SUR LES COMÈTES

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Among the hypotheses which one has proposed on the origin of the comets, the most probable appears to me to be that of Mr. Herschell, which consists in regarding them as small nebulae formed by the condensation of the nebulous material spread with so much profusion in the universe. The comets would be thus, relative to the solar system, that which the aerolites are with respect to the Earth, to which they seem strangers. When these stars become visible for us, they offer a resemblance so perfect with the nebula that one often confounds them with them, and it is only by their movement, or by the knowledge of all the nebulae contained in the part of the sky where they are indicated, that one succeeds in distinguishing them in it. This hypothesis demonstrates in a happy manner the great extension that the heads and the tails of the comets take, in measure as they approach the Sun, and the extreme rarity of these tails which, despite their immense profusion, never weaken sensibly the radiance of the stars that one sees to intersect, so that it is very probable that many have enveloped the Earth without having been perceived.

When the nebulae arrive in this part of space where the attraction of the Sun is predominant, and what we will call *sphere of activity* of this star, it forces them to describe some elliptical or hyperbolic orbits. But their speed being equally possible following all directions, they must be moved indifferently in all senses and in all inclinations to the ecliptic, this which is conformed to that which one observes. If their orbits are elliptic, they are very elongated; since their major axes are at least equal to the radius of the sphere of activity of the Sun; but these orbits can be hyberbolic, and if the axes of these hyperboles are not very great with respect to the mean distance of the Sun to the Earth, the movement of the comets which describe them will seem sensibly hyperbolic. However, out of one hundred comets of which one has the elements already, none has appeared to move itself in an hyperbola, this which forms a plausible objection against the preceding hypothesis, at least that the chances which give a sensible hyperbola are extremely rare with respect to the contrary chances. The conformity of this hypothesis with the phenomena which the comets offer to us has made me suspect that this is so, and, in order to assure myself of it, I have applied to this object the Calculus of probabilities. I have found that in effect there are odds a great number against unity that a nebula which penetrates into the sphere of solar activity, in a manner to be able to be observed, will describe either a very elongated ellipse or a hyperbola which, by the

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. September 17, 2010 magnitude of its axis, will be confounded sensibly with a parabola in the part which one observes. This application of the analysis of probabilities being able to interest geometers and astronomers, I am going to expose here.

The comets are so small that they become visible only if their perihelion distance is not very considerable. Until the present, this distance has surpassed only two times the diameter of the terrestrial orbit, and most often it has been under the radius of this orbit. One imagines that, in order to approach so near to the Sun, their speed at the moment of their entry into its sphere of activity must have a magnitude and a direction contained in some narrow limits. It is necessary therefore to determine what is, within these limits, the ratio of the chances which give a sensible hyperbola to the chances which give an orbit which one can confound with a parabola. It is clear that this ratio depends on the law of possibility of the perihelion distances of the observable comets, and the examination of the table of the elements of the cometary orbits already calculated demonstrate to us that, beyond a perihelion distance equal to the radius of the terrestrial orbit, the possibilities of the perihelion distances diminish with a great rapidity in measure as these distances increase. The law of these possibilities must therefore be subject to this condition; but being, to this nearly, unknown, we are able only to determine the limit of the ratio of which there is question, or its value in the case most favorable to the sensible hyperbolas. If one supposes the radius of the sphere of activity of the Sun equal to one hundred thousand times its distance to the Earth, this which appears to be yet beneath that which indicates the smallness of the parallax of the stars, the analysis gives, in the most favorable case, $\frac{5713}{5714}$ for the probability that a nebula which penetrates into the sphere of solar activity, in a manner to be able to be observed, will describe a hyperbola of which the major axis will equal at least one hundred times the distance of the Sun to the Earth. A similar hyperbola will be confounded sensibly with a parabola; there is thus, in the case most favorable to the sensible hyperbolas, by very nearly odds fifty-six against one that, out of one hundred comets, none must have a sensible hyperbolic movement; it is therefore not surprising that, until here, one has not observed at all similar movement.

