BOOK II
CHAPTER II

DE LA PROBABILITÉ DES ÉVÉNEMENTS COMPOSÉ D'ÉVÉNEMENTS SIMPLES DONT POSSIBILITÉS RESPECTIVES SONT DONNÉES

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ON THE PROBABILITY OF EVENTS COMPOSED OF SIMPLE EVENTS OF WHICH THE RESPECTIVE POSSIBILITIES ARE GIVEN

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To determine the probability to draw thus from the preceding urns x white balls, before bringing forth either x' black balls, or x'' red balls, or, etc. Solution of the problem by the method of combinations. Identity of this problem with the one which consists in determining the lots of a number n of players of whom the respective skills are known, when there are lacking, in order to win the game, x coupts to the first, x' to the second, x'' to the third, etc. . . . . . . Nº 7.

General solution of the preceding problem, by the analysis of generating functions. In the case of two players A and B of whom the respective skills are equal, the problem is the one that Pascal proposed to Fermat and that these two great geometers resolved. It reverts to imagining an urn which contains two balls, one white and the other black, bearing each the nº 1, the white ball being for player A, the black ball for player B. We draw from the urn one ball that we return next there, in order to proceed to a new drawing, and we continue thus until the sum of the drawn values, favorable to one of the players, attains a given number. After a certain number of drawings, there is lacking yet to player A, the number x, and to player B the number x'. The two players agree then to be retired from the game by dividing the stake that they have set in beginning: the concern is to know how this division must be made. That which returns to the players must be evidently proportional to their respective probabilities to win the game. Generalization and solution of the problem: 1° by supposing in the urn one white ball favorable to A and bearing the nº 1 and two black balls favorable to B and the one bearing, the nº 1, and the other, the nº 2; each ball diminishing with its number the number of points which lack to the player to whom it is favorable; 2° by supposing in the urn, two white balls bearing the nºs 1 and 2 and two black balls bearing the same numbers. Nº 8.

Conceiving in an urn, r balls marked with the nº 1, r balls marked with the nº 2, and so forth until nº n; these balls being well mixed in the urn and each drawn successively, we demand the probability that there will exit at least s balls at the rank indicated by their number. General solution of the problem and of the one in which, having i urns each containing the number n of balls, all of different colors and if we draw all successively from each urn, by completing the drawing from one urn, before passing to another urn, we demand the probability that one or many balls of the same color, will exit at the same rank in the complete drawings from the urns. . . . . Nº 9.

Two players A and B of whom the respective skills are p and q and of whom the first has the number a of tokens, and the second the number b, play with this condition, that the one who loses gives a token to his adversary, and that the game ends only when one of the players will have lost all his tokens; we demand the probability that one of the players will win the game before or at the n^{th} coup. Generating function of this probability, whence we deduce the general expression of the probability. Expression of the probability that the game will end before or at the n^{th} coup. That which it becomes, when we suppose a infinite. Very close value of the same expression, when we suppose moreover p and q equals, and when b is a considerable number. If $b = 100$, there is disadvantage to wager one against one that A will win the game in 23780 trials; but there is advantage to wager that he will win it in 23781 trials. . . . . Nº 10.

A number $n + 1$ of players play together with the following conditions. Two of among them play first, and the one who loses is retired after having set a franc into the game, in order to return

§3. If we develop the product $(1+p)(1+p')(1+p'')$.etc. composed of n factors; [189]
this development will contain all the possible combinations of the n letters p, p', p'', $\ldots$, $p^{(n-1)}$, taken one by one, two by two, three by three, etc.[1] to n; and each combination will have for coefficient unity. Thus the combination $pp'p''$ resulting from the product $(1+p)(1+p')(1+p'')$, multiplied by the term 1 of the development of the other factors; its coefficient is evidently unity. Now, in order to have the total number of combinations of n letters taken x by x; we will observe that each of these combinations become p^x, when we suppose p', p'', etc. equal to p. Then the product of the n preceding factors is changed into the binomial $(1+p)^n$; now the coefficient of p^x in the development of this binomial, is

$$\frac{n(n-1)(n-2)\ldots(n-x+1)}{1.2.3\ldots x};$$

this quantity expresses therefore the number of combinations of n letters taken x by x. We will have the total number of combinations of these letters, taken one by one, two by two, etc. to n by n, by making $p=1$, in the binomial $(1+p)^n$, and by subtracting unity from it; that which gives 2^n-1 for this number.

Let us suppose that in each combination, we have regard not only to the number of letters, but further to their situation; we will determine the number of combinations, by observing that, in the combination of two letters pp', we are able to put p' in the second place, and next in the first; that which gives the two combinations pp', $p'p$. By introducing [190]
next a new letter p'' in each of these combinations, we are able to put it in the first, in the second or in the third place; that which gives 2.3 combinations. By continuing thus, we see that in a combination of x letters, we are able to give $1.2.3\ldots x$ different situations; whence it follows that the total number of combinations of n letters, taken x by x, being by that which precedes,

$$\frac{n(n-1)(n-2)\ldots(n-x+1)}{1.2.3\ldots x},$$

the total number of combinations, when we have regard to the different situation of the letters, will be this same function, by suppressing its denominator.[2]

We are able easily, by means of these formulas, to determine the benefits of lotteries. Let us suppose that the number of tickets[3] of a lottery, be n, and that there exits r of them at each drawing; we wish to have the probability that a combination of s of these tickets, will exit in the first drawing.

The total number of combinations of tickets, taken r by r, is by that which precedes,

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r},$$

[1] *Translator's note*: one by one, two by two, etc. In other words, one at a time, two at a time. In modern notation, these are the combinations $\binom{n}{1}$, $\binom{n}{2}$, etc.

