}}{x^\alpha}(\cos gx + \sqrt{-1}\sin gx) = \frac{k}{(1-\alpha)(f - g\sqrt{-1})^{1-\alpha}}, \tag{6}$$

the integral being taken from x null to x infinity.

We represent the fraction $\frac{1}{(f-g\sqrt{-1})^{1-\alpha}}$, by $h(\cos\phi + \sqrt{-1}\sin\phi)$; we will have

$$f - g\sqrt{-1} = h^{\frac{1}{\alpha-1}}\left(\cos\frac{\phi}{1-\alpha} - \sqrt{-1}\sin\frac{\phi}{1-\alpha}\right),$$

that which gives

$$h^{\frac{1}{\alpha-1}}\cos\frac{\phi}{1-\alpha} = f,$$
$$h^{\frac{1}{\alpha-1}}\sin\frac{\phi}{1-\alpha} = g,$$

whence we deduce

$$\tan\frac{\phi}{1-\alpha} = \frac{g}{f},$$
$$h = (f^2+g^2)^{\frac{\alpha-1}{2}}.$$

The first equation gives

$$\phi = (A + r\pi)(1-\alpha),$$

A being the first small positive angle of which $\frac{g}{f}$ is the tangent, and r being a whole number, which we must suppose null, according to that which precedes. This put, equation (6) will give the following two

$$\int \frac{dx\, e^{-fx}\cos gx}{x^\alpha} = \frac{k\cos A}{(1-\alpha)(f^2+g^2)^{\frac{1-\alpha}{2}}}, \tag{7}$$

$$\int \frac{dx\, e^{-fx}\sin gx}{x^\alpha} = \frac{k\sin A}{(1-\alpha)(f^2+g^2)^{\frac{1-\alpha}{2}}}. \tag{8}$$

We have, by taking the integrals from x null to x infinity,

$$\int \frac{dx\, e^{-fx}\cos gx}{x^\alpha} = \int \frac{f}{g}\frac{dx\, e^{-fx}\sin gx}{x^\alpha} + \int \frac{\alpha}{g}\frac{dx\, e^{-fx}\sin gx}{x^{\alpha+1}};$$

we will have therefore

$$\int \frac{dx\, e^{-fx}\sin gx}{x^{\alpha+1}} = \frac{k}{\alpha(1-\alpha)(f^2+g^2)^{\frac{\alpha-1}{2}}}(g\cos A - f\sin A); \tag{9}$$

by supposing f null and $g = 1$, we have

$$\tan\frac{\phi}{1-\alpha} = \frac{1}{0} = \infty,$$

that which gives

$$\frac{\phi}{1-\alpha} = \frac{\pi}{2},$$

and, consequently,

$$A = \frac{\pi}{2}(1-\alpha);$$

then it is clear that equations (7), (8), (9) coincide with equations (3), (4), (5).

On the integration of equations in finite differences, non-linear.

Until the present, the geometers have occupied themselves principally with equations in finite differences, linears; these are, in fact, those which present themselves most frequently in this kind of analysis: but the consideration of the non-linear equations can be useful, I am going to expose here a method to integrate them in many cases.

I have already observed that the integration of the equations in the finite differences is, at base, only an elimination among any number of similar equations. By designating therefore by $x^{(n)}$ and $x^{(n+1)}$ the two variables of an equation given among them, this equation will be changed into an equation in the finite differences. In order to integrate it, we differentiate this equation with respect to the infinitely small differences $dx^{(n)}$ and $dx^{(n+1)}$; we could, by means of the proposed equation and of its differential, arrive to an equation of this form,

$$dx^{(n)}\,\phi(x^{(n)}) = dx^{(n+1)}\,\psi(x^{(n+1)}),$$

and, consequently, to the equation

$$\int dx^{(n+1)}\,\psi(x^{(n+1)}) - \int dx^{(n)}\,\phi(x^{(n)}) = a.$$

This equation is only a transformed from the proposed, but in which the two variables are separated.