The attraction of the planets, and perhaps further the resistance of the ether, ought to change many cometary orbits, into some ellipses of which the major axis is much less than the radius of the sphere of activity of the Sun. One can believe that this change has taken place for the orbit of the comet of 1682, of which the major axis surpasses only thirty-five times the distance of the Sun to the Earth. A change greater still is arrived in the orbit of the comet of 1770, of which the major axis equals only around six times this distance.

A comet loses, at each return to its perihelion, a part of its substance, as the heat and the light of the Sun raise vapors from it and they disperse into space, to a distance from the comet such that its attraction can not make them fall back to its surface. This star must therefore, after many returns, dissipate itself in whole or reduce itself to a fixed core which will present some phases as the planets. The comet of 1682, the only one in which one has until now observed some phases, appears to approach from this state of stability. If this core is too small in order to be perceived, or if the evaporative substances which remain on its surface are too small in quantity to form, by their evaporation, a sensible comet head, the star will disappear forever. Perhaps is this one of the causes which render so rare the reappearances of the comets; perhaps yet has this cause made vanish more often than one had ought expect, many comets of which one was able to follow the trace in space, by means of the elements of their orbits; perhaps finally the same cause has rendered invisible the comet of 1770, which, if it has continued to be moved in an ellipse which it has described during its apparition, is returned, since this period, at least seven times to its perihelion.

Let

V be the speed of a comet at the instant where it penetrates into the sphere of activity of the Sun;

r be the radius vector of the comet at the same instant;

a the semi-major axis of the orbit which it is going to describe around the Sun;

e the eccentricity of this orbit;

D its perihelion distance.

In taking for unity of mass that of the Sun and for unity of distance its mean distance to the Earth, and, moreover, neglecting the masses of the comets and of the planets relative to this star, one will have, as one knows[1]:

$$\frac{1}{a} = \frac{2}{r} - V^2,$$
$$rV \sin \varpi = \sqrt{a(1-e^2)},$$
$$D = a(1-e);$$

ϖ being the angle which the direction of the speed V makes with the radius vector r. These equations give, in eliminating a and e,

$$\sin^2 \varpi = \frac{2D - \frac{2D^2}{r} + D^2V^2}{r^2V^2},$$

whence one draws

$$1 - \cos \varpi = 1 - \frac{\sqrt{1 - \frac{D}{r}}}{rV} \sqrt{r^2V^2 \left(1 + \frac{D}{r}\right) - 2D}.$$

Now, if one imagines a sphere of which the center is the one of the comet and of which the radius is equal to the speed V, this speed will be able to be equally directed towards all the points of the half of this sphere contained in the sphere of activity of the Sun. The probability of a direction forming the angle ϖ with the radius vector will be $2\pi \sin \varpi$, π being the semi-circumference of which the radius is unity; by dividing therefore the integral $2\pi \int d\varpi \sin \varpi$ by the surface of the half-sphere, one will have the probability that the direction of the speed V will be contained within the limits zero and ϖ; this probability is thus $1 - \cos \varpi$. The limits of the perihelion distance which corresponds to these limits of ϖ are zero and D; by supposing therefore all the values

[1] *Oeuvres de Laplace*, T. I., Book II, Chapter IV.

of D equally possible, one has for the probability that the perihelion distance will be contained between zero and D

$$1 - \frac{\sqrt{1-\frac{D}{r}}}{rV}\sqrt{r^2V^2\left(1+\frac{D}{r}\right)-2D}.$$

It is necessary to multiply this value by dV; by integrating it next within some determined limits and dividing the integral by the greatest value of V, a value which we will designate by U, one will have the probability that the value of V will be contained within these limits. This put, the smallest value of V is that which renders null the quantity contained under the preceding radical, this which gives

$$rV = \frac{\sqrt{2D}}{\sqrt{1+\frac{D}{r}}}.$$

We suppose next to the other limit

$$rV = i\sqrt{r};$$

and we seek, within these limits, the value of the integral

$$(a) \qquad \int dV\left[1 - \frac{\sqrt{1-\frac{D}{r}}}{rV}\sqrt{r^2V^2\left(1+\frac{D}{r}\right)-2D}\right]$$