[2] *Translator's note*: That is, the number of permutations.

[3] *Translator's note*: The word is "numéro," or number used in the sense of a label. I have therefore chosen to render it as ticket. Laplace later uses "billet," in this case referring specifically to a lottery ticket.

In order to have among these combinations, the number of those in which the s tickets are comprehended, we will observe that if we subtract these tickets from the total of the tickets, and if we combine $r-s$ by $r-s$, there remains $n-s$, the number of these combinations will be the sought number; because it is clear that by adding the s tickets to each of these combinations, we will have the combinations r by r of the tickets in which are these s tickets.[4] This number is therefore

$$\frac{(n-s)(n-s-1)\dots(n-r+1)}{1.2.3\dots(r-s)};$$

by dividing it by the total number of combinations r by r of the n tickets, we will have for the sought probability

$$\frac{r(r-1)(r-2)\dots(r-s+1)}{n(n-1)(n-2)\dots(n-s+1)}.$$

By dividing this quantity by $1.2.3\dots s$, we will have by that which precedes, the probability [191]
that the s tickets will exit in a determined order among them. We will have the probability that the first s tickets of the drawing, will be those of the proposed combination, by observing that this probability reverts to that of bringing forth this combination, by supposing that there exits only s tickets at each drawing, that which reverts to making $r=s$ in the preceding function which becomes thus

$$\frac{1.2.3\dots s}{n(n-1)\dots(n-s+1)}.$$

Finally, we will have the probability that the s chosen tickets will exit first in a determined order, by reducing the numerator of this fraction, to unity.

The quotients of the stakes divided by these probabilities, are those which the lottery must render to the players: the excess of these quotients over that which it gives, is its benefit. In fact, if we name p the probability of the player, m his stake, and x that which the lottery must render to him, for equality of the game; $x-m$ will be the stake of the lottery; because having received the stake m, and rendering x to the player; it puts into the game only $x-m$. Now for equality of the game, the mathematical hope[5] of each player must be equal to his fear: his hope is the product of the stake $x-m$ of his adversary, by the probability p to obtain it: his fear is the product of his stake m, by the probability $1-p$ of the loss. We have therefore

$$p(x-m)=(1-p)m;$$

that is that for the equality of the game, the stakes must be reciprocal to the probabilities to win. This equation gives

$$x=\frac{m}{p};$$

[4] *Translator's note*: Laplace is here expressing the quantity $\binom{s}{s}\binom{n-s}{r-s}$.

[5] *Translator's note*: hope, the *espérance* or expectation.

thus that which the lottery must render, is the quotient of the stake divided by the probability of the player to win.

§4. A lottery being composed of n numbered tickets of which r exit at each drawing, we require the probability that after i drawings, all the tickets will have exited.

Let us name $z_{n,q}$ the number of cases in which, after i drawings, the totality of the tickets 1, 2, 3, $\dots q$ will have exited. It is clear that this number is equal to the number $z_{n,q-1}$ of cases in which the tickets 1, 2, 3, $\dots q-1$ have exited, less the number of cases in which these tickets being brought out, the ticket q is not drawn; now this last number is evidently the same as the one of the cases in which the tickets 1, 2, 3, $\dots q-1$ would be extracted, if we remove the ticket q from the n tickets of the lottery, and this number is $z_{n-1,q-1}$; we have therefore [192]

$$z_{n,q} = z_{n,q-1} - z_{n-1,q-1}. \qquad (i)$$

Now the number of all possible cases in a single drawing, being $\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}$, the one of all possible cases in i drawings, is

$$\left(\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}\right)^i.$$

The number of all the cases in which the ticket 1 will not exit in these i drawings, is the number of all possible cases, when we subtract this ticket from the n tickets in the lottery; and this number is

$$\left(\frac{(n-1)(n-2)\dots(n-r)}{1.2.3\dots r}\right)^i;$$

the number of cases in which the ticket 1 will exit in i drawings, is therefore

$$\left(\frac{n(n-1)\dots(n-r+1)}{1.2.3\dots r}\right)^i - \left(\frac{(n-1)(n-2)\dots(n-r)}{1.2.3\dots r}\right)^i,$$

or

$$\triangle\left(\frac{(n-1)(n-2)\dots(n-r)}{1.2.3\dots r}\right)^i;$$

this is the value of $z_{n,1}$. This premised, equation (i) will give, by making successively $q=2$, $q=3$, etc.,

$$z_{n,2} = \triangle^2\left(\frac{(n-2)(n-3)\dots(n-r-1)}{1.2.3\dots r}\right)^i,$$
$$z_{n,3} = \triangle^3\left(\frac{(n-3)(n-4)\dots(n-r-2)}{1.2.3\dots r}\right)^i,$$
etc.;

and generally, [193]

$$z_{n,q} = \triangle^q\left(\frac{(n-q)(n-q-1)\dots(n-r-q+1)}{1.2.3\dots r}\right)^i.$$

Thus the probability that the tickets 1, 2, 3, $\dots q$ will exit in i drawings, being equal to $z_{n,q}$ divided by the number of all possible cases, it will be

$$\frac{\triangle^q[(n-q)(n-q-1)\dots(n-r-q+1)]^i}{[n(n-1)(n-2)\dots(n-r+1)]^i}$$

If we make in this expression $q=n$, we will have, s being here the variable which must be supposed null in the result,

$$\frac{\triangle^n[s(s-1)\dots(s-r+1)]^i}{[n(n-1)\dots(n-r+1)]^i}$$

for the expression of the probability that all the tickets of the lottery will exit in i drawings.