If, in the proposed, the two variables $x^{(n)}$ and $x^{(n+1)}$ enter in a manner that we have $\psi(x^{(n+1)}) = \phi(x^{(n+1)})$, the transformed will become

$$\int dx^{(n+1)}\,\psi(x^{(n+1)}) - \int dx^{(n)}\,\phi(x^{(n)}) = a,$$

and, by integrating

$$\int dx^{(n)}\,\phi(x^{(n)}) = an + b,$$

b being the arbitrary constant introduced by the integration; we will have thus $x^{(n)}$ a function of $an + b$. We apply this method to some examples.

We consider first the equation

$$0 = 1 - \beta(x - y) + xy,$$

that which gives

$$\beta = \frac{1 + xy}{x - y};$$

by differentiating, we will have

$$\frac{dx}{1 + x^2} - \frac{dy}{1 + y^2} = 0,$$

and, consequently,

$$\int \frac{dx}{1 + x^2} - \int \frac{dy}{1 + y^2} = a,$$

a being a constant which must be a function of β; because this last equation is only a transformed of the proposed. Now, if we make $x = x^{(n+1)}$, $y = x^{(n)}$, this proposed is changed into the equation in the finite differences,

$$0 = 1 - \beta(x^{(n+1)} - x^{(n)}) + x^{(n+1)}x^{(n)},$$

or

$$0 = 1 + (x^{(n)} - \beta)\Delta x^{(n)} + x^{(n)^2}.$$

Its transformed becomes

$$a = \int \frac{dx^{(n+1)}}{1 + x^{(n+1)^2}} - \int \frac{dx^{(n)}}{1 + x^{(n)^2}},$$

or

$$\Delta \int \frac{dx^{(n)}}{1 + x^{(n)^2}} = a;$$

by integrating it, we will have

$$\int \frac{dx^{(n)}}{1 + x^{(n)^2}} = an + b,$$

b being the arbitrary constant introduced by the integration in the finite differences. The integral $\int \frac{dx^{(n)}}{1+x^{(n)^2}}$ is, as we know, $\arctan x^{(n)}$; thus we have

$$x^{(n)} = \tan(an + b).$$

In order to determine a, we suppose n and b such that $an + b$ is null; we will have $x^{(n)}$ null, and $x^{(n+1)} = \tan a$: now, the preceding equation in the finite differences gives, when $x^{(n)}$ is null,

$$x^{(n+1)} = \frac{1}{\beta} = \tan a;$$

a is therefore the angle of which the tangent is $\frac{1}{\beta}$. The arbitrary b is the angle of which the tangent is $x^{(0)}$.

We consider now the equation

$$0 = 1 - \beta(x^2 + y^2) + 2\gamma xy + x^2y^2,$$

it gives, by differentiating it,

$$\frac{dx^2}{dy^2} = \frac{(x^2y - \beta y + \gamma x)^2}{(xy^2 - \beta x + \gamma y)^2} = \frac{y^2(x^2-\beta)^2 + 2\gamma xy(x^2-\beta) + \gamma^2x^2}{x^2(y^2-\beta)^2 + 2\gamma xy(y^2-\beta) + \gamma^2y^2}.$$

Substituting into the numerator, for x^2 its value $\frac{1-\beta y^2+2\gamma xy}{\beta - y^2}$, and into the denominator, for y^2 its value $\frac{1-\beta x^2+2\gamma xy}{\beta - x^2}$, we will have

$$\frac{dx^2}{dy^2} = \frac{1 - \frac{1+\beta^2-\gamma^2}{\beta}x^2 + x^4}{1 - \frac{1+\beta^2-\gamma^2}{\beta}y^2 + y^4},$$

by making therefore

$$2\alpha = \frac{1+\beta^2-\gamma^2}{\beta},$$

we will have

$$\frac{dx}{\sqrt{1-2\alpha x^2+x^4}} - \frac{dy}{\sqrt{1-2\alpha y^2+y^4}} = 0,$$

and by integrating it, we will have the following equation, which is only a transformed of the proposed equation,

$$\int \frac{dx}{\sqrt{1-2\alpha x^2+x^4}} - \int \frac{dy}{\sqrt{1-2\alpha y^2+y^4}} = a,$$

