MÉMOIRE
SUR
L'INCLINAISON MOYENNE DES ORBITES DES COMÈTES
SUR
LA FIGURE DE LA TERRE ET SUR LES FONCTIONS

P. S. Laplace*

Mémoires de l'Académie royale des Sciences de Paris,
Savants étranges, 1773. T. VII, 1776, p.503–524.
Oeuvres T. VIII, p. 279–324.

One of the most extraordinary phenomena which the system of the world offers to us is the movement of the planets and of their satellites in the same sense and very nearly in the same plane; if we picture to ourselves in fact all these stars describing from west to east some nearly circular orbits and very little inclined to the ecliptic, while the comets seem to be moved indifferently in every sense and with every possible inclination in some very eccentric eclipses, we perceive a quite marked separation between the planets and the comets, such that, in the movement of these great bodies, nature does not follow this gradation by insensible nuances which she always observes when its march is not interrupted by some particular causes.

We count in all six planets and ten satellites; now, if we suppose that they have been launched at random, it is easy to see that the probability that they will rotate all in the same sense is $\frac{1}{2^{15}} = \frac{1}{32768}$, so that there is odds 32767 against unity, that this will not happen. If we multiply the fraction $\frac{1}{32768}$ by that which expresses the probability that the orbits will be contained in a likewise small zone as that which contains them, we will see that the actual disposition of our planetary system would be infinitely less probable if it were due to chance, and that it announces consequently, with a certitude equivalent or even superior to that of a great number of events of which it would seem absurd to us to doubt, the existence of a regular cause which has determined the planets and their satellites to be moved in the same sense and nearly in the same plane; I suppress that analysis, that M. Daniel Bernoulli has given a long time ago, and which besides is quite simple.[1]

Now what is the cause which is able to have determined thus the movement of the planets and of the satellites? Has it been particular to these stars, or indeed has it had an influence on the movement of all those which rotate around the sun? The first of these questions seems to me quite difficult to resolve; and I confess that after having reflected

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. August 20, 2010

[1]Translator's note: 1734 (1735) *Researches physiques et astronomiques sur le probleme propose pour la seconde fois par l'Académie Royale des Sciences de Paris*.

on it a long time, and after having examined with attention all the hypotheses imagined until now in order to explain this phenomenon, I have found nothing satisfying. As for the second question, we can easily answer it; it suffices for this: 1° to calculate the mean inclination of the orbits of all the observed comets, and to see how much it deviates from 45°; because, by supposing the comets launched at random, there is as much to wager that it will be above as below 45°; 2° to know the ratio of the number of direct comets to that of the retrogrades, and to see by how much it deviates from unity; because it is as probable that it will be greater as less. These calculations have been made by M. du Séjour in his excellent work on the comets; this learned author has found that the mean inclination of the sixty-three comets observed to the present was $46^\circ 16'$, which deviates little from 45°, and that the ratio of the direct comets to the retrogrades was $\frac{5}{4}$, which deviates little from unity. Thence he concludes, with reason, that there exists for the comets no cause which determines them to be moved in one sense rather than in another, and very nearly in the same plane, and that in this way that which determines the movement of the planets is entirely independent of the general system of the universe.

This interesting observation of M. du Séjour has suggested the idea of submitting to Analysis the probabilities that the mean inclination of the comets and the ratio of the number of direct to that of the retrogrades, will be contained between some given limits, by supposing that they have been projected at random; this calculation is even necessary in order to give more certitude to this observation; because if, for example, the mean inclination of the comets was $45^\circ + \alpha$, and if it were a very great number, as a million, to wager against unity that it must be below, we could conclude from it with much likelihood that there exists a cause which determines the comets to be moved in one plane rather than in another; it is therefore essential to know the probabilities that the mean inclination will be above or below $45^\circ + \alpha$. The same reasoning can be applied to the ratio of the number of direct comets to that of the retrogrades. It is easy to calculate the probability that this ratio will be between two given limits; it suffices, for this, to raise the binomial $\left(\frac{1}{2} + \frac{1}{2}\right)$ to the power indicated by the number of comets; let n be this number, by developing $\left(\frac{1}{2} + \frac{1}{2}\right)^n$, the term

$$\frac{n(n-1)\cdots(n-\mu+1)}{1.2.3\ldots\mu}\left(\frac{1}{2}\right)^{n-\mu}\left(\frac{1}{2}\right)^{\mu}$$

will express the probability that there will be $n - \mu$ direct comets, and μ retrograde comets; therefore, if we wish to determine the probability that the ratio of the direct to the retrogrades will be contained between the two limits $\frac{n-\mu}{\mu}$ and $\frac{n-\mu'}{\mu'}$, it is necessary to take the sum of the terms of the binomial $\left(\frac{1}{2} + \frac{1}{2}\right)$ raised to the power n, contained between the term

$$\frac{n(n-1)\cdots(n-\mu+1)}{1.2.3\ldots\mu}\left(\frac{1}{2}\right)^{n-\mu}\left(\frac{1}{2}\right)^{\mu}$$

and the term

$$\frac{n(n-1)\cdots(n-\mu'+1)}{1.2.3\ldots\mu'}\left(\frac{1}{2}\right)^{n-\mu'}\left(\frac{1}{2}\right)^{\mu'};$$

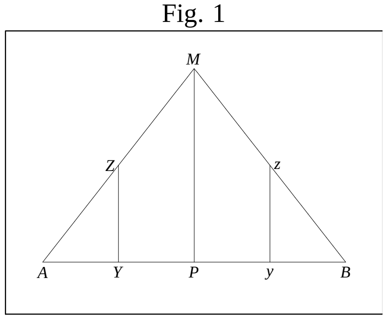
this sum will express the probability demanded; but it is much more difficult to determine the probability that the mean inclination of the orbits will be contained between two given limits; this problem seems to be to me one of the most complicated of all the analysis of chances, especially when one intends, at the same time as I have made it, to find a general formula for any number of comets. I confess that it could have been impossible for me to attain it without the help of a method that I have given elsewhere[2], in order to find directly the general expression of the quantities subjugated to a law which serves to form them. I hope that the application of this method to the problem in question will not be useless in order to make known the nature and the advantages.

II.