If n and i are very great numbers, we will have by the formulas of §40 of the first Book, the value of this probability, by means of a highly convergent series. Let us suppose, for example, that only one ticket exits at each drawing, the preceding probability becomes

$$\frac{\triangle^n s^i}{n^i}.$$

Let us propose to determine the number i of drawings in which this probability is $\frac{1}{k}$, n and i being very great numbers. By following the analysis of the section cited, we will determine first a by the equation

$$0=\frac{i+1}{a}-s-\frac{nc^a}{c^a-1};$$

that which gives

$$a=\frac{i+1}{n+s}\left\{\frac{1-c^{-a}}{1-\frac{sc^{-a}}{n+s}}\right\}.$$

We have next by §40 of the first Book, when c^{-a} is a very small quantity of the order $\frac{1}{i}$, as that takes place in the present question; we have, I say, to the quantities nearly of order $\frac{1}{i^2}$, s being supposed null in the result of the calculation, [194]

$$\frac{\triangle^n s^i}{n^i}=\frac{\left(\frac{i}{i+1}\right)^{i+\frac{1}{2}}c^{na-i}(1-c^{-a})^{n-i}}{\sqrt{1-\left(\frac{i+1}{n}\right)c^{-a}}}.$$

Now we have, to the quantities nearly of the order $\frac{1}{i^2}$,

$$\left(\frac{i}{i+1}\right)^{i+\frac{1}{2}}=c^{-1};$$

by supposing next $c^{-a}=z$, we have

$$(1-c^{-a})^{n-i}=c^{(i-n)z}\left[1+\left(\frac{i-n}{2}\right)z^2\right];$$

moreover, the equation which determines a, gives

$$i+1-na=(i+1)z;$$

whence we deduce

$$c^{na-i-1}=c^{-iz}(1-z);$$

we will have therefore, to the quantities nearly of order $\frac{1}{i^2}$,

$$\frac{\triangle^n s^i}{n^i}=c^{-nz}\left[1+\left(\frac{i-2n+1}{2n}\right)z+\left(\frac{i-n}{2}\right)z^2\right].$$

In order to determine z, let us take up again the equation

$$a=\frac{i+1}{n}-\left(\frac{i+1}{n}\right)c^{-a};$$

we will have by formula (p) of §21 of the second Book of the *Mécanique céleste*,

$$z=c^{-a}=q+\left(\frac{i+1}{n}\right)q^2+\frac{3\left(\frac{i+1}{n}\right)^2}{1.2}q^3+\frac{4^2\left(\frac{i+1}{n}\right)^3}{1.2.3}q^4+\text{etc.};$$

q being supposed equal to $c^{-\left(\frac{i+1}{n}\right)}$. This value of z gives

$$c^{-nz}=c^{-nq}[1-(i+1)q^2];$$

consequently, [195]

$$\frac{\triangle^n s^i}{n^i}=c^{-nq}\left[1+\left(\frac{i+1-2n}{2n}\right)q-\left(\frac{n+i+2}{2}\right)q^2\right].$$

By equating this quantity to the fraction $\frac{1}{k}$, we will have

$$q=\frac{\log k}{n}\left[1+\left(\frac{i+1-2n}{2n^2}\right)-\left(\frac{n+i+2}{2n^2}\right)\log k\right];$$

now we have

$$i+1=-n\log q;$$

we will have therefore very nearly for the expression of the number i of drawings, according to which the probability that all the tickets will have exited is $\frac{1}{k}$,

$$i=(\log n-\log\log k)(n-\tfrac{1}{2}+\tfrac{1}{2}\log k)+\tfrac{1}{2}\log k;$$

we must observe that all these logarithms are hyperbolic.

Let us suppose the lottery composed of ten thousand tickets, or $n = 10000$, and $k = 2$, this formula gives

$$i = 95767,4$$

for the expression of the number of drawings, in which we can wager one against one, that the ten thousand tickets of the lottery will exit; it is therefore odds a little less than one against one that they will exit in 95767 drawings, and odds a little more than one against one that they will exit in 95768 drawings.

We will determine by a similar analysis, the number of drawings in which we are able to wager one against one, that all the tickets of the lottery of France will exit. This lottery is, as one knows, composed of 90 tickets of which five exit at each drawing. The probability that all the tickets will exit in i drawings, is then by that which precedes,

$$\frac{\triangle^n[s'(s'-1)(s'-2)(s'-3)(s'-4)]^i}{[n(n-1)(n-2)(n-3)(n-4)]^i},$$

n being here equal to 90, and s' needing to be supposed null in the result of the calculation. [196]
If we make $s = s' - 2$, this function becomes

$$\frac{\triangle^n[s(s^2-1)(s^2-4)]^i}{[(n-2)(\overline{n-2}^2-1)(\overline{n-2}^2-4)]^i};$$

or by developing in series,

$$\frac{(\triangle^n s^{5i} - 5i\triangle^n s^{5i-2} + \text{etc.})}{(n-2)^{5i}}\left(1 + \frac{5i}{(n-2)^2} + \text{etc.}\right),$$

s needing to be supposed equal to -2 in the result of the calculation.

