BOOK II
CHAPTER III

DES LOIS DE LA PROBABILITÉ QUI RÉSULTENT DE LA MULTIPLICATION INDÉFINIE DES ÉVÉNEMENTS

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ON THE LAWS OF PROBABILITY WHICH RESULT FROM THE INDEFINITE MULTIPLICATION OF EVENTS

p being the probability of the arrival of a simple event at each trial, and $1-p$ that of its non-arrival, to determine the probability that, out of a very great number n of trials, the number of times that the event will take place will be comprehended within some given limits. Solution of the problem. The the most probable number of times, is np. Expression of the probability that this number of times will be comprehended within the limits $np \pm l$. The limits $\pm l$ remaining the same, this probability increases with the number of trials n: the probability remaining the same, the ratio of the interval $2l$ of the limits to the number n, is tightened when n increases, and, in the case of n infinite, this ratio becomes null, and the probability is changed into certitude. The solution of the preceding problem serves further to determine the probability that the value of p, supposed unknown, is comprehended within some given limits, when, out of a very great number n of trials, we know the number i of events corresponding to p which arrived: p is very nearly $\frac{i}{n}$, and generally when, in a trial, there must arrive any one of many simple events, the respective probabilities of these events are very nearly proportional to the number of times that they will arrive in a very great number n of trials. P being the probability of the arrival of an event composed of two simple events, of which p and $1-p$ are the respective probabilities, and $1-P$ being the probability of the non-arrival of this composite event; if out of a very great number n of arrivals and of non-arrivals of the same event, we know the number i of these arrivals, we have the probability that the value of P will be comprehended within some given limits, and, as P is a known function of p, we conclude from it the probability that the value of p will be comprehended within some given limits.N° 16.

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§16. In measure as events are multiplied, their respective probabilities are developed more and more: their mean results and the profits or the losses which depend on them, converge toward some limits which they approach with probabilities always increasing. The determination of these increases and of these limits, is one of the most interesting and most delicate parts of the analysis of chances. [275]

Let us consider first the manner in which the possibilities of two simple events of which one alone must arrive at each trial,[1] is developed when we multiply the number of trials. It is clear that the event of which the facility is greatest, must probably arrive more often in a given number of trials; and we are carried naturally to think that by repeating the trials a very great number of times, each of these events will arrive proportionally to its facility, that we will be able thus to discover by experience. We are going to demonstrate analytically this important theorem.

We have seen in §6 that if p and $1-p$ are the respective probabilities of two events a and b; the probability that in $x+x'$ trials, the event a will arrive x times and the event b, x' times, is equal to

$$\frac{1.2.3\ldots(x+x')}{1.2.3\ldots x.1.2.3\ldots x'}p^x(1-p)^{x'};$$

this is the $(x'+1)^{\text{st}}$ term of the binomial $[p+(1-p)]^{x+x'}$. Let us consider the greatest of these terms that we will designate by k. The anterior term will be $\frac{kp}{1-p}\cdot\frac{x'}{x+1}$, and the following term will be $k\frac{1-p}{p}\cdot\frac{x}{x'+1}$. In order that k be the greatest term, it is necessary that we have at the same time [276]

$$\frac{p}{1-p} < \frac{x+1}{x'} > \frac{x}{x'+1};$$

it is easy to conclude from it that if we make $x+x'=n$, we will have

$$x < (n+1)p > (n+1)p-1;$$

thus x is the greatest whole number comprehended within $(n+1)p$; by making therefore

$$x = (n+1)p - s,$$

that which gives

$$p = \frac{x+s}{n+1}, \quad 1-p = \frac{x'+1-s}{n+1}, \quad \frac{p}{1-p} = \frac{x+s}{x'+1-s},$$

s will be less than unity. If x and x' are very great numbers, we will have very nearly,

$$\frac{p}{1-p} = \frac{x}{x'},$$

that is that the exponents of p and of $1-p$, in the greatest term of the binomial, are quite nearly in the ratio of these quantities; so that of all the combinations which can take place

[1] Herein trial translates *coup*.

in a very great number n of trials, the most probable is that in which each event is repeated proportionally to its probability.

The l^{th} term, after the greatest, is

$$\frac{1.2.3\ldots n}{1.2.3\ldots(x-l).1.2.3\ldots(x'+l)}p^{x-l}(1-p)^{x'+l}.$$

We have, by §32 of the first Book,

$$1.2.3\ldots n = n^{n+\frac{1}{2}}c^{-n}\sqrt{2\pi}\left\{1+\frac{1}{12n}+\text{etc.}\right\},$$

that which gives

$$\begin{aligned}\frac{1}{1.2.3\ldots(x-l)} &= (x-l)^{l-x-\frac{1}{2}}\frac{c^{x-l}}{\sqrt{2\pi}}\left\{1-\frac{1}{12(x-l)}-\text{etc.}\right\},\\ \frac{1}{1.2.3\ldots(x'+l)} &= (x'+l)^{-x'-l-\frac{1}{2}}\frac{c^{x'+l}}{\sqrt{2\pi}}\left\{1-\frac{1}{12(x'+l)}-\text{etc.}\right\}.\end{aligned}$$ [277]

Let us develop the term $(x-l)^{l-x-\frac{1}{2}}$. Its hyperbolic logarithm is

$$\left(l-x-\tfrac{1}{2}\right)\left[\log x+\log\left(1-\tfrac{l}{x}\right)\right];$$

now we have

$$\log\left(1-\frac{l}{x}\right) = -\frac{l}{x}-\frac{l^2}{2x^2}-\frac{l^3}{3x^3}-\frac{l^4}{4x^4}-\text{etc.};$$

we will neglect the quantities of order $\frac{1}{n}$, and we will suppose that l^2 does not surpass at all the order n; then we will be able to neglect the terms of order $\frac{l^4}{x^3}$, because x and x' are of order n. We will have thus

$$\begin{aligned}&\left(l-x-\tfrac{1}{2}\right)\left[\log x+\log\left(1-\tfrac{l}{x}\right)\right]\\ &\quad=\left(l-x-\frac{1}{2}\right).\log x+l+\frac{l}{2x}-\frac{l^2}{2x}-\frac{l^3}{6x^2},\end{aligned}$$

that which gives, by passing again from the logarithms to the numbers,

$$(x-l)^{l-x-\frac{1}{2}} = c^{l-\frac{l^2}{2x}}x^{l-x-\frac{1}{2}}\left(1+\frac{l}{2x}-\frac{l^3}{6x^2}\right);$$

we will have similarly

$$(x'+l)^{-l-x'-\frac{1}{2}} = c^{-l-\frac{l^2}{2x'}}x'^{-l-x'-\frac{1}{2}}\left(1-\frac{l}{2x'}+\frac{l^3}{6x'^2}\right).$$