Theorem 1. *I suppose an indefinite number of bodies launched at random into space and circulating about the Sun; the question is to find the probability that the mean inclination of their orbits on a given plane, such as the ecliptic, will be contained between two given limits, as* 40° *and* 50°.

By *mean inclination*, I intend the sum of all the inclinations divided by the number of orbits.

Fig. 1



In order to solve this problem, I consider first only two bodies M and N, and I suppose that the straight line AB (*fig.* 1) represents 90° or the greatest mean inclination of the two orbits; I begin by tracing a line $AZMB$, of which each ordinate is proportional to the probability that the mean inclination will be equal to the corresponding abscissa AY; I will name this line *curve of the probabilities*. Now, if we make $AY = x$ and $YZ = y$, y will be proportional to $2x$, from A to the middle P of the straight line AB; because if the mean inclination of the two orbits is x, x being less than $\frac{1}{2}a$, it is clear that this can happen in as many ways as there are points in the straight line $2x$; in fact, the inclination of the orbit of M can, in this case, be equally either 0, or dx, or $2dx$, or $3dx$, or etc. as far as $2x$, by representing by dx the infinitely small increase of the inclination of this orbit. We can therefore make $YZ = 2AY$; and hence, AZM will be a straight line, and APM a right triangle such that $PM = 2AP = a$.

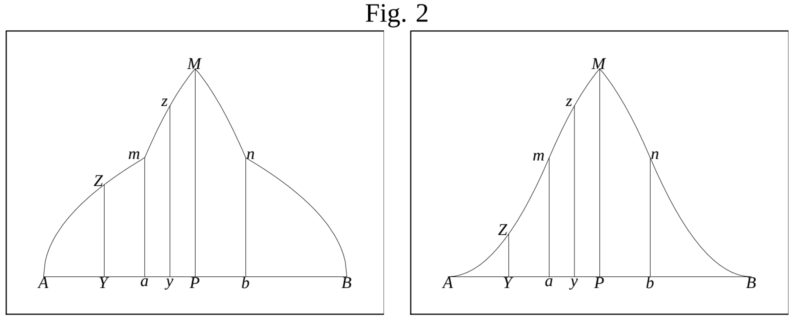
[2] *Oeuvres de Laplace*, T. VIII, p. 97. This is the paper "Recherchés sur l'integration of equations différentielles aux différences finies et sur leur usage dans la théorie des hasards." This occupies pages 69 to 197. All further page references are to this memoir.

Presently, the line BM must be entirely equal to the straight line AM, because, at equal distance from points A and B, the ordinates must be equal, seeing that it is as probable that the mean inclination approach to the limit A as to the limit B; the line AMB will be therefore composed of two equal straight lines AM and BM, such that $PM = a$.

If we wish to see now the probability that the mean inclination will be contained between two limits Y and y, it will be necessary to divide the area $YZMzy$ by the entire area AMB, and the quotient will represent this probability.

III.

Fig. 2



We suppose that there be three bodies M, N and P; let the straight line $AB = a$ be divided (*fig.* 2)[3] into three equal parts, Aa, ab, bB;and we seek the probability that the mean inclination will be equal to any abscissa AY, or, what comes to the same, we trace the curve $AmMnB$ of the probabilities; let $AY = x$, x being supposed first less than Aa or $\frac{1}{3}a$. I suppose that any one of the three bodies, M for example, has an inclination that I designate by f; it is necessary consequently that the mean inclination of the two others is $\frac{3x-f}{2}$, since, by hypothesis, the mean inclination of the three bodies is x; now, $\frac{3x-f}{2}$ being less than $\frac{a}{2}$, it is easy to see, by the preceding article, that the number of cases in which this can happen is $3x - f$. It is necessary to multiply now this quantity by df, and to take the integral, from $f = 0$ to $f = 3x$, in order to have the total number of cases in which the mean inclination of the three bodies can be x, and we will find $\frac{9}{2}x^2$ for this number; we can therefore, from A to a, suppose the ordinate YZ equal to $\frac{9}{2}\frac{x^2}{a}$; which gives

$$ay = \frac{9}{2}x^2$$

for the equation of the curve AZM, and hence likewise for that of the curve Bn, by making x to begin at the point B.

We determine now the nature of the curve mMn; I observe first that it must be composed of two entirely equal parts, mM and Mn, P being the middle of the straight

[3] *Translator's note*: Figures 2, 3 and 4 of Laplace are incorrect. A reproduction of Figures 2 and 3 are placed on the left. The corrected figures using the formulas of Laplace are placed to the right. Figure 4 as given by Laplace is retained. One should expect, as the number of observations increase, the distribution of the mean approach more nearly the normal distribution.

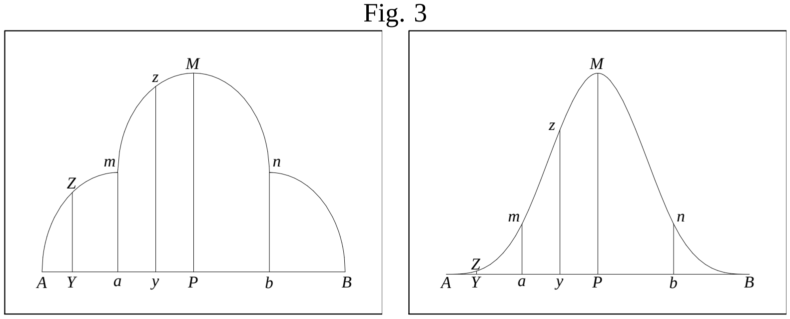
line AB; let $ay = z$ (*fig.* 2), or $Ay = \frac{1}{3}a + z$, and let f be the inclination of the orbit of the body M; the two other bodies N and P will have therefore together the inclination $a + 3z - f$; now, let $3z - f = u$, so that the inclination of these two bodies is $a + u$, and hence their mean inclination $\frac{a}{2} + \frac{u}{2}$; the number of cases in which this can happen is, by the preceding article, $a - u$ or $a + f - 3z$; it is necessary therefore to multiply this quantity by df and to integrate, from $f = 0$ to $f = 3z$, in order to have the number of cases which take place in this interval; we will have thus $3az - \frac{9}{2}z^2$ for the number of these cases; it is necessary now to determine the number of cases which take place from $f = 3z$ to $f = a$, and for this I make $f = 3z + s$; the total inclination of the two bodies N and P will be therefore $a - s$, and, hence, their mean inclination $\frac{a}{2} - \frac{s}{2}$; now, the number of cases in which this can happen is, by the preceding article, $a - s$; multiplying therefore this quantity ds, and integrating it from $s = 0$ to $s = a - 3z$, we will have $\frac{1}{2}a^2 - \frac{9}{2}z^2$, for the number of cases which take place from $f = 3z$ to $f = a$. Reassembling therefore all these cases, we will have $\frac{1}{2}a^2 + 3az - 9z^2$, for the number of those which give the mean inclination of three bodies equal to $\frac{1}{3}a + z$. Thus, one can suppose the ordinate yz equal to $\frac{\frac{1}{2}a^2 + 3az - 9z^2}{a}$, and the equation of the curve mMn will be