We have by §40 of the first Book, by neglecting the terms of order $\frac{1}{i^2}$, and supposing c^{-a} very small of order $\frac{1}{i}$,

$$\frac{\triangle^n s^{5i}}{(n-2)^{5i}} = \frac{\left(\frac{5i+1}{a}\right)^{5i}\left(\frac{5i}{5i+1}\right)^{5i} c^{(n-2)a-5i}(1-c^{-a})^n}{(n-2)^{5i}\sqrt{1+\frac{1}{5i}-\frac{na^2c^{-a}}{5i(1-c^{-a})^2}}},$$

a being given by the equation

$$a = \frac{(5i+1)(1-c^{-a})}{(n-2)\left(1+\frac{2c^{-a}}{n-2}\right)}.$$

We have thus, by neglecting the terms of order $\frac{1}{i^2}$,

$$\frac{\triangle^n s^{5i}}{(n-2)^{5i}} = \frac{\left(1+\frac{2c^{-a}}{n-2}\right)^{5i}}{(1-c^{-a})^{5i}}(1-c^{-a})^n c^{1-(5i+1)c^{-a}-\frac{10ic^{-a}}{n-2}}$$
$$\times\left(\frac{5i}{5i+1}\right)^{5i}\left(1-\frac{1}{10i}+\frac{na^2c^{-a}}{10i}\right);$$

now we have

$$\left(1+\frac{2c^{-a}}{n-2}\right)^{5i} = c^{\frac{10ic^{-a}}{n-2}},$$
$$(1-c^{-a})^{-5i} = c^{5ic^{-a}}\left(1+\frac{5i}{2}c^{-2a}\right),$$
$$\left(\frac{5i}{5i+1}\right)^{5i} = c^{-1}\left(1+\frac{1}{10i}\right);$$

we will have therefore to the quantities nearly of order $\frac{1}{i^2}$,

$$\frac{\triangle^n s^i}{(n-2)^{5i}} = (1-c^{-a})^n\left(1-c^{-a}+\frac{5i}{2}c^{-2a}+\frac{na^2c^{-a}}{10i}\right).$$

By substituting for a its value, and observing that i is very little different from $n-2$, in the present case, as we will see hereafter; we have very nearly, [197]

$$\frac{na^2c^{-a}}{10i} = \frac{5i+12}{2(n-2)}c^{-a}.$$

I keep for greater exactitude, the term $\frac{12c^{-a}}{2(n-2)}$, although of order $\frac{1}{i^2}$, because of the size of its factor 12; we will have therefore

$$\frac{\triangle^n s^{5i}}{(n-2)^{5i}} = (1-c^{-a})^n\left(1+\frac{5i-2n+16}{2(n-2)}c^{-a}+\frac{5i}{2}c^{-2a}\right).$$

If we change in this equation $5i$ into $5i-2$, we will have that of $\frac{\triangle^n s^{5i-2}}{(n-2)^{5i-2}}$; but the value of a will no longer be the same. Let a' be this new value, we will have

$$a' = \frac{(5i-1)(1-c^{-a'})}{(n-2)\left(1+\frac{2c^{-a'}}{n-2}\right)},$$

that which gives, very nearly,

$$a' = a - \frac{2}{n-2}.$$

In that case we have

$$1-c^{-a'} = 1-c^{-a}-\frac{2c^{-a}}{n-2};$$

whence we deduce, by neglecting the quantities of order $\frac{1}{i}$,

$$(1-c^{-a'})^n = (1-c^{-a})^n;$$

consequently we have, by neglecting the quantities of order $\frac{1}{i}$,

$$\frac{\triangle^n s^{5i-2}}{(n-2)^{5i-2}} = (1-c^{-a})^n.$$

We will have therefore, to the quantities nearly of order $\frac{1}{i^2}$,

$$\begin{aligned}&\frac{\triangle^n[s(s^2-1)(s^2-4)]^i}{[n(n-1)(n-2)(n-3)(n-4)]^i}\\&\quad=(1-c^{-a})^n\left[1+\frac{(5i-2n+16)}{2(n-2)}c^{-a}+\frac{5i}{2}c^{-2a}\right].\end{aligned}$$

This quantity must, by the condition of the problem, be equal to $\frac{1}{2}$, that which gives [198]

$$1-c^{-a}=\sqrt[n]{\frac{1}{2}}\left[1-\frac{(5i-2n+16)}{2n(n-2)}c^{-a}-\frac{5i}{2n}c^{-2a}\right];$$

whence we deduce

$$c^{-a}=\left(1-\sqrt[n]{\frac{1}{2}}\right)\left[1+\frac{(5i-2n+16)}{2n(n-2)}+\frac{5i}{2n}c^{-a}\right];$$

consequently we have by hyperbolic logarithms,

$$a=\log\left(\frac{\sqrt[n]{2}}{\sqrt[n]{2}-1}\right)-\frac{(5i-2n+16)}{2n(n-2)}-\frac{5i}{2n}c^{-a};$$

now we have, to the quantities nearly of order $\frac{1}{i^2}$,

$$a=\frac{5i+1}{(n-2)\sqrt[n]{2}};$$

we will have therefore

$$i=\frac{n-2}{5}\sqrt[n]{2}\left[1-\frac{1}{2n}-\frac{16}{10in}-\frac{1}{2}(\sqrt[n]{2}-1)\right]\log\left(\frac{\sqrt[n]{2}}{\sqrt[n]{2}-1}\right).$$

By substituting for n its value 90, we find

$$i=85,53;$$

so that there is odds a little less than one to one that all the tickets will exit in 85 drawings, and odds a little more than one to one that they will exit in 86 drawings.