We have next by that which precedes, $p=\frac{x+s}{n+1}$, s being less than unity; by making therefore $p=\frac{x-z}{n}$, z will be contained within the limits $\frac{x}{n+1}$ and $-\frac{n-x}{n+1}$, and consequently it will be, setting aside the sign, below unity. The value of p gives $1-p=\frac{x'+z}{n}$; we will have by the preceding analysis, [278]

$$p^{x-l}(1-p)^{x'+l}=\frac{x^{x-l}x'^{x'+l}}{n^n}\left(1+\frac{nzl}{xx'}\right);$$

thence we deduce

$$\frac{1.2.3\ldots n}{1.2.3\ldots(x-l).1.2.3\ldots(x'+l)}p^{x-l}(1-p)^{x'+l}$$
$$=\frac{\sqrt{n}c^{-\frac{nl^2}{2xx'}}}{\sqrt{\pi}\sqrt{2xx'}}\left(1+\frac{nzl}{xx'}+\frac{l(x'-x)}{2xx'}-\frac{l^3}{6x^2}+\frac{l^3}{6x'^2}\right).$$

We will have the term anterior to the greatest term, and which is extended from it at the distance l, by making l negative in this equation; by uniting next these two terms, their sum will be

$$\frac{2\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}}c^{-\frac{nl^2}{2xx'}}.$$

The finite integral

$$\sum\frac{2\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}}c^{-\frac{nl^2}{2xx'}},$$

taken from $l=0$ inclusively, will express therefore the sum of all the terms of the binomial $[p+(1-p)]^n$, comprehended between the two terms, of which the one has p^{x+l} for factor, and the other has p^{x-l} for factor, and which are thus equidistant from the greatest term; but it is necessary to subtract from this sum, the greatest term which is evidently contained twice.

Now, in order to have this finite integral, we will observe that we have, by §10 of the first Book, y being function of l,

$$\sum y=\frac{1}{c^{\frac{dy}{dl}}-1}=\left(\frac{dy}{dl}\right)^{-1}-\frac{1}{2}\left(\frac{dy}{dl}\right)^{0}+\frac{1}{12}\frac{dy}{dl}+\text{etc.};$$

whence we deduce by the preceding section,

$$\sum y=\int y\,dl-\frac{1}{2}y+\frac{1}{12}\frac{dy}{dl}+\text{etc.}+\text{ constant.}$$

y being here equal to $\frac{2\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}}c^{-\frac{nl^2}{2xx'}}$, the successive differentials of y acquire for factor $\frac{nl}{2xx'}$ [279]
and its powers; thus l being supposed to not be able to be more than order $\sqrt{n}$, this factor is of order $\frac{1}{\sqrt{n}}$, and consequently its differentials divided by the respective powers of dl, decrease more and more; by neglecting therefore, as we have done previously, the terms of

order $\frac{1}{n}$, we will have, by starting with l the two finite and infinitely small integrals, and designating by Y the greatest term of the binomial,

$$\sum y = \int ydl - \frac{1}{2}y + \frac{1}{2}Y.$$

The sum of all the terms of the binomial $[p + (1-p)]^n$ contained between the two terms equidistant from the greatest term by the number l, being equal to $\sum y - \frac{1}{2}Y$, it will be

$$\int ydl - \frac{1}{2}y;$$

and if we add there the sum of these extreme terms, we will have for the sum of all these terms,

$$\int ydl + \frac{1}{2}y.$$

If we make

$$t = \frac{l\sqrt{n}}{\sqrt{2xx'}},$$

this sum becomes

$$\frac{2}{\sqrt{\pi}}\int dt\, c^{-t^2} + \frac{\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}}c^{-t^2}. \qquad (o)$$

The terms that we have neglected being of the order $\frac{1}{n}$, this expression is so much more exact, as n is greater: it is rigorous, when n is infinity. It would be easy, by the preceding analysis, to have regard to the terms of order $\frac{1}{n}$, and of the superior orders.

We have, by that which precedes, $x = np + z$, z being a number smaller than unity; we [280] have therefore

$$\frac{x+l}{n} - p = \frac{l+z}{n} = \frac{t\sqrt{2xx'}}{n\sqrt{n}} + \frac{z}{n};$$

thus formula (o) expresses the probability that the difference between the ratio of the number of times that the event a must arrive, to the total number of trials, and the facility p of this event, is comprehended within the limits

$$\pm\frac{t\sqrt{2xx'}}{n\sqrt{n}} + \frac{z}{n}. \qquad (l)$$

$\sqrt{2xx'}$ being equal to

$$n\sqrt{2p(1-p) + \frac{2z}{n}(1-2p) - \frac{2z^2}{n^2}};$$

we see that the interval comprehended between the preceding limits is of order $\frac{1}{\sqrt{n}}$.

If the limit of t, that we will designate by T, is supposed invariable, the probability determined by the function (o), remains very nearly the same; but the interval comprehended between the limits (l), diminishes without ceasing in measure as the trials are repeated, and it becomes null, when their number is infinite.

This interval being supposed invariable; when the events are multiplied, T increases without ceasing, and quite nearly as the square root of the number of trials. But when T is considerable, formula (o) becomes, by §27 of the first Book,

$$1 - \frac{c^{-T^2}}{2T\sqrt{\pi}} \cfrac{1}{1 + \cfrac{q}{1 + \cfrac{2q}{1 + \cfrac{3q}{1 + \text{etc.}}}}} + \frac{c^{-T^2}}{\sqrt{2n\pi\left[p(1-p) + \frac{z}{n}(1-2p) - \frac{z^2}{n^2}\right]}},$$

q being equal to $\frac{1}{2T^2}$. When we make T increase, c^{-T^2} diminishes with an extreme rapidity, [281]
and the preceding probability approaches rapidly to unity to which it becomes equal, when the number of trials is infinite.

There are here two sorts of approximations: one of them is relative to the limits taken on both sides of the facility of the event a; the other approximation is related to the probability that the ratio of the arrivals of this event, to the total number of trials, will be contained within these limits. The indefinite repetition of the trials increases more and more this probability, the limits remaining the same: it narrows more and more the interval of these limits, the probability remaining the same. Into infinity, this interval becomes null, and the probability is changed into certitude.

The preceding analysis unites to the advantage of demonstrating this theorem, the one to assign the probability that in a great number n of trials, the ratio of the arrivals of each event will be comprehended within some given limits. Let us suppose, for example, that the facilities of the births of boys and of girls are in the ratio of 18 to 17, and that there are born in one year, 14000 infants; we demand the probability that the number of boys will not surpass 7363, and will not be less than 7037.

In this case, we have

$$p = \frac{18}{35}, \quad x = 7200, \quad x' = 6800, \quad n = 14000, \quad l = 163;$$

formula (o) gives quite nearly $0,994303$ for the sought probability.