$$ay = \frac{1}{2}a^2 + 3az - 9z^2.$$

If we wish now to have the probability that the mean inclination of three orbits will be contained between two given limits, we will seek the area contained between these limits, we will divide it by the entire area of the curve AMB; the quotient will express the demanded probability.

IV.

Fig. 3



We suppose now four bodies $M,\ N,\ P,\ Q$, and we divide the straight line AB (*fig.* 3) into four equal parts $Aa,\ aP,\ Pb$ and bB; the curve $AmMnB$ will be composed of four parts $Am,\ mM,\ Mn$ and nB, such, however, that we have Am equal to Bn, and mM equal to nM.

We determine the nature of these curves, and, for this, let as above $AY = x$, x being less than $\frac{1}{4}a$, $YZ = y$; let moreover f be the inclination of the orbit of the body M; the sum of the inclinations of the orbits of the three other bodies $N,\ P$ and Q will be $4x-f$, and hence their mean inclination will be $\frac{4x-f}{3}$; now, by the preceding article, the number of cases in which this can happen is

$$\frac{9}{2}\left(\frac{4x-f}{3}\right)^2=\frac{1}{2}(4x-f)^2.$$

If we multiply this quantity by df, and if we integrate from $f=0$ to $f=4x$, we will have $\frac{32}{3}x^3$ for the number of cases in which the mean inclination of the four orbits can be x; hence, we can suppose that from A to a, the equation of the curve Am is

$$a^2y=\frac{32}{3}x^3.$$

In order to have the equation of the curve mM, I suppose $ay=z$, hence $Ay=\frac{1}{4}a+z$; let f be the inclination of the body M, the sum of the inclinations of the three other bodies will be therefore $a+4z-f$; hence, their mean inclination will be $\frac{a+4z-f}{3}$; now, as long as $4z-f$ is a positive quantity, the number of cases in which this inclination is possible, is (art. preceding)

$$\frac{1}{2}a^2+3a\left(\frac{4z-f}{3}\right)-9\left(\frac{4z-f}{3}\right)^2=\frac{1}{2}a^2+a(4z-f)-(4z-f)^2;$$

if we multiply this quantity by df, and if we integrate from $f=0$ to $f=4z$, we will have $2a^2z+8az^2-\frac{64}{3}z^3$, for the number of cases which can take place in this interval.

In order to have the number of those which correspond to the interval contained between $f=4z$ and $f=a$, I make $f-4z=s$; $\frac{a+4z-f}{3}$ becomes therefore $\frac{a-s}{3}$; let $a-s=u$, we will have $\frac{u}{3}$ for the mean inclination of the three orbits; now the number of cases in which this can happen is, by the preceding article, $\frac{1}{2}u^2$ or $\frac{1}{2}(a-s)^2$; multiplying this quantity by ds and integrating it, from $s=0$ to $s=a-4z$, we will have $\frac{1}{6}a^3-\frac{32}{3}z^3$, for the number of all possible cases from $f=4z$ to $f=a$; therefore, the number of all the cases in which the mean inclination of the four orbits can be $\frac{1}{4}a+z$ is

$$\frac{1}{6}a^3+2a^2z+8az^2-32z^3;$$

we can thus suppose that, from a to P, the equation of the curve mM is

$$a^2y=\frac{1}{6}a^3+2a^2z+8az^2-32z^3.$$

V.

If there were five bodies M, N, P, Q and R, by dividing the straight line AB into five equal parts, we will obtain the curves corresponding to each of these parts, by means of the curves relative to four bodies, as we just concluded this one, by means of the curves relative to three bodies. Thence we can infer generally that the curves relative to n bodies can always be deduced from those which are relative to $n-1$ bodies. In order to establish in a general manner the relation which exists between these different curves, we suppose the straight line AB (*fig.* 3) divided into n equal parts, and we determine the equation of the curve relative to the r^{th} part; let $\frac{r-1}{n}a + z$ be the distance of one of their ordinates to the point A, z being less than $\frac{a}{n}$; let further $\frac{{}_r y_{n,z}}{a^{n-2}}$ be this ordinate, or, what amounts to the same, let ${}_r y_{n,z}$ be the number of cases in which it can happen that the mean inclination of the n bodies is $\frac{r-1}{n}a + z$. This put, if we designate by f the inclination of the body M, the sum of the inclinations of the $n-1$ other bodies will be $(r-1)a + nz - f$; hence, their mean inclination will be

$$\frac{(r-1)a + nz - f}{n-1};$$

now it can happen that $nz - f$ is positive or negative; I suppose it first positive; the number of cases in which it can happen that the mean inclination of the $n-1$ bodies is $\frac{(r-1)a+nz-f}{n-1}$ is

$${}_r y_{n-1, \frac{nz-f}{n-1}}.$$

By multiplying this quantity by df, an integrating it from $f = 0$ to $f = nz$, we will have, for the number of cases which correspond to this interval,

$$\int_0^{nz} {}_r y_{n-1, \frac{nz-f}{n-1}}\, df.$$

If $nz - f$ is a negative quantity, let $nz - f = -s$, we will have $\frac{(r-1)a-s}{n-1}$ for the mean inclination of the $n-1$ bodies; now

$$\frac{(r-1)a - s}{n-1} = \frac{r-2}{n-1}a + \frac{a-s}{n-1};$$

and the number of cases in which this is possible is ${}_{r-1} y_{n-1, \frac{a-s}{n-1}}$; therefore we have

$$\int_0^{a-nz} {}_{r-1} y_{n-1, \frac{a-s}{n-1}}\, ds;$$

for the number of cases from $s = 0$ to $s = a - nz$ or, what amounts to the same, from $f = nz$ to $f = a$; hence

$$(\sigma) \qquad {}_r y_{n,z} = \int_0^{nz} {}_r y_{n-1, \frac{nz-f}{n-1}}\, df + \int_0^{a-nz} {}_{r-1} y_{n-1, \frac{a-s}{n-1}}\, ds;$$

such is the general equation by means from which, when we know the curves relative to $n-1$ bodies, we can determine those which are relative to n bodies.