A quite simple and very close way to obtain the value of i, is to suppose $\frac{\triangle^n s^i}{n^i}$, or the series

$$1-n\left(\frac{n-1}{n}\right)^i+\frac{n(n-1)}{2}\left(\frac{n-2}{n}\right)^i-\text{etc.}$$

equal to the development

$$1-n\left(\frac{n-1}{n}\right)^i+\frac{n(n-1)}{1.2}\left(\frac{n-1}{n}\right)^{2i}-\text{etc.}$$

of the binomial $\left[1-\left(\frac{n-1}{n}\right)^i\right]^n$. In reality, the two series have the first two terms equal respectively. Their third terms are also, more of less, equal between them; for we have quite nearly $\left(\frac{n-2}{n}\right)^i$ equal to $\left(\frac{n-1}{n}\right)^{2i}$. In fact, their hyperbolic logarithms are, by neglecting the terms of order $\frac{i}{n^2}$, both equal to $-\frac{i}{n}$. We will see in the same way, that the fourth terms, the fifth, etc., are very little different, when n and i are very great numbers; but the difference increases without ceasing, in measure as the terms move away from the first, that which must in the end, produce in them an evident difference between the series themselves. In order to estimate it, let us determine the value of i concluded from the equality of the two series. By equating to $\frac{1}{k}$, the binomial $\left[1-\left(\frac{n-1}{n}\right)^i\right]^n$, we will have [199]

$$i=\frac{\log\left(1-\sqrt[n]{\frac{1}{k}}\right)}{\log\left(\frac{n-1}{n}\right)},$$

these logarithms being able to be, at will, hyperbolic or tabulated. Let $\sqrt[n]{\frac{1}{k}}=1-z$. We will have by taking the hyperbolic logarithms of each member of this equation,

$$\frac{1}{n}\log k=-\log(1-z)=z+\frac{z^2}{2}+\text{etc.},$$

that which gives very nearly,

$$z=\frac{\log k}{n}\left(1-\frac{\log k}{2n}\right);$$

we will have therefore in hyperbolic logarithms,

$$\log\left(1-\sqrt[n]{\frac{1}{k}}\right)=\log z=\log\log\ k-\log\ n-\frac{\log\ k}{2n}.$$

We have next [200]

$$\log\frac{n-1}{n}=-\frac{1}{n}-\frac{1}{2n^2}-\text{etc.}$$

The preceding expression for i becomes thus very nearly,

$$i=n(\log n-\log\log k)\left(1-\frac{1}{2n}\right)+\tfrac{1}{2}\log k;$$

the excess of the value found previously for i, over this one, is

$$\frac{\log k}{2}(\log n-\log\log k);$$

this excess becomes infinite, when n is infinite; but a very great number is necessary in order to render it very evident; and in the case of $n=10000$ and of $k=2$, it is still only three units.

If we consider likewise the development

$$1 - n\left(\frac{n-5}{n}\right)^i + \text{etc.}$$

of the expression $\frac{\triangle^n [s'(s'-1)(s'-2)(s'-3)(s'-4)]^i}{[n(n-1)(n-2)(n-3)(n-4)]^i}$, as the one of the binomial $\left[1 - \left(\frac{n-5}{n}\right)^i\right]^n$, we will have in order to determine the number i of coups in which we can wager one against one, that all the tickets will exit, the equation

$$\left[1 - \left(\frac{n-5}{n}\right)^i\right]^n = \frac{1}{2};$$

that which gives

$$i = \frac{\log\left(\frac{\sqrt[n]{2}}{\sqrt[n]{2}-1}\right)}{\log\left(\frac{n}{n-5}\right)}.$$

These logarithms can be tabulated. By making $n = 90$, we find

$$i = 85,204,$$

that which differs very little from the value $i = 85,53$ that we have found above. [201]

§5. An urn being supposed to contain the number x of balls, we draw from it a part or the totality, and we demand the probability that the number of extracted balls will be even.

The sum of the cases in which this number is unity, equals evidently x; since each of the balls can equally be extracted. The sum of the cases in which this number equals 2, is the sum of the combinations of x balls taken two by two, and this sum is, by §3, equal to $\frac{x(x-1)}{1.2}$. The sum of the cases in which the same number equals 3, is the sum of the combinations of balls taken three by three, and this sum is $\frac{x(x-1)(x-2)}{1.2.3}$, and so forth. Thus the successive terms of the development of the function $(1+1)^x - 1$, will represent all the cases in which the number of extracted balls, is successively 1, 2, 3, etc. to x; whence it is easy to conclude that the sum of all the cases relative to the odd numbers, is $\frac{1}{2}(1+1)^x - \frac{1}{2}(1-1)^x$, or 2^{x-1}; and that the sum of all the cases relative to the even numbers, is $\frac{1}{2}(1+1)^x + \frac{1}{2}(1-1)^x - 1$, or $2^{x-1} - 1$. The union of these two sums is the number of all the possible cases; this number is therefore $2^x - 1$; thus the probability that the number of extracted balls will be even, is $\frac{2^{x-1}-1}{2^x-1}$, and the probability that this number will be odd, is $\frac{2^{x-1}}{2^x-1}$; there is therefore advantage to wager with equality, on an odd number.

If the number x is unknown, and if we know only that it can not exceed n, and that this number and all the lesser are equally possible; we will have the number of all the possible cases relative to the odd numbers, by making the sum of all the values of 2^{x-1}, from $x = 1$ to $x = n$, and it is easy to see that this sum is $2^n - 1$. We will likewise have the sum of all the possible cases relative to the even numbers, by summing the function $2^{x-1} - 1$, from

$x = 1$ to $x = n$, and we find this sum equal to $2^n - n - 1$; the probability of an even number is therefore then $\frac{2^n - n - 1}{2^{n+1} - n - 2}$, and that of an odd number is $\frac{2^n - 1}{2^{n+1} - n - 2}$. [202]

Let us suppose now that the urn contains the number x of white balls, and the same number of black balls; we ask the probability that by drawing any even number of balls, we will bring forth as many white balls as black balls, all the even numbers being able to be brought forth equally.