If we make now $x = x^{(n+1)}$, $y = x^{(n)}$; the proposed will be changed into the equation in the finite differences,

$$0 = 1 - \beta(x^{(n+1)^2} + x^{(n)^2}) + 2\gamma x^{(n+1)}x^{(n)} + x^{(n+1)^2}x^{(n)^2},$$

and its transformed will become

$$\Delta \int \frac{dx^{(n)}}{\sqrt{1-2\alpha x^{(n)^2} + x^{(n)^4}}} = a,$$

whence we deduce, by integrating in the finite differences,

$$\int \frac{dx^{(n)}}{\sqrt{1-2\alpha x^{(n)^2} + x^{(n)^4}}} = an + b,$$

b being an arbitrary constant, which is equal to

$$\int \frac{dx^{(0)}}{\sqrt{1-2\alpha x^{(0)^2} + x^{(0)^4}}}.$$

In order to determine a, we will designate by $\psi(x^{(n)})$ the integral

$$\int \frac{dx^{(n)}}{\sqrt{1 - 2\alpha x^{(n)^2} + x^{(n)^4}}},$$

and a by $\psi(q)$; we will have

$$\psi(x^{(n+1)}) - \psi(x^{(n)}) = \psi(q).$$

We suppose that $\psi(x)$ is null, when x is null; we will have, by making $x^{(0)}$ null,

$$\psi(x^{(1)}) = \psi(q) \quad \text{or} \quad q = x^{(1)};$$

now, the proposed gives then $x^{(1)} = \frac{1}{\sqrt{\beta}}$, therefore

$$q = \frac{1}{\sqrt{\beta}},$$

hence

$$\psi(x^{(n)}) = n\psi\left(\frac{1}{\sqrt{\beta}}\right) + \psi(x^{(0)}).$$

In deducing from this equation the value of $x^{(n)}$, we will have the integral of the proposed. The value of $\psi(x^{(n)})$, in algebraic, circular or logarithmic quantities, is impossible in finite terms: it is therefore impossible to represent alternately than by a characteristic the expression of $x^{(n)}$; but it is remarkable that it depends on the rectification of the conic sections.

We can similarly integrate by a transcendent quadrature the general equation in the finite differences,

$$\begin{aligned} 0 = a &+ b(x^{(n+1)} + x^{(n)}) + c(x^{(n+1)^2} + x^{(n)^2}) + fx^{(n+1)}x^{(n)} \\ &+ gx^{(n+1)}x^{(n)}(x^{(n+1)} + x^{(n)}) + hx^{(n+1)^2}x^{(n)^2}, \end{aligned}$$

because, if we make

$$x^{(n)} = \frac{lx'^{(n)} + p}{x'^{(n)} + q},$$

we will have a differential equation in $x'^{(n)}$ in the same form as the preceding; and, by determining conveniently the three arbitraries $l,\ p$ and q, we could make the coefficients of $(x'^{(n+1)} + x'^{(n)})$ and of $x'^{(n+1)}x'^{(n)}(x'^{(n+1)} + x'^{(n)})$ vanish, and render equal the constant coefficient and the one of $x'^{(n)^2}x'^{(n+1)^2}$. The differential equation is then reduced to the form of that which we just integrated.

On the reduction of functions into Tables.

In order to reduce to Tables the values of a function in one variable alone, we give to this variable some successive numeric values, and such that its increments are very small and equal among themselves. We place next beside each increment the corresponding value of the function. A Table thus formed is named *Table in simple entry*. It gives not only the values of the function, corresponding to the indicated increments of the variable, but again those which correspond to the intermediate increments: a simple proportion, or, if we wish more exactitude, the method of the differences make known the intermediate values of the function.