VI.

It is necessary now, in the manner of equation (σ), to find the general expression of ${}_r y_{n,z}$; for this, I observe that ${}_r y_{n,z}$ has a value of this form

$$(i) \qquad {}_r y_{n,z} = {}_r A_n z^{n-1} + {}_r B_n z^{n-2} + {}_r C_n z^{n-3} + \cdots + {}_r G_n z + {}_r H_n,$$

where ${}_rA_n$, ${}_rB_n$,... are some functions of r and n which the question is to determine; in order to attain it, I will make use of a method that I have exposed elsewhere (*see* page 97 of this volume); the preceding expression of ${}_ry_{n,z}$ gives

$$ {}_ry_{n-1,\frac{nz-f}{n-1}} = {}_rA_{n-1}\left(\frac{nz-f}{n-1}\right)^{n-2} + {}_rB_{n-1}\left(\frac{nz-f}{n-1}\right)^{n-3} + \cdots + {}_rG_{n-1}\left(\frac{nz-f}{n-1}\right)^{0}; $$

therefore we will have

$$ \begin{aligned} \int_0^{nz} {}_ry_{n-1,\frac{nz-f}{n-1}}\,df = {}_rA_{n-1}\left(\frac{n}{n-1}\right)^{n-1} z^{n-1} + {}_rB_{n-1}\frac{n-1}{n-2}\left(\frac{n}{n-1}\right)^{n-2} z^{n-2} \\ + {}_rC_{n-1}\frac{n-1}{n-3}\left(\frac{n}{n-1}\right)^{n-3} z^{n-3} + \cdots + {}_rG_{n-1}nz; \end{aligned} $$

we will have similarly

$$ {}_{r-1}y_{n-1,\frac{a-s}{n-1}} = {}_{r-1}A_{n-1}\left(\frac{a-s}{n-1}\right)^{n-2} + {}_{r-1}B_{n-1}\left(\frac{a-s}{n-1}\right)^{n-2} + \cdots + {}_{r-1}G_{n-1}. $$

Therefore

$$ \begin{aligned} \int_0^{a-ns} {}_ry_{n-1,\frac{a-s}{n-1}}\,ds = {}_{r-1}A_{n-1}\left[\left(\frac{a}{n-1}\right)^{n-1} - \left(\frac{n}{n-1}\right)^{n-1} z^{n-1}\right] \\ + \frac{n-1}{n-2}\,{}_{r-1}B_{n-1}\left[\left(\frac{a}{n-1}\right)^{n-2} - \left(\frac{n}{n-1}\right)^{n-2} z^{n-2}\right] + \cdots \\ + {}_{r-1}G_{n-1}(a-nz). \end{aligned} $$

The equation (σ) will give therefore

$$ \begin{aligned} {}_ry_{n,z} =& z^{n-1}\left(\frac{n}{n-1}\right)^{n-1}\left({}_rA_{n-1} - {}_{r-1}A_{n-1}\right) \\ &+ z^{n-2}\frac{n-1}{n-2}\left(\frac{n}{n-1}\right)^{n-2}\left({}_rB_{n-1} - {}_{r-1}B_{n-1}\right) \\ &+ z^{n-3}\frac{n-1}{n-3}\left(\frac{n}{n-1}\right)^{n-3}\left({}_rC_{n-1} - {}_{r-1}C_{n-1}\right) \\ &+ \cdots \\ &+ \left(\frac{a}{n-1}\right)^{n-1} {}_{r-1}A_{n-1} + \frac{n-1}{n-2}\,{}_{r-1}B_{n-1}\left(\frac{a}{n-1}\right)^{n-2} + \cdots \end{aligned} $$

By comparing this equation with equation (i), we will have the following:

$$(\Psi)\quad \begin{cases} {}_rA_n = \left(\dfrac{n}{n-1}\right)^{n-1} ({}_rA_{n-1} - {}_{r-1}A_{n-1}), \\ {}_rB_n = \dfrac{n-1}{n-2}\left(\dfrac{n}{n-1}\right)^{n-2} ({}_rB_{n-1} - {}_{r-1}B_{n-1}), \\ {}_rC_n = \dfrac{n-1}{n-3}\left(\dfrac{n}{n-1}\right)^{n-3} ({}_rC_{n-1} - {}_{r-1}C_{n-1}), \\ \cdots \\ {}_rG_n = n({}_rG_{n-1} - {}_{r-1}G_{n-1}), \\ {}_rH_n = \left(\dfrac{a}{n-1}\right)^{n-1} {}_{r-1}A_{n-1} + \dfrac{n-1}{n-2}\left(\dfrac{a}{n-1}\right)^{n-2} {}_{r-1}B_{n-1} + \cdots \end{cases}$$

These equations are in finite partial differences, excepting the last which gives without any integration the value of ${}_rH_n$ when we know ${}_rA_n$, ${}_rB_n$,. . ..

One can determine further ${}_rH_n$ by the following consideration: it is clear that

$${}_ry_{n,0} = {}_{r-1}y_{n,\frac{a}{n}},$$

that is that the ordinate of the curve of probabilities, which corresponds to the extremity of the $(r-1)$st part of the straight line AB divided into n equal parts, is the same as the ordinate which corresponds to the beginning of the r^{th} part; therefore we have

$$(\Gamma)\qquad {}_rH_n = {}_{r-1}A_n\left(\frac{a}{n}\right)^{n-1} + {}_{r-1}B_n\left(\frac{a}{n}\right)^{n-2} + \cdots + {}_{r-1}H_n$$

or

$${}_rH_n - {}_{r-1}H_n = {}_{r-1}A_n\left(\frac{a}{n}\right)^{n-1} + {}_{r-1}B_n\left(\frac{a}{n}\right)^{n-2} + \cdots.$$

Hence, by integrating with respect to r alone, we have

$${}_rH_n = \sum\left[{}_{r-1}A_n\left(\frac{a}{n}\right)^{n-1} + {}_{r-1}B_n\left(\frac{a}{n}\right)^{n-2} + \cdots\right]$$

the characteristic $\sum$ being the sign of integration for the finite differences. We determine presently ${}_rA_n$, ${}_rB_n$, ...