The number of cases in which one white ball from the urn can be combined with a black ball, is evidently $x.x$. The number of cases in which two white balls can be combined with two black balls, is $\frac{x(x-1)}{1.2}\frac{x(x-1)}{1.2}$, and so forth. The number of cases in which we will bring forth as many white balls as black balls, is therefore the sum of the squares of the terms of the development of the binomial $(1+1)^x$, less unity. In order to have this sum, we will observe that it is equal to a term independent of a, in the development of $\left(1 + \frac{1}{a}\right)^x (1+a)^x$. This function is equal to $\frac{(1+a)^{2x}}{a^x}$. The term independent of a, in its development, is thus the coefficient of the middle term of the binomial $(1+a)^{2x}$; this coefficient is $\frac{1.2.3\ldots 2x}{(1.2.3\ldots x)^2}$; the number of cases in which we can draw from the urn as many white balls as black balls, is therefore

$$\frac{1.2.3\ldots 2x}{(1.2.3\ldots x)^2} - 1.$$

The number of all possible cases is the sum of the odd terms in the development of the binomial $(1+1)^{2x}$, less the first, or unity. This sum is $\frac{1}{2}(1+1)^{2x} + \frac{1}{2}(1-1)^{2x}$; the number of possible cases is therefore $2^{2x-1} - 1$, which gives for the expression of the probability sought

$$\frac{\frac{1.2.3\ldots 2x}{(1.2.3\ldots x)^2} - 1}{2^{2x-1} - 1}.$$

In the case where x is a large number, this probability is reduced by §33 of the first Book, to $\frac{2}{\sqrt{x\pi}}$, π being the semi-circumference of which 1 is the radius. [203]

§6. Let us consider a number $x + x'$ of urns, of which the first contains p white balls and q black balls; the second, p' white balls and q' black balls; the third, p'' white balls and q'' black balls, and so forth. Let us suppose that we draw successively one ball from each urn. It is clear that the number of all the possible cases in the first drawing, is $p + q$; in the second drawing, each of the cases of the first being able to be combined with the $p' + q'$ balls of the second urn, we will have $(p+q)(p'+q')$ for the number of all the possible cases relative to the first two drawings. In the third drawing, each of these cases can be combined with the $p'' + q''$ balls of the third urn; that which gives $(p+q)(p'+q')(p''+q'')$ for the number of all the possible cases relative to the three drawings, and thus of the rest. This product for the totality of the urns, will be composed of $x + x'$ factors; and the sum of all the terms of its development, in which the letter p, with or without accent, is repeated x times, and consequently the letter q, x' times, will express the number of cases in which we can draw from the urns, x white balls and x' black balls.

If p', p'', etc. are equal to p, and if q', q'', etc. are equal to q; the preceding product becomes $(p+q)^{x+x'}$. The term multiplied by $p^x q^{x'}$ in the development of this binomial is

$$\frac{(x+x')(x+x'-1)\dots(x+1)}{1.2.3\dots x'} p^x q^{x'}$$

or

$$\frac{1.2.3\dots(x+x')}{1.2.3\dots x.1.2.3\dots x'} p^x q^{x'}.$$

Thus this quantity expresses the number of cases in which we can bring forth x white balls and x' black balls. The number of all the possible cases being $(p+q)^{x+x'}$, the probability to bring forth x white balls and x' black balls is

$$\frac{1.2.3\dots(x+x')}{1.2.3\dots x.1.2.3\dots x'} \left(\frac{p}{p+q}\right)^x \left(\frac{q}{p+q}\right)^{x'},$$

where we must observe that $\frac{p}{p+q}$ is the probability of drawing a white ball from one of the [204]
urns, and that $\frac{q}{p+q}$ is the probability of drawing from it a black ball.

It is clear that it is perfectly equal to draw x white balls and x' black balls, from $x+x'$ urns which each contain p white balls and q black balls, or one alone of these urns, provided that we replace into the urn the ball extracted at each drawing.

Let us consider now a number $x+x'+x''$ urns of which the first contains p white balls, q black balls, and r red balls, of which the second contains p' white balls, q' black balls and r' red balls, and so forth. Let us suppose that we draw one ball from each of these urns. The number of all the possible cases will be the product of the $x+x'+x''$ factors,

$$(p+q+r)(p'+q'+r')(p''+q''+r'').\text{etc.}$$

The number of cases in which we will bring forth x white balls, x' black balls, and x'' red balls, will be the sum of all the terms of the development of this product, in which the letter p will be repeated x times; the letter q, x' times, and the letter r, x'' times. If all the accented letters p', q', etc., are equal to their non-accented correspondents, the preceding product is changed into the trinomial $(p+q+r)^{x+x'+x''}$. The term of its development which has for factor $p^x q^{x'} r^{x''}$, is

$$\frac{1.2.3\dots(x+x'+x'')}{1.2.3\dots x.1.2.3\dots x'.1.2.3\dots x''} p^x q^{x'} r^{x''};$$

thus the number of all the possible cases being $(p+q+r)^{x+x'+x''}$, the probability to bring forth x white balls, x' black balls, and x'' red balls, will be

$$\frac{1.2.3\dots(x+x'+x'')}{1.2.3\dots x.1.2.3\dots x'.1.2.3\dots x''} \left(\frac{p}{p+q+r}\right)^x \left(\frac{q}{p+q+r}\right)^{x'} \left(\frac{r}{p+q+r}\right)^{x''},$$

whence we must observe that $\frac{p}{p+q+r}$, $\frac{q}{p+q+r}$, $\frac{r}{p+q+r}$ are the respective probabilities of drawing from each urn one white ball, one black ball, and one red ball.