If we know the number of times that out of n trials, the event a arrived; formula (o) will give the probability that its facility p supposed unknown, will be comprehended within the given limits. In fact, if we name i this number of times, we will have, by that which precedes, the probability that the difference $\frac{i}{n} - p$ will be comprehended within the limits $\pm\frac{T\sqrt{2xx'}}{n\sqrt{n}} + \frac{z}{n}$; consequently, we will have the probability that p will be comprehended within the limits

$$\frac{i}{n} \mp \frac{T\sqrt{2xx'}}{n\sqrt{n}} - \frac{z}{n}.$$

The function $\frac{T\sqrt{2xx'}}{n\sqrt{n}}$ being of the order $\frac{1}{\sqrt{n}}$, we are able by neglecting the quantities of order [282] $\frac{1}{n}$, to substitute there i instead of x, and $n-i$ instead of x'; the preceding limits become thus, by neglecting the terms of order $\frac{1}{n}$,

$$\frac{i}{n} \mp \frac{T\sqrt{2i(n-i)}}{n\sqrt{n}};$$

and the probability that the facility of the event a is comprehended within these limits, is equal to

$$\frac{2}{\sqrt{\pi}} \int dt\, c^{-t^2} + \frac{\sqrt{n}c^{-T^2}}{\sqrt{\pi}\sqrt{2i(n-i)}}. \qquad (o')$$

We see thus that in measure as the events are multiplied, the interval of the limits is narrowed more and more, and the probability that the value of p falls within these limits, approaches more and more unity or certitude. It is thus that the events, in being developed, make known their respective probabilities.

We arrive directly to these results, by considering p as a variable which can be extended from zero to unity, and by determining, after the observed events, the probability of its diverse values, as we will see it when we will treat the probability of causes, deduced from observed events.

If we have three or a greater number of events a, b, c, etc., of which one alone must arrive at each trial; we will have, by that which precedes, the probability that in a very great number n of trials, the ratio of the number x of times that one of these events, a for example, will arrive, to the number n, will be comprehended within the limits $p \pm \alpha$, α being a very small fraction; and we see that in the extreme case of the number n infinite, the interval 2α of these limits can be supposed null, and the probability can be supposed equal to certitude, so that the numbers of arrivals of each event will be proportional to their respective facilities.

Sometimes the events, instead of making known directly the limits of the value of p, give those of a function of this value; then we conclude from it the limits of p, by the [283] resolution of equations. In order to give a quite simple example of it, let us consider two players A and B, of whom the respective skills are p and $1-p$, and playing together with this condition, that the game is won by the one of the two players who, out of three trials, will have vanquished twice his adversary, the third trial not being played, as useless, when one of the players is vanquished in the first two trials.

The probability of A to win the game, is the sum of the first two terms of the binomial $[p+(1-p)]^3$; it is consequently equal to $p^3+3p^2(1-p)$. Let P be this function; by raising the binomial $P+(1-P)$ to the power n, we will have, by the preceding analysis, the probability that, out of the number n of games, the number of games won by A will be comprehended within the given limits. It suffices for that to change p into P in formula (o).

If we name i the number of games won by A, formula (o') will give the probability that P will be comprehended within the limits

$$\frac{i}{n} \pm \frac{T\sqrt{2i(n-i)}}{n\sqrt{n}}.$$

Let therefore p' be the real and positive root of the equation

$$p^3 + 3p^2(1-p) = \frac{i}{n};$$

by designating by $p' \mp \delta p$ the limits of p, the corresponding limits of P will be very nearly $3p'^2 - 2p'^3 \mp 6p'(1-p')\delta p$; by equating these limits to the preceding, we will have

$$\delta p = \frac{T\sqrt{2i(n-i)}}{6p'(1-p')n\sqrt{n}};$$

thus formula (o') will give the probability that p will be comprehended within the limits

$$p' \mp \frac{T\sqrt{2i(n-i)}}{6p'(1-p')n\sqrt{n}}.$$

The number n of games does not determine the number of trials, since we are able to have some games of two trials, and others of three trials. We will have the probability [284] that the number of games of two trials, will be comprehended within the given limits, by observing that the probability of a game with two trials, is $p^2 + (1-p)^2$; Let us designate this function by P'. By elevating the binomial $P' + (1-P')$ to the power n, formula (o) will give the probability that the number of games of two trials will be comprehended within the limits $nP' \pm l$; now the number of games of two trials being $nP' \pm l$, the number of games with three trials will be $n(1-P') \mp l$; the total number of trials will be therefore $3n - nP' \mp l$; formula (o) will give therefore the probability that the number of trials will be comprehended within the limits

$$2n(1 + p - p^2) \mp T\sqrt{2nP'(1-P')}.$$

§17. Let us consider an urn A containing a very great number n of white and black balls, and let us suppose that at each drawing, we draw one ball from the urn, and that we replace it with a black ball. We demand the probability that after r drawings, the number of white balls will be x.

Let us name $y_{x,r}$ this probability. After a new drawing, it becomes $y_{x,r+1}$. But in order that there are x white balls after $r+1$ drawings, it is necessary that there are either $x+1$ white balls after the drawing r, and that the following drawing makes a white ball exit, or x white balls after the drawing r, and that the following drawing makes a black ball exit. The probability that there will be $x+1$ white balls after r drawings, is $y_{x+1,r}$, and the probability that then the following drawing will make a white ball exit, is $\frac{x+1}{n}$; the probability of the composite event is therefore $\frac{x+1}{n}y_{x+1,r}$; this is the first part of $y_{x,r+1}$. The probability that there will be x white balls after the drawing r, is $y_{x,r}$; and the probability that then there will exit a black ball, is $\frac{n-x}{n}$, because the number of black balls in the urn is $n-x$; the probability of the composite event is therefore $\frac{n-x}{n}y_{x,r}$; this is the second part of $y_{x,r+1}$. Thus we have

$$y_{x,r+1} = \frac{x+1}{n}y_{x+1,r} + \frac{n-x}{n}y_{x,r}.$$

If we make

$$x = nx', \quad r = nr', \quad y_{x,r} = y'_{x',r'},$$

this equation becomes [285]

$$y'_{x',r'+\frac{1}{n}} = \left(x' + \tfrac{1}{n}\right) y'_{x'+\frac{1}{n},r'} + (1-x')y'_{x',r'};$$

n being supposed a very great number, we are able to reduce into convergent series $y_{x,r'+\frac{1}{n}}$ and $y_{x'+\frac{1}{n},r'}$; we will have therefore, by neglecting the squares and the superior powers of $\frac{1}{n}$,

$$\frac{1}{n}\cdot\frac{dy'_{x',r'}}{dr'} = \frac{x'}{n}\cdot\frac{dy'_{x',r'}}{dx'} + \frac{1}{n}y'_{x',r'};$$

the integral of this equation in partial differences is

$$y'_{x',r'} = c^{r'}\phi(x'c^{r'}),$$

$\phi(x'c^{r'})$ being an arbitrary function of $x'c^{r'}$, that it is necessary to determine through the value of $y'_{x,0}$.