The first of the equations (Ψ) gives

$${}_1A_n = \left(\frac{n}{n-1}\right)^{n-1} {}_1A_{n-1};$$

and by integrating, by the method of page 74 of this volume, we will have

$${}_1A_n = H\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\cdots\left(\frac{n}{n-1}\right)^{n-1},$$

H being an arbitrary constant; now, putting $n = 2$, we have ${}_1A_2 = 2$; therefore $H = 1$; we have besides

$$\left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \cdots \left(\frac{n}{n-1}\right)^{n-1} = \frac{n^{n-1}}{1.2.3\ldots(n-1)} = \frac{n^{n-1}}{\nabla(n-1)},$$

by designating, as I have elsewhere (*see* page 74 of this volume), the product $1.2.3\ldots(n-1)$ by $\nabla(n-1)$; we will have therefore

$${}_1A_n = \frac{n^{n-1}}{\nabla(n-1)};$$

hence,

$${}_2A_n = \left(\frac{n}{n-1}\right)^{n-1} \left[{}_2A_{n-1} - \frac{(n-1)^{n-2}}{\nabla(n-2)}\right];$$

let

$${}_2A_n = -\frac{n^{n-1}}{\nabla(n-1)}u_n;$$

we will have

$$u_n = u_{n-1} - 1;$$

therefore

$$u_n = -n + H;$$

hence,

$${}_2A_n = \frac{n^{n-1}}{\nabla(n-1)}(n - H);$$

now, putting $n = 2$, we have ${}_2A_n = -2$, because we have, by article II,

$${}_2y_{2,z} = -2z + a;$$

therefore

$$H = 1 \quad \text{and} \quad {}_2A_n = -\frac{n^{n-1}}{\nabla(n-1)}(n-1);$$

hence,

$${}_2A_n = \left(\frac{n}{n-1}\right)^{n-1} \left[{}_3A_{n-1} + \frac{(n-1)^{n-2}}{\nabla(n-2)}(n-2)\right];$$

let

$${}_3A_n = \frac{n^{n-1}}{\nabla(n-1)}u_n,$$

and we will have

$$u_n = u_{n-1} + (n-2);$$

whence we deduce

$$u_n = \frac{(n-1)(n-2)}{1.2} + H;$$

now, putting $n=2$, we have ${}_3A_n=0$; therefore

$$H=0 \quad \text{and} \quad {}_3A_n=-\frac{n^{n-1}}{\nabla(n-1)}\frac{(n-1)(n-2)}{1.2}.$$

By continuing to operate thus, we will find

$${}_rA_n=\pm\frac{n^{n-1}}{\nabla(n-1)}\frac{(n-1)(n-2)\cdots(n-r+1)}{1.2.3\cdots(r-1)}$$

or

$${}_rA_n=\pm\frac{n^{n-1}}{\nabla(r-1)\nabla(n-r)}$$

the $+$sign having place if r is odd, and the $-$sign if it is even.

I observe here, relative to the product $\frac{1}{\nabla(n-1)}$, that we have

$$\frac{1}{\nabla(n-r)}=1,$$

when $n-r=0$ and when $n-r=1$; in fact

$$\frac{1}{\nabla(n-r)}=\frac{n(n-1)\cdots(n-r+1)}{1.2.3\cdots n}.,$$

Now, this last quantity is equal to 1, when $n-r=1$ and when $n-r=0$; if r is greater than n, these two numbers being supposed positive and whole, we have

$$\frac{1}{\nabla(n-r)}=0,$$

because then we have evidently

$$n(n-1)\cdots(n-r+1)=0.$$

We determine now ${}_rB_n$.

It is easy to see, by the preceding articles, that we have

$${}_1B_n=0 \quad {}_1C_n=0, \quad \ldots$$

Next the second of the equations (Ψ) give

$${}_2B_n=\frac{n-1}{n-2}\left(\frac{n}{n-1}\right)^{n-2}{}_2B_{n-1};$$

whence I deduce, by integrating

$${}_2B_n=\frac{Hn^{n-2}}{\nabla(n-2)},$$

H being an arbitrary constant. In order to determine it, I observe that the differential equation in ${}_2B_n$ begins to exist only when $n=3$, so that, in order to have H, it is necessary to know ${}_2B_2$; now it is clear that ${}_2B_2$ is the only constant term in the expression of ${}_2y_{2,z}$, and hence, the last of the equations (Ψ) gives

$${}_2B_2 = {}_1A_1 a = 2a;$$

therefore

$$H = a \quad \text{and} \quad {}_2B_n = \frac{n^{n-2}a}{\nabla(n-2)}.$$

Thence we will have

$${}_3B_n = -\frac{n^{n-2}a}{\nabla(n-2)}(n+H),$$

H being an arbitrary constant; now, putting $n = 2$, we have ${}_3B_n = 0$; therefore

$$H = -2$$

and

$${}_3B_n = -\frac{n^{n-2}}{\nabla(n-2)}a(n-2).$$

We will have, in the same manner

$${}_4B_n = \frac{n^{n-2}a}{\nabla(n-2)}\left[\frac{(n-2)(n-3)}{1.2} + H\right];$$

now, putting $n = 2$, ${}_4B_n = 0$; therefore

$$H = 0.$$

By continuing to operate thus, we will find generally

$${}_rB_n = \mp\frac{n^{n-2}a}{\nabla(n-2)}\frac{(n-2)(n-3)\cdots(n-r+1)}{1.2.3\cdots(r-2)}$$