We see generally that if the urns contain each the same number of colors, p being the [205]
number of balls of the first color; q the one of the balls of the second color; r, s, etc., those

of the balls of the third, the fourth, etc.; $x + x' + x'' + x''' +$ etc. being the number of urns; the probability to bring forth x balls of the first color, x' balls of the second, x'' of the third, x''' of the fourth, etc., will be

$$\frac{1.2.3\ldots(x + x' + x'' + x''' + \text{etc.})}{1.2.3\ldots x.1.2.3\ldots x'.1.2.3\ldots x''.1.2.3\ldots x'''.\text{etc.}}\left(\frac{p}{p+q+r+s+\text{etc.}}\right)^x$$
$$\times\left(\frac{q}{p+q+r+s+\text{etc.}}\right)^{x'}\left(\frac{r}{p+q+r+s+\text{etc.}}\right)^{x''}\left(\frac{s}{p+q+r+s+\text{etc.}}\right)^{x'''}.\text{etc.}$$

§7. Let us determine now the probability of drawing from the preceding urns, x white balls, before bringing forth either x' black balls, or x'' red balls, etc. It is clear that n expressing the number of the colors, this must happen at the latest after $x + x' + x'' +$ etc. $- n + 1$ drawings. Because when the number of extracted white balls is equal or less than x, the one of the extracted black balls, less than x', the one of the extracted red balls, less than x'', etc.; the total number of the extracted balls, and consequently, the number of drawings is equal or less than $x + x' + x'' +$ etc. $- n + 1$; we can therefore consider here only $x + x' + x'' +$ etc. $- n + 1$ urns.

In order to have the number of cases in which we can bring forth x white balls at the $(x + i)^{\text{th}}$ drawing, it is necessary to determine all the cases in which $x - 1$ white balls will have come forth at the drawing $x + i - 1$. This number is the term multiplied by p^{x-1} in the development of the polynomial $(p + q + r + \text{etc.})^{x+i-1}$, and this term is

$$\frac{1.2.3\ldots(x + i - 1)}{1.2.3\ldots(x - 1)1.2.3\ldots i}p^{x-1}(q + r + \text{etc.})^i.$$

By combining it with the p white balls of the urn $x + i$, we will have a product which it will be necessary further to multiply by the number of all the possible cases relative to the $x' + x'' +$ etc. $- n - i + 1$ following drawings, and this number is

$$(p + q + r + \text{etc.})^{x'+x''+\text{etc.}-n-i+1};$$

we will have therefore [206]

$$\frac{1.2.3\ldots(x + i - 1)}{1.2.3\ldots(x - 1)1.2.3\ldots i}p^x(q + r + \text{etc.})^i(p + q + r + \text{etc.})^{x'+x''+\text{etc.}-n-i+1}; \qquad (a)$$

for the number of cases in which the event can happen precisely at the drawing $x + i$. It is necessary however to exclude from it the cases in which q is raised to the power x', those in which r is raised to the power x'', etc.; because in all these cases, it has already happened in the drawing $x + i - 1$, either x' black balls, or x'' red balls, or etc. Thus in the development of the polynomial $(q + r + \text{etc.})^i$, it is necessary to have regard only to the terms multiplied by $q^f r^{f'} s^{f''}$.etc., in which f is less than x', f' is less than x'', f'' is less than x''', etc. The term multiplied by $q^f r^{f'} s^{f''}$.etc., in this development, is

$$\frac{1.2.3\ldots i}{1.2.3\ldots f.1.2.3\ldots f'.1.2.3\ldots f''.\text{etc.}}q^f r^{f'} s^{f''}.\text{etc.}$$

All the terms that we must consider in the function (a) are therefore represented by

$$\frac{1.2.3\ldots(x+f+f'+\text{etc.}-1)}{1.2.3\ldots(x-1).1.2.3\ldots f.1.2.3\ldots f'.\text{etc.}} p^x q^f r^{f'}.\text{etc.} \\ \times (p+q+r+\text{etc.})^{x'+x''+\text{etc.}-f-f'-\text{etc.}-n+1}; \qquad (b)$$

because i is equal to $f + f' +$ etc. Thus by giving in this last function, to f all the integral values from $f = 0$ to $f = x' - 1$, to f' all the values from $f' = 0$ to $f' = x'' - 1$, and so forth, the sum of all these terms will express the number of cases in which the proposed event can happen in $x + x' +$ etc. $- n + 1$ drawings. It is necessary to divide this sum by the number of all the possible cases, that is by $(p + q + r + \text{etc.})^{x+x'+x''+\text{etc.}-n+1}$. If we designate by p' the probability of drawing a white ball from any one of the urns; by q' that of drawing from it a black ball; by r' that of drawing a red ball, etc., we will have

$$p' = \frac{p}{p+q+r+\text{etc.}}, \quad q' = \frac{q}{p+q+r+\text{etc.}}, \quad r' = \frac{r}{p+q+r+\text{etc.}}, \quad \text{etc.};$$

the function (b) divided by $(p + q + r + \text{etc.})^{x+x'+x''+\text{etc.}-n+1}$; will become thus, [207]

$$\frac{1.2.3\ldots(x+f+f'+\text{etc.}-1)}{1.2.3\ldots\overline{x-1}.1.2.3\ldots f.1.2.3\ldots f'.\text{etc.}} p'^x q'^f r'^{f'}.\text{etc.}$$

The sum of the terms which we will obtain by giving to f all the values from $f = 0$ to $f = x' - 1$, to f' all the values from $f' = 0$ to $f' = x'' - 1$, etc., will be the sought probability to bring forth x white balls before x' black balls, or x'' red balls, or, etc.