Let

$$\sqrt{r^2V^2\left(1+\frac{D}{r}\right)-2D} = \left(rV\sqrt{1+\frac{D}{r}}-z\right);$$

we will have

$$rV = \frac{2D+z^2}{2z\sqrt{1+\frac{D}{r}}};$$

formula (a) becomes thus

$$V + \frac{\sqrt{1-\frac{D}{r}}}{r}\int dz\left(\frac{1}{2} - \frac{4D}{2D+z^2} + \frac{D}{z^2}\right).$$

By integrating, it becomes

$$(b) \qquad V + \frac{\sqrt{1-\frac{D}{r}}}{r}\left(\frac{z}{2} - 2\sqrt{2D}\arctan\frac{z}{\sqrt{2D}} - \frac{D}{z}\right) + C.$$

C being an arbitrary constant. In order to determine it, we will observe that the two limits of rV being, by that which precedes,

$$\frac{\sqrt{2D}}{\sqrt{1+\frac{D}{r}}},\ i\sqrt{r},$$

the corresponding limits of z are

$$\sqrt{2D},\ i\sqrt{r}\sqrt{1+\frac{D}{r}}\left[1-\sqrt{1-\frac{2D}{i^2 r\left(1+\frac{D}{r}\right)}}\right]$$

This last limit is

$$\frac{D}{i\sqrt{r}}\left[1-\frac{D}{2r}\left(1-\frac{1}{i^2}\right)+\cdots\right].$$

In determining therefore C in a manner that formula (b) is null at the first limit and extends to the second, this formula becomes

$$\frac{(\pi-2)\sqrt{2D}}{2r}-\frac{D}{ir\sqrt{r}}.$$

If one divides this function by U, we will have

$$\frac{(\pi-2)\sqrt{2D}}{2rU}-\frac{D}{iUr\sqrt{r}}.$$

for the probability that the perihelion distance of a star which enters into the sphere of activity of the Sun will be contained within the limits zero and D, the value of V^2 not exceeding $\frac{i^2}{r}$. This value is $\frac{2}{r}-\frac{1}{a}$; one has therefore

$$\frac{1}{a}=\frac{2-i^2}{r};$$

the orbit is elliptical or parabolic when i^2 is inferior or equal to 2, it is hyperbolic when i^2 surpasses 2. If one supposes, for example, $a=-100$, one will have

$$i^2=\frac{r+200}{100},$$

and the probability that the perihelion distance being contained between zero and D, the orbit will be either elliptical, or parabolic, or a hyperbola of which the semi-major axis will be at least equal to 100, is

$$\frac{(\pi-2)\sqrt{2D}}{2rU}-\frac{10D}{rU\sqrt{r(r+200)}}.$$

The probability of a value of i more considerable, or of a hyperbola of which the semi-major axis will be less than 100, is equal to

$$\frac{10D}{rU\sqrt{r(r+200)}},$$

because, by supposing i infinite, one will have $\frac{(\pi-2)\sqrt{2D}}{2rU}$ for the probability that the perihelion distance will be contained between zero and D. If one subtracts from it the probability that the orbits will be ellipses, or parabolas, or hyperbolas with a semi-major axis equal or superior to 100, one will have

$$\frac{10D}{rU\sqrt{r(r+200)}},$$

for the probability of the hyperbolas with a major axis below this value. Thus the perihelion distance being supposed contained between zero and D, the probability that the orbit will be either an ellipse, or a parabola, or a hyperbola with a semi-major axis at least equal to 100, is to the probability that it will be a hyperbola with a semi-major axis inferior, as

$$\frac{(\pi-2)}{10}\sqrt{\frac{r}{2D}(r+200)}-1:1.$$

If one supposes $r = 100000$ and $D = 2$, of the greater perihelion distances being so rare that one can set them aside, this ratio becomes the one of $5712,7$ to unity; there is therefore by very nearly odds of fifty-six against one that, out of one hundred observable cometary orbits, none must be a hyperbola with a semi-major axis inferior to 100.