or

$${}_rB_n = \mp\frac{n^{n-2}a}{\nabla(r-2)\nabla(n-r)}.$$

The third of the equations (Ψ) gives

$${}_2C_n = \frac{n-1}{n-3}\left(\frac{n}{n-1}\right)^{n-3} {}_2C_{n-1};$$

whence I deduce, by integrating,

$${}_2C_n = \frac{n^{n-3}H}{\nabla(n-3)}.$$

In order to determine H, I observe that the differential equation in ${}_2C_n$ begins to exist only when $n = 4$; it is necessary therefore, in order to have H, to know ${}_2C_3$; now it is clear that ${}_2C_3$ is the only constant term of the expression of ${}_2y_{3,z}$; hence, the last of the equations (Ψ) gives

$$ {}_2C_3 = {}_1A_2 \left(\frac{a}{2}\right)^2; $$

therefore

$$ {}_2C_3 = \frac{a^2}{2} \quad \text{and} \quad H = \frac{a^2}{1.2}; $$

thus

$$ {}_2C_n = \frac{n^{n-3}a^2}{1.2.\nabla(n-3)}. $$

Thence we will deduce

$$ {}_3C_n = -\frac{n^{n-3}a^2}{1.2.\nabla(n-3)}(n+H). $$

In order to determine H, it is necessary to know ${}_3C_3$; now this quantity is the only constant term of the expression ${}_3y_{3,z}$; thus the last of the equations (Ψ) gives

$$ {}_3C_3 = {}_2A_2 \left(\frac{a}{2}\right)^2 + 2\cdot {}_2B_2\frac{a}{2} = \frac{a^2}{2}; $$

hence,

$$ H = -4 \quad \text{and} \quad {}_3C_n = -\frac{n^{n-3}a^2}{1.2.\nabla(n-3)}\left(\frac{n-3}{1} - 1\right). $$

Thence I deduce

$$ {}_4C_n = \frac{n^{n-3}a^2}{1.2.\nabla(n-3)}\left[\frac{(n-3)(n-4)}{1.2} - \frac{n-3}{1} + H\right]; $$

now we have ${}_4C_3 = 0$; therefore $H = 0$. By continuing to operate thus, we will find generally

$$ {}_rC_n = \mp\frac{n^{n-3}a^2}{1.2.\nabla(n-3)}\left[\frac{(n-3)(n-4)\cdots(n-r)}{1.2.3\cdots(r-2)} - \frac{(n-3)\cdots(n-r+1)}{1.2\cdots(r-3)}\right] $$

or

$$ {}_rC_n = \mp\frac{n^{n-3}a^2}{1.2}\left[\frac{1}{\nabla(r-2)\nabla(n-r-1)} - \frac{1}{\nabla(r-3)\nabla(n-r)}\right], $$

the upper sign having place if r is odd, and the lower if it is even. I have found, in the same manner,

$$ {}_rD_n = \mp\frac{n^{n-4}a^3}{1.2.3}\left[\frac{1}{\nabla(r-2)\nabla(n-r-2)} - \frac{4}{\nabla(r-3)\nabla(n-r-1)} + \frac{1}{\nabla(r-4)\nabla(n-r)}\right], $$

the upper sign always having place if r is odd, and the lower if it is even.

VII.

We will have thus, by the preceding method, the law of each term, whatever be r and n, but this does not yet suffice; it is necessary, moreover, to have the law of these terms with respect to one another, that is, the law of the q^{th} term of the sequence

$$ {}_{r}A_{n} z^{n-1} + {}_{r}B_{n} z^{n-2} + \cdots $$

We name ${}_{r}\overset{q}{T}_{n} z^{n-q}$ this term; ${}_{r}\overset{q}{T}_{n}$ will be a function of q, of r and of n; we can already know, by that which precedes, in what manner it is a function of r and of n; it is necessary presently to determine in what manner it is a function of q; for that, I take the terms already found

$$ \begin{aligned} {}_{r}A_{n} &= \pm \frac{n^{n-1}}{\nabla(r-1)\nabla(n-r)}, \\ {}_{r}B_{n} &= \mp \frac{n^{n-2}a}{\nabla(r-2)\nabla(n-r)} \\ {}_{r}C_{n} &= \mp \frac{n^{n-3}a^2}{1.2}\left[\frac{1}{\nabla(r-2)\nabla(n-r-1)} - \frac{1}{\nabla(r-3)\nabla(n-r)}\right] \end{aligned} $$

$$ \begin{aligned} {}_{r}D_{n} = \mp \frac{n^{n-4}a^3}{1.2.3}\Bigg[&\frac{1}{\nabla(r-2)\nabla(n-r-2)} \\ &- \frac{4}{\nabla(r-3)\nabla(n-r-1)} + \frac{1}{\nabla(r-4)\nabla(n-r)}\Bigg], \end{aligned} $$

the upper sign having place if r is odd, and the lower if it is even. Thence, I conclude that we have generally

$$ \begin{aligned} {}_{r}\overset{q}{T}_{n} = \mp \frac{n^{n-q}a^{q-1}}{\nabla(q-1)}\Bigg[&\frac{1}{\nabla(r-2)\nabla(n-r-q+2)} \\ &+ \frac{M_q}{\nabla(r-3)\nabla(n-r-q+3)} \\ &+ \frac{{}^{1}M_{q}}{\nabla(r-4)\nabla(n-r-q+4)} + \cdots + \frac{{}^{q-3}M_{q}}{\nabla(r-q)\nabla(n-r)}\Bigg], \end{aligned} $$

an expression in which it is necessary to determine M_q, ${}^{1}M_{q}$, ... In order to attain this, I observe that this value of ${}_{r}\overset{q}{T}_{n}$ begins to exist only when $n = q$; now we have

$$ {}_{3}\overset{q}{T}_{q} = -\frac{a^{q-1}M_q}{\nabla(q-1)}; $$

moreover, the equation (Γ) gives

$$\begin{aligned}
{}_{3}\overset{q}{T}{}_{q} &= {}_{2}A_{q}\left(\frac{a}{q}\right)^{q-1} + {}_{2}B_{q}\left(\frac{a}{q}\right)^{q-2} + \cdots + {}_{2}\overset{q}{T}{}_{q} \\
&= a^{q-1}\left[-\frac{1}{\nabla(q-2)} + \frac{1}{\nabla(q-2)} + \frac{1}{1.2.\nabla(q-3)} + \cdots + \frac{1}{\nabla(q-1)}\right] \\
&= \frac{a^{q-1}}{\nabla(q-1)}\left[1 + \frac{q-1}{1} + \frac{(q-1)(q-2)}{1.2} + \cdots + 1 - q\right] \\
&= \frac{a^{q-1}}{\nabla(q-1)}(2^{q-1} - q);
\end{aligned}$$