Let us suppose that urn A has been replenished in this manner. We project a right prism of which the base being a regular polygon of $p+q$ sides, is narrow enough so that the prism never falls on it. On the $p+q$ lateral faces, p are white and q are black, and we put into urn A, at each projection, a ball of the color of the face on which the prism falls. After n projections, the number of white balls will be quite nearly, by the preceding section, $\frac{np}{p+q}$, and the probability that it will be $\frac{np}{p+q}+l$, is, by the same section,

$$\frac{p+q}{\sqrt{2npq\pi}}c^{-\frac{(p+q)^2l^2}{2npq}}.$$

If we make

$$x = \frac{np}{p+q} + l, \quad \frac{(p+q)^2}{2pq} = i^2,$$

this function becomes

$$\frac{i}{\sqrt{\pi n}}c^{-\frac{i^2}{n}\left(x-\frac{np}{p+q}\right)^2};$$

this is the value of $y_{x,0}$, or of $y'_{x',0}$; but the preceding value of $y'_{x',r'}$, gives [286]

$$y_{x,0} = \phi\left(\frac{x}{n}\right);$$

we have therefore

$$\phi\left(\frac{x}{n}\right) = \frac{i}{\sqrt{n\pi}} c^{-i^2 n\left(\frac{x}{n}-\frac{p}{p+q}\right)^2};$$

hence,

$$y'_{x',r'} = \frac{ic^{r'}}{\sqrt{n\pi}} c^{-i^2 n\left(\frac{xc^{r'}}{n}-\frac{p}{p+q}\right)^2}$$

whence we deduce

$$y_{x,r} = \frac{ic^{\frac{r}{n}}}{\sqrt{n\pi}} c^{-\frac{i^2}{n}\left(xc^{\frac{r}{n}}-\frac{np}{p+q}\right)^2}.$$

The most probable value of x is that which renders $xc^{\frac{r}{n}} - \frac{np}{p+q}$ null, and consequently it is equal to

$$\frac{np}{(p+q)c^{\frac{r}{n}}};$$

the probability that the value of x will be contained within the limits

$$\frac{np}{(p+q)c^{\frac{r}{n}}} \pm \frac{\mu\sqrt{n}}{c^{\frac{r}{n}}},$$

is

$$2\int \frac{i\,d\mu}{\sqrt{\pi}} c^{-i^2\mu^2},$$

the integral being taken from $\mu = 0$.

Let us seek now the mean value of the number of white balls contained within urn A, after r drawings. This value is the sum of all the possible numbers of white balls, multiplied by their respective probabilities; it is therefore equal to

$$\frac{2np}{(p+q)c^{\frac{r}{n}}} \int \frac{i\,d\mu}{\sqrt{\pi}} c^{-i^2\mu^2},$$

the integral being taken from $\mu = 0$ to $\mu = \infty$. This value is thus [287]

$$\frac{np}{(p+q)c^{\frac{r}{n}}};$$

consequently, it is the same as the most probable value of x.

Let us consider now two urns A and B containing each the number n of balls, and let us suppose that in the total number $2n$ of balls, there are as many white as black. Let us imagine that we draw at the same time, one ball, from each urn, and that next one puts into one urn, the ball extracted from the other. Let us suppose that we repeat this operation, any number r times, by agitating at each time the urns, in order to well mix the balls; and let us seek the probability that after this number r of operations, there will be x white balls in urn A.

Let $z_{x,r}$ be this probability. The number of possible combinations in r operations, is n^{2r}; because at each operation, the n balls of urn A are able to be combined with each of n balls from urn B, that which produces n^2 combinations; $n^{2r}z_{x,r}$ is therefore the number of combinations in which it is possible to have x white balls in urn A after these operations. Now, it can happen that the $(r+1)^{\text{st}}$ operation makes a white ball exit from urn A, and makes a white ball return; the number of cases in which this can arrive, is the product of $n^{2r}z_{x,r}$ by the number x of white balls of urn A, and by the number $n-x$ of white balls which must be then in urn B, since the total number of white balls of the two urns, is n. In all these cases, there remains x white balls in urn A; the product $x(n-x)n^{2r}z_{x,r}$ is therefore one of the parts of $n^{2r+2}z_{x,r+1}$.

It can happen further that the $(r+1)^{\text{st}}$ operation makes exit and return into urn A, a black ball, that which conserves in this urn x white balls. Thus $n-x$ being, after the r^{th} operation, the number of black balls of urn A, and x being the one of black balls of urn B, $(n-x)xn^{2r}x_{x,r}$ is further a part of $n^{2r+2}z_{x,r+1}$.

If there are $x-1$ white balls in urn A after the r^{th} operation, and if the operation [288]
following makes a black ball exit from it, and makes a white ball return there; there will be x white balls in urn A after the $(r+1)^{\text{st}}$ operation; the number of cases in which that can happen, is the product of $n^{2r}z_{x-1,r}$ by the number $n-x+1$ of the black balls of urn A after the r^{th} drawing, and by the number $n-x+1$ of white balls of urn B, after the same operation; $(n-x+1)^2n^{2r}z_{x-1,r}$ is therefore again a part of $n^{2r+2}z_{x,r+1}$.

Finally, if there are $x+1$ white balls in urn A after the r^{th} operation, and if the operation following makes a white ball exit from it, and makes a black ball return there; there will be again, after this last operation, x white balls in the urn. The number of cases in which that can arrive, is the product of $n^{2r}z_{x+1,r}$ by the number $x+1$ of white balls of urn A, and by the number $x+1$ of black balls of urn B after the r^{th} operation; $(x+1)^2n^{2r}z_{x+1,r}$ is therefore further part of $n^{2r+2}z_{x,r+1}$.

By reuniting all these parts, and by equating their sum to $n^{2r+2}z_{x,r+1}$, we will have the equation in partial finite differences

$$z_{x,r+1} = \left(\frac{x+1}{n}\right)^2 z_{x+1,r} + \frac{2x}{n}\left(1-\frac{x}{n}\right) z_{x,r} + \left(1-\frac{x-1}{n}\right)^2 z_{x-1,r}.$$

Although this equation is in differences of the second order with respect to the variable x, however its integral contains only one arbitrary function which depends on the probability of the diverse values of x in the initial state of urn A. In fact, it is clear that if we knew the values of $z_{x,0}$ corresponding to all the values of x, from $x=0$ to $x=n$; the preceding equation will give all the values of $z_{x,1}$, $z_{x,2}$, etc., by observing that the negative values of x being impossible, $z_{x,r}$ is null when x is negative.

If n is a very great number, this equation is transformed into an equation in partial differences that we obtain thus. we have then, very nearly,

$$z_{x+1,r} = z_{x,r} + \left(\frac{dz_{x,r}}{dx}\right) + \frac{1}{2}\left(\frac{d^2 z_{x,r}}{dx^2}\right),$$
$$z_{x-1,r} = z_{x,r} - \left(\frac{dz_{x,r}}{dx}\right) + \frac{1}{2}\left(\frac{d^2 z_{x,r}}{dx^2}\right),$$
$$z_{x,r+1} = z_{x,r} + \left(\frac{dz_{x,r}}{dx}\right).$$

Let [289]

$$x = \frac{n + \mu\sqrt{n}}{2}, \quad r = nr', \quad z_{x,r} = U;$$

the preceding equation in the partial finite differences will become, by neglecting the terms of order $\frac{1}{n^2}$,

$$\left(\frac{dU}{dr'}\right) = 2U + 2\mu\left(\frac{dU}{d\mu}\right) + \left(\frac{d^2U}{d\mu^2}\right).$$