If the function contains two variables x and y, then, by giving to x a determined value, we make y increment successively, and we will place the corresponding value of the function beside each increment. We will form thus, for each value of x, a Table in simple entry, and the union of these Tables corresponding to the successive increments of x, will form a Table in *double entry*, which will represent the proposed function, in x and y.

The Table of Pythagoras, which gives the product xy of the two numbers x and y, is the simplest case of this kind of Tables; and by prolonging it to a considerable number, it will give the products of the large numbers; but then it would be encumbering by its excessive extent: we would facilitate therefore extremely the numerical calculus, by reducing it to a Table in simple entry.

In order to arrive to it, it would be necessary to be able to reduce xy to one or many functions of the from $\phi(X+Y)$, X being a function of x and Y being a function of y. Then we would have X, by means of the value of x, by a Table in simple entry; the same Table would give again Y, by means of the values of y; because, in the present case, Y is a function of y, entirely similar to that of X in x. Finally, a Table in simple entry would give again xy, by means of the values of $X+Y$.

We see now if this reduction of xy is possible. We suppose

$$xy = \phi(X+Y).$$

By differentiating this equation with respect to x, we will have

$$y\,\frac{dx}{dX} = \phi'(X+Y),$$

by designating $\frac{d\,\phi(x)}{dz}$ by $\phi'\,(z)$. We will have similarly, by differentiating with respect to y,

$$x\,\frac{dy}{dY} = \phi'\,(X+Y).$$

The comparison of these two equations give

$$\frac{dx}{x\,dX} = \frac{dy}{y\,dY},$$

the first member of this equation being a function of x alone, and the second member being a function of y alone; it is clear that the two variables x and y being independents, each of these members must be equal to one same constant which we will indicate by q; we will have therefore

$$\frac{dx}{x\,dX} = q = \frac{dy}{y\,dY}.$$

The integrals of these equations are evidently

$$x = Ae^{qX}, \qquad y = Be^{qY}.$$

A and B being two arbitrary constants, because it is clear that we have

$$dx = Ae^{qX}(e^{q dX} - 1) = x(e^{q dX} - 1),$$

that which gives

$$\frac{dx}{x\,dX} = q,$$

if we have

$$e^{q dX} - 1 = q\,dX \qquad \text{or} \qquad e = (1 + q\,dX)^{\frac{1}{q\,dX}}.$$

By developing the second member into series, by the known theorem of the binomial, and neglecting unity, having regard to $\frac{1}{q\,dX}$, we will have

$$e = 2 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \cdots = 2.71821;$$

we will have thus the three equations

$$x = Ae^{qX}, \quad y = Be^{qY}, \quad xy = ABe^{q(X+Y)};$$

and it is clear that the Tables in simple entry, of which we have spoken above, will be reduced to one alone, if we make $A = B = 1$; and then X and Y are nulls, when x and y are equal to unity.

The Table in simple entry, which we obtain in this manner, is a Table of logarithms, X being the logarithm of x. The logarithms are hyperbolic, if q is equal to unity, that is to say if the infinitely small increment of the logarithm X is equal to the one of the number x, when x is equal to unity. The logarithms are those which we name *tabular*, if q is such that we have $e^q = 10$. This value of q offers the advantage of giving the logarithms of the numbers ten, one hundred, one thousand, etc. times greater or lesser, by adding or subtracting from these logarithms 1, or 2, or 3, ...

If we employ two functions to represent xy, if, for example, we suppose

$$xy = \phi(X + Y) - \phi(X - Y),$$

we will have

$$y\frac{d^2x}{dX^2} = x\frac{d^2y}{dY^2} = \phi''(X + Y) - \phi''(X - Y),$$

$\phi''(z)$ being equal to $\frac{d\phi'(z)}{dz}$. We will have therefore

$$\frac{d^2x}{dX^2} + a^2x = 0,$$
$$\frac{d^2y}{dY^2} + a^2y = 0;$$

a being any constant. The simplest case is the one of a null, and then we can suppose $x = X$, $y = Y$; that which gives