by comparing this expression of ${}_{3}\overset{q}{T}{}_{q}$ with the preceding, we will have

$$-M_q = 2^{q-1} - q.$$

In order to find ${}^{1}M_{q}$, I observe that we have

$${}_{4}\overset{q}{T}{}_{q} = \frac{{}^{1}M_{q}a^{q-1}}{\nabla(q-1)}.$$

Moreover,

$${}_{4}\overset{q}{T}{}_{q} = {}_{3}A_{q}\left(\frac{a}{q}\right)^{q-1} + {}_{3}B_{q}\left(\frac{a}{q}\right)^{q-2} + \cdots + {}_{3}T_{q},$$

which gives

$$\begin{aligned}
{}_{4}\overset{q}{T}{}_{n} = a^{q-1}\Bigg[&+\frac{1}{1.2.\nabla(q-3)} - \frac{1}{\nabla(q-3)} \\
&- \frac{1}{1.2.\nabla(q-4)} + \frac{2^2-3}{1.2.\nabla(q-3)} \\
&- \frac{1}{1.2.3.\nabla(q-5)} + \frac{2^3-4}{1.2.3.\nabla(q-4)} \\
&- \cdots \\
&- \frac{1}{\nabla(q-2)} + \frac{2^{q-2}-(q-1)}{\nabla(q-2)} + \frac{2^{q-1}-q}{\nabla(q-1)}\Bigg].
\end{aligned}$$

By summing this quantity, we will have

$${}_{4}\overset{q}{T}{}_{q} = \frac{a^{q-1}}{\nabla(q-1)}\left[3^{q-1} - 2^{q-1}q + \frac{q(q-1)}{1.2}\right].$$

By comparing this value of ${}_{4}\overset{q}{T}{}_{q}$ with the preceding, we will find

$${}^{1}M_{q} = 3^{q-1} - 2^{q-1}q + \frac{q(q-1)}{1.2}.$$

I have found, in the same manner

$$-{}^{2}M_{q}=4^{q-1}-3^{q-1}q+2^{q-1}\frac{q(q-1)}{1.2}-\frac{q(q-1)(q-2)}{1.2.3}.$$

It is useless to seek new terms, because their law is clear, so that we have generally

$$\begin{aligned}{}^{s}M_{q}=\pm\Bigg[&(s+2)^{q-1}-(s+1)^{q-1}q+s^{q-1}\frac{q(q-1)}{1.2}\\&-(s-1)^{q-1}\frac{q(q-1)(q-2)}{1.2.3}+\cdots\Bigg]\end{aligned}$$

s cannot exceed $q-3$, and the upper sign takes place if s is odd, and the lower if it is zero or odd; we have therefore

$$\begin{aligned}{}_{r}\overset{q}{T}_{n}=\mp\frac{n^{n-q}a^{q-1}}{\nabla(q-1)}\Bigg[&\frac{1}{\nabla(r-2)\nabla(n-r-q+2)}\\&-\frac{2^{q-1}-q}{\nabla(r-3)\nabla(n-r-q+3)}\\&+\frac{3^{q-1}-2^{q-1}q+\frac{q(q-1)}{1.2}}{\nabla(r-4)\nabla(n-r-q+4)}\\&-\cdots\\&\mp\frac{(q-1)^{q-1}-(q-2)^{q-1}q+\cdots}{\nabla(r-q)\nabla(n-r)}\Bigg];\end{aligned}$$

hence,

$$\begin{aligned}{}_{r}y_{n,z}=&\pm\frac{(nz)^{n-1}}{\nabla(r-1)\nabla(n-r)}\mp\frac{a(nz)^{n-2}}{\nabla(r-2)\nabla(n-r)}\\&\mp\frac{a^{2}(nz)^{n-3}}{1.2}\left[\frac{1}{\nabla(r-2)\nabla(n-r-1)}-\frac{1}{\nabla(r-3)\nabla(n-r)}\right]\\&\dots\\&\mp\frac{a^{q-1}(nz)^{n-q}}{\nabla(q-1)}\Bigg[\frac{1}{\nabla(r-2)\nabla(n-r-q+2)}\\&\qquad-\frac{2^{q-1}-q}{\nabla(r-3)\nabla(n-r-q+3)}\\&\qquad+\frac{3^{q-1}-2^{q-1}q+\frac{q(q-1)}{1.2}}{\nabla(r-4)\nabla(n-r-q+4)}\\&\qquad-\cdots\\&\qquad\mp\frac{(q-1)^{q-1}-(q-2)^{q-1}q+\cdots}{\nabla(r-q)\nabla(n-r)}\Bigg]\\&+\cdots\\&+\frac{a^{n-1}}{\nabla(n-1)}\left[(r-1)^{n-1}-\frac{n}{1}(r-2)^{n-1}+\frac{n(n-1)}{1.2}(r-3)^{n-1}-\cdots\right];\end{aligned}$$

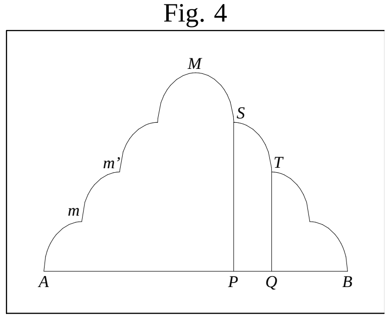
the upper sign having place if r is odd, and the lower if it is even, except for the term

$$\mp\frac{[(q-1)^{q-1}-(q-2)^{q-1}q+\cdots]}{\nabla(r-q)\nabla(n-r)},$$

for which the upper sign has place when q is odd, and the lower when it is even.

VIII.