In order to integrate this equation which, as we are able to be assured by the method that I have given for this object, in the *Mémoires de l'Académie des Sciences*, of the year 1773,[2] is integrable in finite terms, only by means of definite integrals, let us make

$$U = \int \phi\, dt c^{-\mu t},$$

ϕ being a function of t and of r'. We will have

$$2\mu\left(\frac{dU}{d\mu}\right) = 2c^{-\mu t} t\phi - 2\int c^{-\mu t}(\phi\, dt + td\phi),$$
$$\left(\frac{ddU}{d\mu^2}\right) = \int c^{-\mu t} t^2 \phi\, dt;$$

the equation in the partial differentials in U, become thus

$$\int c^{-\mu t}\left(\frac{dU}{dr'}\right) dt = 2c^{-\mu t} t\phi + \int c^{-\mu t} dt\left[t^2\phi - 2t\frac{d\phi}{dt}\right].$$

By equating between them the terms affected of the $\int$ sign, we will have the equation in the partial differentials

$$\left(\frac{d\phi}{dr'}\right) = t^2\phi - 2t\left(\frac{d\phi}{dt}\right).$$

[2]This must refer to his "Recherches sur l'integration des équations différentielles aux différences finies, et sur leur usage dans la théorie des hasards." *Mémoires de l'Académie royale des Sciences de Paris (Savants étrangers)* [**?**].

The term outside the $\int$ sign, equated to zero, will give for the equation in the limits of the integral,

$$0 = t\phi c^{-\mu t}.$$

The integral of the preceding equation in the partial differentials of ϕ, is

$$\phi = c^{\frac{1}{4}t^2}\psi\left(\frac{t}{c^{2r'}}\right),$$

$\psi\left(\frac{t}{c^{2r'}}\right)$ being an arbitrary function of $\frac{t}{c^{2r'}}$; we have therefore [290]

$$U = \int dt c^{-\mu t + \frac{1}{4}t^2}\psi\left(\frac{t}{c^{2r'}}\right).$$

Let there be

$$t = 2\mu + 2s\sqrt{-1},$$

the expression of U will take this form,

$$U = c^{-\mu^2}\int ds\, c^{-s^2}\Gamma\left(\frac{s-\mu\sqrt{-1}}{c^{2r'}}\right). \tag{A}$$

It is easy to see that the preceding equation, to the limits of the integral, requires that the limits of the integral relative to s, are taken from $s = -\infty$ to $s = \infty$. By taking the radical $\sqrt{-1}$, with the $-$ sign, we will have for U an expression of this form

$$U = c^{-\mu^2}\int ds\, c^{-s^2}\Pi\left(\frac{s+\mu\sqrt{-1}}{c^{2r'}}\right),$$

the arbitrary function $\Pi(s)$ being able to be different from $\Gamma(s)$. The sum of these two expressions of U will be its complete value. But it is easy to be assured that the integrals being taken from $s = -\infty$ to $s = \infty$, the addition of this new expression of U adds nothing to the generality of the first, in which it is comprehended.

Let us develop now the second member of equation (A), according to the powers of $\frac{1}{c^{2r'}}$, and let us consider one of the terms of this development, such as

$$\frac{H^{(i)}c^{-\mu^2}}{c^{4ir'}}\int ds\, c^{-s^2}(s-\mu\sqrt{-1})^{2i};$$

this term becoming, after the integrations,

$$\frac{1.3.5\ldots(2i-1)}{2^i}\sqrt{\pi}\frac{H^{(i)}c^{-\mu^2}}{c^{4ir'}} \times\left[1 - \frac{i(2\mu)^2}{1.2} + \frac{i(i-1)(2\mu)^4}{1.2.3.4} - \frac{i(i-1)(i-2)(2\mu)^6}{1.2.3.4.5.6} + \text{etc.}\right]$$

Let us consider further one term of this development, relative to the odd powers of $\frac{1}{c^{2r'}}$, [291]
such as

$$\frac{L^{(i)}\sqrt{-1}c^{-\mu^2}}{c^{(4i+2)r'}}\int ds\, c^{-s^2}(s-\mu\sqrt{-1})^{2i+1},$$

This term becomes, after the integrations,

$$\frac{1.3.5\ldots(2i+1)L^{(i)}\sqrt{\pi}\mu c^{-\mu^2}}{2^i c^{(4i+2)r'}}\left[1-\frac{i(2\mu)^2}{1.2.3}+\frac{i(i-1)(2\mu)^4}{1.2.3.4.5}-\text{etc.}\right].$$

Thus we will have therefore the general expression of the probability U, developed into a series ordered according to the powers of $\frac{1}{c^{2r'}}$, a series which becomes very convergent, when r' is a considerable number. This expression must be such, that $\int U dx$ or $\frac{1}{2}\int U d\mu\sqrt{n}$ be equal to unity, the integrals being extended to all the values of x and of μ, that is from x null to $x=n$, and from $\mu=-\sqrt{n}$ to $\mu=\sqrt{n}$; because it is certain that one of the values of x needing to take place, the sum of the probabilities of all these values must be equal to unity. By taking the integral $\int c^{-\mu^2}d\mu$ within the limits of μ, we have the same result to very nearly, as by taking it from $\mu=-\infty$ to $\mu=\infty$: the difference is only of the order $\frac{c^{-n}}{\sqrt{n}}$; and seeing the extreme rapidity with which c^{-n} diminishes in measure as n increases, we see that this difference is insensible when n is a great number. This premised, let us consider in the integral $\frac{1}{2}\int U d\mu\sqrt{n}$, the term

$$\frac{1.3.5\ldots(2i-1)H^{(i)}\sqrt{n\pi}}{2^i c^{4ir'}}$$
$$\times\int d\mu c^{-\mu^2}\left[1-\frac{i(2\mu)^2}{1.2}+\frac{i(i-1)(2\mu)^4}{1.2.3.4}-\text{etc.}\right].$$

By extending the integral from $\mu=-\infty$ to $\mu=\infty$, this term becomes

$$\frac{1.3.5\ldots(2i-1)\frac{1}{2}H^(i)\pi\sqrt{n}}{2^i c^{4ir'}}\left[1-i+\frac{i(i-1)}{1.2}-\frac{i(i-1)(i-2)}{1.2.3}+\text{etc.}\right].$$