$$0 = \phi''(X + Y) - \phi''(X - Y).$$

Thus, $\phi''(X+Y)$ is equal to a constant and, consequently, $\phi(X+Y)$ is of the form $b(X+Y)^2 + p(X+Y) + q$; b, p and q being some constants. The preceding expression of xy will determine these constants, and it will give

$$xy = \frac{1}{2}[(x+y)^2 - (x-y)^2].$$

By forming a Table in simple entry, of the function $\frac{1}{2}t^2$, the difference of the two numbers which correspond in this Table to $t = x+y$ and $t = x-y$ or $y-x$, according as X will be greater or lesser than Y; this difference, I say, will be the product xy.

By making $a = 1$, the equations

$$\frac{d^2x}{dX^2} + x = 0, \qquad \frac{d^2y}{dY^2} + y = 0$$

will be satisfied, by making $x = \sin X$, $y = \sin Y$; and then we will have

$$xy = \frac{1}{2}[\cos(X-Y) - \cos(X+Y)];$$

we can therefore, by means of a Table of sines and of cosines, determine the product of the two numbers x and y; we will determine the angles X and Y by means of their sines x and y, and by taking in the Table the cosines of the angles $X-Y$ and $X+Y$, their semi-difference will be the product xy. This ingenious manner to make the Tables of sines serve to the multiplication of numbers was imagined and employed around a century before the invention of the logarithms, which, as we have just seen, depends only on a single function $\phi(X+Y)$, is much simpler and renders very easy the division of numbers, their elevation to the powers and the extraction of their roots; because we have

$$\frac{x}{y} = e^{q(X-Y)} \quad \text{and} \quad x^n = e^{qnX};$$

thus division is reduced to a subtraction; the elevation to the powers is reduced to a multiplication and the extraction of the roots to a division.

The facility of all these calculations renders the logarithms one of the most powerful instruments of the human mind and, when the metric system will be generally adopted, they will become of common usage in the Society, to which they will be as useful as our arithmetic scale, of which this system is the complement. We must therefore multiply, as much as is possible, the uses of logarithms and by their means reduce to Tables in simple entry the Tables in double entry. This is that which I have done in regard to the Table of the astronomic refractions, published by the Bureau of Longitudes, and in which the formula of the refractions, which I have given in the tenth Book of the *Mécanique céleste*, is reduced in this manner to some Tables of simple entry. Mr. Oltmans has made next the same thing in regard to the formula of the elevations concluded from the barometric observations.

We can generalize the preceding analysis, by considering any function $\phi(X+Y)$. We suppose that we have generally

$$u = \phi(X+Y),$$

by differentiating, we will have

$$\frac{\partial u}{\partial x}\,\frac{dx}{dX} = \phi'(X+Y), \qquad \frac{\partial u}{\partial y}\,\frac{dy}{dY} = \phi'(X+Y),$$

hence

$$\frac{\frac{dx}{dX}}{\frac{dy}{dY}} = \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}.$$

It is necessary, therefore, in order that the reduction be possible, that the quotient of $\frac{\partial u}{\partial x}$, divided by $\frac{\partial u}{\partial y}$, be of the form $\frac{\mathbf{S}}{\mathbf{T}}$, **S** being a function of x, and **T** a function of y. The differential equation

$$0 = \mathbf{S}\,dx + \mathbf{T}\,dy$$

has for integral

$$\text{const.} = \int \mathbf{S}\,dx + \int \mathbf{T}\,dy,$$

it has therefore also for integral $u =$const.; thus every finite equation in x and y, which, moreover, containing an arbitrary, satisfies the preceding equation, gives for the expression of this constant a function of x and y, of which we can determine the values by means of a Table in simple entry.