Fig. 4



If we make, as previously, $AB=a=90^\circ$ (*fig.* 4), and if we divide this straight line into n equal parts, we will have, for the equation of the curve corresponding to the r^{th} part,

$$a^{n-2}y = {}_r y_{n,z}.$$

If we wish next to determine the probability that the mean inclination of the n orbits is contained between any two points P and Q, we will determine the area $STPQ$, and the quotient of this area divided by the entire area $Amm'MSTB$ will express the demanded probability. Thus we see that the entire area of the curve is an essential element to know. In order to attain it, I observe that the area contained between the two abscissas $\frac{r-1}{n}a$ and $\frac{r}{n}a$ is

$$\frac{1}{a^{n-2}}\left[\frac{{}_rA_n}{n}\left(\frac{a}{n}\right)^n+\frac{{}_rB_n}{n-1}\left(\frac{a}{n}\right)^{n-1}+\cdots\right];$$

I designate by ${}_rK_n$ this area; now the last of the equations (Ψ) of article VI gives

$${}_{r+1}H_{n+1}=n\left[\frac{{}_rA_n}{n}\left(\frac{a}{n}\right)^n+\frac{{}_rB_n}{n-1}\left(\frac{a}{n}\right)^{n-1}+\cdots\right];$$

therefore we will have

$${}_rK_n=\frac{{}_{r+1}H_{n+1}}{na^{n-2}};$$

hence,

$${}_rK_n=\frac{a^2}{n\nabla(n)}\left[r^n-\frac{n+1}{1}(r-1)^n+\cdots\right].$$

Presently, the entire area of the curve is equal to

$$ {}_{n}K_{n} + {}_{n-1}K_{n} + \cdots $$

Naming therefore S this area, we will have

$$\begin{aligned} S = \frac{a^2}{n\nabla(n)} \Big[\quad & n^n - \frac{n+1}{1}(n-1)^n + \frac{(n+1)n}{1.2}(n-2)^n - \cdots \\ & + (n-1)^n - \frac{n+1}{1}(n-2)^n + \cdots \\ & +(n-2)^n - \frac{n+1}{1}(n-3)^n + \cdots \Big] \\ = \frac{a^2}{n\nabla(n)} & \left[n^n - n(n-1)^n + \frac{(n+1)n}{1.2}(n-2)^n - \cdots \right]. \end{aligned}$$

Now, by designating by the character Δ the finite difference of a quantity, we have, as we know

$$ n^n - n(n-1)^n + \cdots = \Delta^n o^n; $$

moreover, we have generally

$$ \Delta x^n = \nabla(n); $$

hence,

$$ S = \frac{a^2}{n}. $$

Remark. — The area contained between the two abscissas $\frac{r-1}{n}a$ and $\frac{r}{n}a$ must be equal to the area contained between the two abscissas $\frac{n-r+1}{n}a$ and $\frac{n-r}{n}a$; that is to say that we must have

$$ {}_{r}K_{n} = {}_{n-r+1}K_{n}, $$

because these two areas are equally situated with respect to the extremities A and B; we must therefore have

$$ (\mu) \qquad \left\{ \begin{aligned} & r^n - \frac{n+1}{1}(r-1)^n + \frac{(n+1)n}{1.2}(r-2)^n - \cdots \\ & \quad = (n-r+1)^n - \frac{n+1}{1}(n-r)^n + \cdots, \end{aligned} \right. $$

continuing in it both members of this equation, until we arrive to a term which is null. We can be assured moreover of the truth of this equation, by observing that we have

$$\begin{aligned} & r^n - (n+1)(r-1)^n + \cdots \\ & \quad \mp (n+1)(r-n)^n \pm (n-r+1)^n = \Delta^{n+1}(r-n-1)^n, \end{aligned}$$

the $+$sign having place if n is odd, and the $-$sign if it is even; now $\Delta^{n+1} r^n = 0$, whence it is easy to conclude the equation (μ).

IX.

In order to apply the preceding theory to Nature, it would be necessary to suppose $n = 63$, because there exist now sixty-three comets of which we have calculated the orbits; but this calculation would be painful because of its length; thus, abandoning to those who will wish to undertake it, I will satisfy myself to suppose here $n = 12$; I imagine therefore the straight line AB, divided into twelve equal parts, of which each is consequently $7^\circ \frac{1}{2}$; we will find, by the preceding article, that the probability that the mean inclination of the twelve orbits will be contained between $45^\circ - 7^\circ \frac{1}{2}$ and 45°, or between $45^\circ + 7^\circ \frac{1}{2}$ and 45°, is equal to

$$\frac{6^{12}}{\nabla(12)}\left[1 - 13\left(\frac{5}{6}\right)^{12} + \frac{13.12}{1.2}\left(\frac{4}{6}\right)^{12} - \frac{13.12.11}{1.2.3}\left(\frac{3}{6}\right)^{12} + \frac{13.12.11.10}{1.2.3.4}\left(\frac{2}{6}\right)^{12} - \frac{13.12.11.10.9}{1.2.3.4.5}\left(\frac{1}{6}\right)^{12}\right].$$

Now, by making the calculation, I find this quantity equal to 0.339; whence it follows: 1° that there is odds of 839 to 161, that the mean inclination of twelve orbits will be over $37^\circ \frac{1}{2}$; 2° that the odds are as much that it will be below $52^\circ \frac{1}{2}$; 3° that there is odds of 678 against 322, that it will be between the limits $37^\circ \frac{1}{2}$ and $52^\circ \frac{1}{2}$.

Now, if we add together the information of the last twelve comets observed of which the table is here:

Comet of the years	Inclination of the orbits
1774	$82^\circ\ 48'\ 0''$
1773	61.25.21
1772	18.59.40
1771	11.15.29
1771	31.25.55
1770	1.44.30
1769	40.42.30
1766	8.20.00
1766	40.50.20
1764	53.54.19
1763	73.39.29
1762	85.03.02

we will find that their mean inclination is $42^\circ\ 31'$. In order to suspect in these comets a cause which tends to make them move in the plane of the ecliptic, it would be necessary that there be odds of a very great number against unity that, if they were launched at random, their mean inclination would surpass $42^\circ\ 30'$; now we just found that there is odds of 839 against 161, that which is not odds of six against one that it will be above $37^\circ \frac{1}{2}$, and it is considerably less odds that it will be above $42^\circ\ 30'$.

[Section X, “La figure de la terre” concerns the figure of the earth. In Section XI, “Sur les fonctions,” Laplace discusses a memoir of Lagrange, “Sur une nouvelle espèce de calcul relatif à l’intégration et à la différentiation des quantités variables,” which had been published in 1772. These sections have been omitted.]