The factor $1-i+\frac{i(i-1)}{1.2}-$ etc. is equal to $(1-1)^i$; it is therefore null, except in the case [292]
of $i=0$, where it is reduced to unity. It is clear that the terms of the expression of U which contain the odd powers of μ, give a null result in the integral $\frac{1}{2}\int U d\mu\sqrt{n}$, extended from $\mu=-\infty$ to $\mu=\infty$; because these terms have for factor $c^{-\mu^2}$, and we have generally within these limits

$$\int\mu^{2i+1}d\mu c^{-\mu^2}=0.$$

There is therefore only the first term of the expression of U, a term that we will represent by $Hc^{-\mu^2}$, which can give a result in the integral $\frac{1}{2}\int U d\mu\sqrt{n}$, and this result is $\frac{1}{2}H\sqrt{n\pi}$; we have therefore

$$\frac{1}{2}H\sqrt{n\pi}=1;$$

consequently,

$$H = \frac{2}{\sqrt{n\pi}}.$$

The general expression of U has thus the following form,

$$U = \frac{2c^{-\mu^2}}{\sqrt{n\pi}} \left\{ \begin{array}{l} 1 + \dfrac{Q^{(1)}(1-2\mu^2)}{c^{4r'}} + \dfrac{Q^{(2)}(1-4\mu^2+\frac{4}{3}\mu^4)}{c^{8r'}} + \text{etc.} \\ + \dfrac{L^{(0)}\mu}{c^{2r'}} + \dfrac{L^{(1)}\mu(1-\frac{2}{3}\mu^2)}{c^{6r'}} + \dfrac{L^{(2)}\mu(1-\frac{4}{3}\mu^2+\frac{4}{15}\mu^4)}{c^{10r'}} + \text{etc.} \end{array} \right\}; \qquad (k)$$

$Q^{(1)}$, $Q^{(2)}$, etc., $L^{(0)}$, $L^{(1)}$, etc. being some indeterminate constants which depend on the initial value of U.

Let us suppose that U becomes X when r is null; X being a given function of μ. We have generally these two theorems,

$$0 = Q^{(i)} \int \mu^{2q} d\mu\, U_i c^{-\mu^2},$$
$$0 = L^{(i)} \int \mu^{2q+1} d\mu\, U'_i c^{-\mu^2},$$

when q is less than i; U_i and U'_i being functions of μ, by which $\frac{2Q^{(i)}c^{-\mu^2}}{\sqrt{n\pi}c^{4ir'}}$ and $\frac{2L^{(i)}c^{-\mu^2}}{\sqrt{n\pi}c^{(4i+2)r'}}$ are multiplied in the expression of U. In order to demonstrate these theorems, we will observe that, by that which precedes, $\frac{2Q^{(i)}c^{-\mu^2}U_i}{\sqrt{n\pi}}$ is equal to [293]

$$(\sqrt{-1})^{2i} H^{(i)} c^{-\mu^2} \int ds\, c^{-s^2}(\mu + s\sqrt{-1})^{2i};$$

it is necessary therefore to show that we have

$$0 = \iint \mu^{2q} ds\, d\mu\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i};$$

the integrals being taken from μ and s equal to $-\infty$ to μ and s equal to $+\infty$. By integrating first with respect to μ, this term becomes

$$\frac{2q-1}{2} \iint \mu^{2q-2} d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i}$$
$$+ i \iint \mu^{2q-1} d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i-1}.$$

By continuing to integrate thus by parts relatively to μ, we arrive finally to some terms of the form

$$k \iint d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2e},$$

e not being zero, and by that which precedes, these terms are null.

We will prove in the same manner, that we have

$$0 = L^{(i)} \int \mu^{2q+1}\, d\mu\, U'_i\, c^{-\mu^2}.$$

Thence it follows that we have generally

$$0 = \int U_i\, U_{i'} d\mu\, c^{-\mu^2}, \quad 0 = \int U'_i\ U'_{i'}\ d\mu\, c^{-\mu^2},$$

i and i' being different numbers. Because if, for example, i' is greater than i, all the powers of μ in U_i, are less than $2i'$; each of the terms of U_i will give therefore, by that which precedes, a result null in the integral $\int U_i\, U_{i'} d\mu\, c^{-\mu^2}$. The same reasoning holds for the integral $\int U'_i\ U'_{i'}\ d\mu c^{-\mu^2}$.

But these integrals are not nulls, when $i = i'$. We will obtain them in this case, in this [294] manner. We have, by that which precedes,

$$U_i = \frac{2^i(\sqrt{-1})^{2i} \int ds\, c^{-s^2}(\mu + s\sqrt{-1})^{2i}}{1.3.5\ldots(2i-1)\sqrt{\pi}}.$$

The term which has for factor μ^{2i} in this expression, is

$$\frac{2^i(\sqrt{-1})^{2i}\mu^{2i}}{1.3.5\ldots(2i-1)};$$

now, we are able to consider only this term in the first factor U_i of the integral $\int U_i U_i d\mu\, c^{-\mu^2}$; because the inferior powers of μ, in this factor, give a null result in the integral. we have therefore

$$\int U_i U_i d\mu\, c^{-\mu^2} = \frac{2^{2i}}{[1.3.5\ldots(2i-1)]^2\sqrt{\pi}} \iint \mu^{2i} d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i}.$$

We have, by integrating with respect to μ, from $\mu = -\infty$ to $\mu = \infty$,

$$\begin{aligned}&\iint \mu^{2i} d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i}\\ =\ &\frac{2i-1}{2}\iint \mu^{2i-2} d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i}\\ &+\frac{2i}{2}\iint \mu^{2i-1} d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i-1}\end{aligned}$$

The first term of the second member of this equation is null by that which precedes; this member is reduced therefore to its second term. We find in the same manner, that we have

$$\begin{aligned}&\iint \mu^{2i-1} d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i-1}\\ =\ &\frac{2i-1}{2}\iint \mu^{2i-2} d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i-2},\end{aligned}$$

and thus consecutively; we have therefore

$$\iint \mu^{2i} d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i} = \frac{1.2.3\ldots 2i\pi}{2^{2i}};$$

consequently,

$$\int U_i U_i d\mu\, c^{-\mu^2} = \frac{2.4.6\ldots 2i\sqrt{\pi}}{1.3.5\ldots(2i-1)}.$$

We will find in the same manner, [295]

$$\int U_i' U_i' d\mu\, c^{-\mu^2} = \frac{1}{2}\,\frac{2.4.6\ldots 2i\sqrt{\pi}}{1.3.5\ldots(2i+1)}.$$

We have evidently

$$\int U_i U_{i'}' d\mu\, c^{-\mu^2} = 0,$$

in the same case where i and i' are equal, because the product $U_i U_{i'}'$ contains only odd powers of μ. This premised.

The general expression of U gives for its initial value, that we have designated by X,

$$X = \frac{2c^{-\mu^2}}{\sqrt{n\pi}} \left\{ \begin{array}{c} 1 + Q^{(1)}\left(1 - 2\mu^2\right) + \text{etc.} \\ +L^{(0)}\mu + L^{(1)}\mu\left(1 - \frac{3}{2}\mu^2\right) + \text{etc.} \end{array} \right\}.$$

If we multiply this equation by $U_i d\mu$, and if we take the integrals from $\mu = -\infty$ to $\mu = \infty$, we will have, by virtue of the preceding theorems,

$$\int X U_i d\mu = \frac{2}{\sqrt{n\pi}} Q^{(i)} \int U_i U_i . d\mu . c^{-\mu^2},$$

whence we deduce

$$Q^{(i)} = \frac{1.3.5\ldots(2i-1)\frac{1}{2}\sqrt{n}}{2.4.6\ldots 2i} \int X U_i . d\mu;$$

we will find in the same manner,

$$L^{(i)} = \frac{1.3.5\ldots(2i+1)\sqrt{n}}{2.4.6\ldots 2i} \int X U_i' . d\mu.$$

We will have therefore thus the successive values of $Q^{(1)}$, $Q^{(2)}$, etc.; $L^{(0)}$, $L^{(1)}$, etc., by means of definite integrals, when X or the initial value of U will be given.