We have seen previously that the equation

$$0 = 1 - \beta(x^2 + y^2) + 2\gamma xy + x^2y^2$$

gives the following,

$$0 = \frac{dx}{\sqrt{1 - 2\alpha x^2 + x^4}} - \frac{dy}{\sqrt{1 - 2\alpha y^2 + y^4}},$$

in which

$$\alpha = \frac{1 + \beta^2 - \gamma^2}{2\beta},$$

and as α is given as function of the two constants β and γ, the finite preceding equation contains an arbitrary constant and, consequently, it is the complete integral of the differential equation. In fact, if we suppose $\frac{1}{\beta} = a^2$, the finite equation becomes

$$0 = a^2 - (x^2 + y^2) + 2xy\sqrt{1 - 2\alpha a^2 + a^4} + a^2s^2y^2,$$

a being an arbitrary constant which is met not at all in the differential equation. This equation gives

$$a = \frac{x\sqrt{Y} - y\sqrt{X}}{1 - x^2y^2},$$

Y being equal to $1 - 2\alpha y^2 + y^4$, and X being $1 - 2\alpha x^2 + x^4$. We have moreover, by that which precedes,

$$\psi(x) = \psi(y) + \psi(a),$$

$\psi(x)$ being the integral $\int \frac{dx}{\sqrt{X}}$, this integral commencing with x; by forming therefore a Table in simple entry of the values of $\psi(x)$, this Table will give the values of $\frac{x\sqrt{Y}-y\sqrt{X}}{1-x^2y^2}$ or of a; because the difference $\psi(x)-\psi(y)$ being equal to $\psi(a)$, this Table will give the value of a. We can likewise, by means of a second Table in simple entry, which gives the values of any function $\Gamma(x)$ of x, have that of $\Gamma\left(\frac{x\sqrt{Y}-y\sqrt{X}}{1-x^2y^2}\right)$.

If we make $A = 1-2\alpha a^2 + a^4$, the preceding algebraic equation will give

$$x = \frac{y\sqrt{A}+a\sqrt{Y}}{1-a^2y^2},$$

by changing x into $x^{(n)}$ and y into $x^{(n-1)}$, we will have

$$x^{(n)} = \frac{x^{(n-1)}\sqrt{A}+a\sqrt{X^{(n-1)}}}{1-a^2x^{(n-1)^2}}.$$

$X^{(n)}$ being that which X becomes when we change x into $x^{(n)}$ in it. We will have, by means of this equation, the value of $x^{(n)}$ in $x^{(0)}$ and a; because it will give the value of $x^{(1)}$ as a function of these two quantities; next it gives $x^{(2)}$ as a function of $x^{(1)}$ and of a, and, consequently, as function of $x^{(0)}$ and of a, by substituting for $x^{(1)}$ its value, and thus in sequence. We will have therefore $x^{(n)}$ as a function of $x^{(0)}$, a and n; now we have, by that which we have seen above,

$$\psi(x^{(n)}) = n\psi(a) + \psi(x^{(0)});$$

by designating therefore the reverse sign $\rotatebox[origin=c]{180}{ψ}$ the value of $x^{(n)}$ in $\psi(x^{(n)})$, such that we have

$$x^{(n)} = \rotatebox[origin=c]{180}{ψ}[\psi(x^{(n)})];$$

we will have

$$x^{(n)} = \rotatebox[origin=c]{180}{ψ}[n\psi(a)+\psi(x^{(0)})].$$

The Table in simple entry which gives $\psi(x)$ in x will give therefore the value of $x^{(n)}$.

We suppose $\alpha = -1$; we will have

$$X = (1+x^2)^2, \quad x^{(n)} = \frac{x^{(n-1)}+a}{1-ax^{(n-1)}}, \quad \psi(x) = \int \frac{dx}{1+x^2} = \arctan x;$$

$\psi(x)$ will be therefore $\arctan x$ and, consequently, $\rotatebox[origin=c]{180}{ψ}(x)$ will be $\tan x$; we will have therefore

$$x^{(n)} = \tan(n\arctan a + \arctan x^{(0)}),$$

thus the Table of tangents will give generally the value of $x^{(n)}$ or the value of the integral of the equation in finite differences,

$$0 = a - (x^{(n+1)} - x^{(n)}) + ax^{(n+1)}x^{(n)}.$$