The preceding analysis supposes all the values of D contained between 0 and 2, equally possible relative to the comets that one can perceive. However, the examination of the table of the elements of the cometary orbits already calculated shows that the perihelion distances which surpass unity are in much smaller number than those which are below. We name $\phi(D)$ the probability of a perihelion distance D relative to an observable comet. One have just seen that the probability that the perihelion distance of an observable comet will be contained between zero and D, D being quite small with respect to r, is, in the case where all these distances are equally possible, equal to

$$\frac{(\pi-2)\sqrt{2D}}{2rU};$$

and that the probability that the semi-major axis will be inferior to $\frac{r}{2-i^2}$ is

$$\frac{D}{iUr\sqrt{r}}.$$

In order to have the ratio of these probabilities, in the case where these distances are not equally possible, it is necessary, according to the analysis of the probabilities, to differentiate these two quantities with respect to D and to multiply the differentials by $\phi(D)$; then, according to this analysis, the preceding probabilities will be respectively as the integrals of these products, or as

$$\frac{(\pi-2)}{2rU}\int \phi(D)d\sqrt{2D};\ \frac{1}{iUr\sqrt{r}}\int dD\phi(D);$$

the integrals being taken from $D = 0$ to its limit, that one here supposes infinity, because $\phi(D)$ is null when D surpasses 5. Thus the probability that the semi-major axis of the orbit will be inferior to $\frac{r}{2-i^2}$, is

$$(q) \qquad \frac{2\int dD\phi(D)}{(\pi-2)i\sqrt{r}\int \frac{dD\,\phi(D)}{\sqrt{2D}}}$$

In the case of $\phi(D)$ constant, the preceding function becomes

$$\frac{\sqrt{2D}}{(\pi-2)i\sqrt{r}},$$

this which is conformed to that which precedes; but, if $\phi(D)$ diminishes when D increases, then the formula (q) diminishes. In order to show it, it suffices to prove that, in this case, one has

$$\frac{2\int dD\phi(D)}{\int\frac{dD\,\phi(D)}{\sqrt{2D}}} < \sqrt{2D}$$

or

$$2\int dD\phi(D) < \sqrt{2D}\int\frac{dD\,\phi(D)}{\sqrt{2D}},$$

and, by differentiating,

$$\phi(D) < \frac{1}{\sqrt{2D}}\int\frac{dD\,\phi(D)}{\sqrt{2D}} = \phi(D) - \frac{1}{\sqrt{2D}}\int dD\,\sqrt{2D}\frac{d\phi(D)}{dD}.$$

Now this inequality is evident, because, $\phi(D)$ diminishes when D increases, $\frac{d\phi(D)}{dD}$ is a negative quantity.

In examining the Table of elements of the cometary orbits already calculated, one sees that one will depart little from the truth by making $\phi(D) = kc^{-D^2}$, c being the number of which the hyperbolic logarithm is unity. Then the formula (q) becomes

$$\frac{\sqrt{\pi}}{(\pi-2)i\sqrt{2r}\int s^{\frac{1}{4}}ds\,e^{-s}}.$$

In supposing, as above, $r = 100000$, and observing that one has

$$\log 10\int s^{\frac{1}{4}}ds\,e^{-s} = 0,9573211$$

the preceding fraction becomes

$$\frac{1}{8264,3};$$

there is then, quite nearly, odds 8263 against one that one nebula which penetrates into the sphere of activity of the Sun will describe an orbit of which the semi-major axis will be at least equal to 100. Thus one can regard the assumption of $\phi(D)$ constant, and extending only to $D = 2$, as the limit of the assumptions favorable to the sensible hyperbolic movements, so that there are odds at least 56 against unity that, out of one hundred observable comets, none will have a sensible movement.