In the case where X is equal to $\frac{2i}{\sqrt{n\pi}} c^{-i^2\mu^2}$, the general expression of U takes a very simple form. Then the arbitrary function $\Gamma\left(\frac{s-\mu\sqrt{-1}}{c^{2r'}}\right)$ of formula (A) is of the form $kc^{-\beta\left(\frac{s-\mu\sqrt{-1}}{c^{2r'}}\right)^2}$. [296]
In order to determine the constants β and k, we will observe that by supposing

$$\beta' = \frac{\beta}{c^{4r'}},$$

we will have

$$U = kc^{-\frac{\mu^2}{1+\beta'}} \int ds\, c^{-(1+\beta')\left(s-\frac{\beta'\mu\sqrt{-1}}{1+\beta'}\right)^2}.$$

By making next

$$\sqrt{1+\beta'}\left(s - \frac{\beta'\mu\sqrt{-1}}{1+\beta'}\right) = s',$$

and observing that the integral relative to s must be taken from $s = -\infty$ to $s = \infty$, the integral relative to s' must be taken within the same limits, we will have

$$U = \frac{k\sqrt{\pi}}{\sqrt{1+\beta'}} c^{-\frac{\mu^2}{1+\beta'}}.$$

By comparing this expression to the initial value of U, which is

$$U = \frac{2i}{\sqrt{n\pi}} c^{-i^2\mu^2};$$

and observing that β is the initial value of β', we will have

$$i^2 = \frac{1}{1+\beta};$$

whence we deduce

$$\beta = \frac{1-i^2}{i^2}, \quad \beta' = \frac{1-i^2}{i^2 c^{4r'}}.$$

We must have next

$$\frac{k\sqrt{\pi}}{\sqrt{1+\beta}} = \frac{2i}{\sqrt{n\pi}};$$

that which gives

$$k\sqrt{\pi} = \frac{2}{\sqrt{n\pi}},$$

a value that we obtain next, by the condition that $\frac{1}{2}\int U d\mu \sqrt{n} = 1$, the integral being taken [297]
from $\mu = -\infty$ to $\mu = \infty$; we will have, for the expression of U, whatever be r',

$$U = \frac{2}{\sqrt{n\pi(1+\beta')}} c^{-\frac{\mu^2}{1+\beta'}}.$$

We find, indeed, that this value of U, substituted into the equation in the partial differentials in U, satisfies it.

β' diminishing without ceasing when r' increases, the value of U varies without ceasing, and becomes in its limit, when r' is infinity,

$$U = \frac{2}{\sqrt{n\pi}} c^{-\mu^2}.$$

In order to give an application of these formulas, let us imagine, in an urn C, a very great number m of white balls, and a parallel number of black balls. These balls having been mixed, let us suppose that we draw from the urn, n balls that we put into urn A. Let us suppose next that we put into urn B, as many white balls, as there are black balls in urn A, and as many black balls, as there are white balls in the same urn. It is clear that the number of cases in which there will be x white balls, and consequently $n-x$ black balls in urn A, is equal to the product of the number of combinations of the m white balls of urn C, taken x by x, by the number of combinations of the m black balls of the same urn, taken $n-x$ by $n-x$. This product is, by §3, equal to

$$\frac{m(m-1)(m-2)\cdots(m-x+1)}{1.2.3\dots x}\,\frac{m(m-1)(m-2)\cdots(m-n+x+1)}{1.2.3\dots(n-x)}$$

or to

$$\frac{(1.2.3\dots m)^2}{1.2.3\dots x.1.2.3\dots(n-x).1.2.3\dots(m-x).1.2.3\dots(m-n+x)}.$$

The number of all possible cases is the number of combinations of the $2m$ balls from urn C, taken n by n; this number is

$$\frac{1.2.3\dots 2m}{1.2.3\dots n.1.2.3\dots(2m-n)};$$

by dividing the preceding fraction by that here, we will have, for the probability of x or for [298]
the initial value of U,

$$\frac{(1.2.3\dots m)^2 1.2.3\dots n.1.2.3\dots(2m-n)}{1.2.3\dots x.1.2.3\dots(m-x).1.2.3\dots(n-x).1.2.3\dots(m-n+x).1.2.3\dots 2m}:$$

Now, if we observe that we have very nearly, when s is a great number,

$$1.2.3\dots s = s^{s+\frac{1}{2}}c^{-s}\sqrt{2\pi};$$

we will find easily after all the reductions, by making

$$x = \frac{n+\mu\sqrt{n}}{2},$$

and by neglecting the quantities of order $\frac{1}{n}$, which are not multiplied by μ^2,

$$U = \frac{2}{\sqrt{n\pi}}\sqrt{\frac{m}{2m-n}}c^{-\frac{m\mu^2}{2m-n}};$$

by making therefore

$$i^2 = \frac{m}{2m-n};$$

we will have

$$U = \frac{2i}{\sqrt{n\pi}} c^{-i^2\mu^2}.$$

If the number m is infinite, then $i^2 = \frac{1}{2}$, and the initial value of U is

$$U = \frac{\sqrt{2}}{\sqrt{n\pi}} c^{-\frac{1}{2}\mu^2}.$$

Its value, after any number of drawings, is

$$U = \frac{2}{\sqrt{n\pi\left(1 + c^{-\frac{4r}{n}}\right)}} c^{-\frac{\mu^2}{1+c^{-\frac{4r}{n}}}}.$$

The case of m infinite returns to the one in which the urns A and B would be filled, by projecting n times a coin which would bring forth indifferently *heads* or *tails*, and putting into urn A, a white ball, each time that *heads* would arrive, and a black ball, each time that [299] *tails* would arrive; and making the inverse for urn B. Because it is clear that the probability of drawing a white ball from urn C, is then $\frac{1}{2}$, as that to bring forth *heads* or *tails*.

By taking the integral $\int U dx$, or $\frac{1}{2}\int U d\mu\sqrt{n}$, from $\mu = -a$ to $\mu = a$, we will have the probability that the number of white balls of urn A, will be comprehended within the limits $\pm a\sqrt{n}$.

We are able to generalize the preceding result, by supposing the urn A filled as at the beginning of this section, by the projection of a prism of $p + q$ lateral faces, of which p are white and q are black. We have seen that then if we make

$$i^2 = \frac{(p+q)^2}{2pq},$$

we have at the origin, or when r is null,

$$U = \frac{i}{\sqrt{n\pi}} c^{-\frac{i^2}{n}\left(x - \frac{np}{p+q}\right)^2}.$$

Let us suppose p and q very little different, so that we have

$$p = \frac{p+q}{2}\left(1 + \frac{a}{\sqrt{n}}\right),$$
$$q = \frac{p+q}{2}\left(1 - \frac{a}{\sqrt{n}}\right),$$

we will have

$$i^2 = \frac{2}{1 - \frac{a^2}{n}},$$

or very nearly $i^2 = 2$; therefore

$$U = \frac{2}{\sqrt{2n\pi}} c^{-\frac{2}{n}\left(x - \frac{n}{2} - \frac{a\sqrt{n}}{2}\right)^2}.$$

By making therefore

$$x = \frac{n + \mu\sqrt{n}}{2};$$

we will have

$$U = \frac{2}{\sqrt{2n\pi}} c^{-\frac{1}{2}(\mu - a)^2}.$$

Let us suppose now that after any number whatsoever of drawings, we have [300]

$$U = \frac{2}{\sqrt{n\beta\pi}} c^{-\frac{(\mu-\alpha)^2}{\beta}},$$

β and α being some functions of r'. If we substitute this value into the equation in the partial differences in U, we will have

$$-\left(\frac{d\beta}{dr'}\right)\left[1 - \frac{2(\mu - \alpha)^2}{\beta}\right] + 4\left(\frac{d\alpha}{dr'}\right)(\mu - \alpha)$$
$$= 4(\beta - 1)\left[1 - \frac{2(\mu - \alpha)^2}{\beta}\right] - 8\alpha(\mu - \alpha),$$

whence we deduce the two following equations,

$$\frac{\left(\frac{d\beta}{dr'}\right)}{\beta - 1} = -4, \quad \left(\frac{d\alpha}{dr'}\right) = -2\alpha.$$

By integrating them, and observing that at the origin of r', $\alpha = a$ and $\beta = 2$, we will have

$$\beta = 1 + c^{-4r'}, \quad \alpha = ac^{-2r'};$$

that which gives

$$U = \frac{2}{\sqrt{n\pi(1 + c^{-4r'})}} c^{-\frac{(\mu - ac^{-2r'})^2}{1 + c^{-4r'}}}.$$

Let us seek now the mean value of the number of white balls contained in urn A, after r drawings. This value is the sum of the products of the diverse numbers of white balls, multiplied by their respective probabilities; it is therefore equal to the integral

$$\int \frac{n + \mu\sqrt{n}}{2}.U.\frac{d\mu\sqrt{n}}{2},$$

taken from $\mu = -\infty$ to $\mu = \infty$. By substituting for U its value given by formula (k), we will have, by virtue of the preceding theorems, for this integral,

$$\frac{1}{2}n + \frac{\sqrt{n}}{4}L^{(0)}c^{-\frac{2r}{n}}.$$

At the origin where r is null, this value is $\frac{1}{2}n + \frac{\sqrt{n}}{2}L^{(0)}$; thus we will have $L^{(0)}$ by means of the number of white balls that urn A contains at this origin. [301]

We are able to obtain quite simply in the following manner, the mean value of the number of white balls, after r drawings. Let us imagine that each white ball has a value that we will represent by unity, the black balls being supposed to have no value. It is clear that the value of urn A will be the sum of the products of all the possible numbers of white balls which are able to exist in the urn, multiplied by their respective probabilities; this value is therefore that which we have named *mean value of the number of white balls*. Let us name it z, after the r^{th} drawing. At the following drawing, if there exits a white ball, this value diminishes by one unit; now if we suppose that x is the number of white balls contained in the urn after the r^{th} drawing, the probability of extracting a white ball from it will be $\frac{x}{n}$; by naming therefore U the probability of this supposition, the integral $\int \frac{Uxdx}{n}$, extended from $x = 0$ to $x = n$, will be the diminution of z, resulting from the probability to extract a white ball, from the urn. If we make, as above, $\frac{r}{n} = r'$, and if we designate the very small fraction $\frac{1}{n}$ by dr', this diminution will be equal to zdr'; because z is equal to $\int Ux\,dx$, a sum of the products of the numbers of white balls, by their respective probabilities. The value of urn A is increased, if we extract a white ball from urn B, in order to put it into urn A; now, x being supposed the number of white balls of urn A, $n - x$ will be the one of the white balls of urn B, and the probability to extract a white ball from this last urn, will be $\frac{n-x}{n}$; by multiplying this probability by the probability U of x, the integral $\int U\frac{n-x}{n}\,dx$, taken from x null to $x = n$, will be the increase of z. $\int U.(n-x)dx$ is the value of urn B; by naming therefore z' this value, $z'dr'$ will be the increase of z: we will have therefore

$$dz = z'dr' - zdr'.$$

The sum of the values of the two urns is evidently equal to n, number of white balls that they contain, that which gives $z' = n - z$; substituting this value of z' into the preceding equation, it becomes [302]

$$dz = (n - 2z)dr';$$

whence we deduce by integrating,

$$z = \frac{1}{2}n + \frac{L^{(0)}}{4c^{2r'}},$$

$L^{(0)}$ being an arbitrary constant; that which is conformed to that which precedes.

We can extend all this analysis, to the case of any number whatsoever of urns: we will limit ourselves here to seek the mean value of the number of white balls that each urn contains after r drawings.

Let us consider a number e of urns, disposed circularly, and each containing the number n of balls, some white, and the others black; n being supposed a very great number. Let us suppose that after r drawings, z_0, z_1, z_2, $\dots z_{e-1}$ are the respective values of the diverse urns. Each drawing consists in extracting at the same time, one ball from each urn, and to put it into the following, by departing from one of them in a determined sense. If we make $\frac{r}{n} = r'$ and $\frac{1}{n} = dr'$; we will have, by the reasoning that we have just made relatively to two urns,

$$dz_i = (z_{i-1} - z_i)dr';$$

this equation holds from $i = 1$ to $i = e - 1$. In the case of $i = e$, we have

$$dz_0 = (z_{e-1} - z_0)dr';$$

by integrating these equations, and supposing that at the origin the respective values of each urn, or the numbers of white balls that they contain, are

$$\lambda_0,\ \lambda_1,\ \dots,\ \lambda_{e-1}.$$

We arrive to this result which holds from $i = 0$ to $i = e - 1$,

$$z_i = \frac{1}{e}\mathrm{S}c^{-\left(1-\cos\frac{2s\pi}{e}\right)r'}\left\{\begin{array}{l} \quad\lambda_0\cos\left(\dfrac{2si\pi}{e} - ar'\right) \\ +\lambda_1\cos\left(\dfrac{2s(i-1)\pi}{e} - ar'\right) \\ +\lambda_2\cos\left(\dfrac{2s(i-2)\pi}{e} - ar'\right) \\ \quad\dots\dots\dots\dots\dots\dots\dots\dots \\ +\lambda_{e-1}\cos\left(\dfrac{2s(i-e+1)\pi}{e} - ar'\right) \end{array}\right\} \qquad [303]$$

the sign S extending to all the values of s, from $s = 1$ to $s = e$, and a being equal to $\sin\frac{2s\pi}{e}$. The term of this expression, corresponding to $s = e$, is independent of r', and equal to $\frac{1}{e}(\lambda_0 + \lambda_1 + \cdots + \lambda_{e-1})$; that is, the entire sum of the white balls of the urns, divided by their number. This term is the limit of the expression of z_i; whence it follows that after an infinite number of drawings, the values of each urn are equal among them.