We can, after this analysis, determine the lot of a number n of players A, B, C, etc., of whom p', q', r', etc. represent the respective skills, that is, their probabilities to win a coup, when in order to win the game, there lack x coups to player A, x' coups to player B, x'' coups to player C, and so forth; because it is clear that relatively to player A, this reverts to determining the probability to bring forth x white balls before x' black balls, or x'' red balls, etc.; by drawing successively a ball from a number $x + x' + x'' +$ etc. $- n + 1$ from urns which contain each p white balls, q black balls, r red balls, etc., p, q, r, etc. being respectively equal to the numerators of the fractions p', q', r', etc. reduced to the same denominator.

§8. The preceding problem can be resolved in a quite simple manner, by the analysis of the generating functions. Let us name $y_{x,x',x'',\text{ etc.}}$ the probability of player A to win the game. At the following coup, this probability is changed into $y_{x-1,x',x'',\text{ etc.}}$, if A wins this coup, and the probability for this is p'. The same probability is changed into $y_{x,x'-1,x'',\text{ etc.}}$, if the coup is won by player B, and the probability for this is q'; it is changed into $y_{x,x',x''-1,\text{ etc.}}$ if the coup is won by player C, and the probability for this is r', and so forth; we have therefore the equation in the partial differences

$$y_{x,x',x'',\text{ etc.}} = p' y_{x-1,x',x'',\text{ etc.}} + q' y_{x,x'-1,x'',\text{ etc.}} + r' y_{x,x',x''-1,\text{ etc.}} + \text{etc.}$$

Let u be a function of t, t', t'', etc., such that $y_{x,x',x'',\text{ etc.}}$ is the coefficient of $t^x t'^{x'} t''^{x''}$.etc. in its development; the preceding equation in the partial differences will give, by passing from the coefficients to the generating functions, [208]

$$u = u(p't + q't' + r't'' + \text{etc.});$$

whence we deduce

$$1 = p't + q't' + r't'' + \text{etc.};$$

consequently,

$$\frac{1}{t} = \frac{p'}{1 - q't' - r't'' - \text{etc.}};$$

that which gives

$$\frac{u}{t^x} = \frac{up'^x}{(1 - q't' - r't'' - \text{etc.})^x} = up'^x \left\{ \begin{aligned} &1 + x(q't' + r't'' + \text{etc.}) \\ &\quad + \frac{x(x+1)}{1.2}(q't' + r't'' + \text{etc.})^2 \\ &\quad + \frac{x(x+1)(x+2)}{1.2.3}(q't' + r't'' + \text{etc.})^3 \\ &\quad + \text{etc.} \end{aligned} \right\}.$$

Now the coefficient of $t^0 t'^{x'} t''^{x''}$.etc. in $\frac{u}{t^x}$ is $y_{x,x',x'',\text{ etc.}}$; and the same coefficient in any term of the last member of the preceding equation, such as $ku.p'^x t'^{l'} t''^{l''}$, etc., is $kp'^x y_{0,x'-l',x''-l'',\text{ etc.}}$; the quantity $y_{0,x'-l',x''-l'',\text{ etc.}}$ is equal to unity, since then player A lacks no coup. Moreover, it is necessary to reject all the values of $y_{0,x'-l',x''-l'',\text{ etc.}}$ in which l' is equal or greater than x', l'' is equal or greater than x'', and so forth, because these terms are not able to be given by the equation in the partial differences, the game being finite, when any one of the players B, C, etc. have no more coups to play; it is necessary therefore to consider in the last member of the preceding equation, only the powers of t' less than x', only the powers of t'' less than x'', etc. The preceding expression of $\frac{u}{t^x}$ will give thus, by passing again from the generating functions to the coefficients,

$$y_{x,x',x'',\text{ etc.}} = p'^x \left\{ \begin{aligned} &1 + x(q' + r' + \text{etc.}) \\ &\quad + \frac{x(x+1)}{1.2}(q' + r' + \text{etc.})^2 \\ &\quad + \frac{x(x+1)(x+2)}{1.2.3}(q' + r' + \text{etc.})^3 \\ &\quad + \text{etc.} \end{aligned} \right\},$$

provided that we reject the terms in which the power of q' surpasses $x' - 1$, those in which the power of r' surpasses $x'' - 1$, etc. The second member of this equation is developed into one sequence of terms comprehended in the general formula [209]

$$\frac{1.2.3\ldots(x + f + f' + \text{etc.} - 1)}{1.2.3\ldots(x-1).1.2.3\ldots f.1.2.3\ldots f'.\text{etc.}} p'^x q'^f r'^{f'}.\text{etc.}$$

The sum of these terms relative to all the values of f, from f null to $f = x' - 1$, to all the values of f', from f' null to $f' = x'' - 1$, etc., will be the probability $y_{x,x',x'',\,\text{etc.}}$; that which is conformed to that which precedes.

In the case of two players A and B, we will have for the probability of player A,

$$p'^x \left\{ 1 + xq' + \frac{x(x+1)}{1.2} q'^2 \cdots + \frac{x(x+1)(x+2)\dots(x+x'-2)}{1.2.3\dots(x'-1)} q'^{x'-1} \right\}.$$

By changing p' into q', and x into x', and reciprocally, we will have

$$q'^{x'} \left\{ 1 + x'p' + \frac{x'(x'+1)}{1.2} p'^2 \cdots + \frac{x'(x'+1)(x'+2)\dots(x+x'-2)}{1.2.3\dots(x-1)} p'^{x-1} \right\}$$

for the probability that player B will win the game. The sum of these two expressions must be equal to unity, that which we see evidently by giving them the following forms. The first expression can, by §37 of the first Book, be transformed into this one

$$p'^{x+x'-1} \left\{ \begin{array}{l} 1 + \dfrac{(x+x'-1)}{1} \cdot \dfrac{q'}{p'} + \dfrac{(x+x'-1)(x+x'-2)}{1.2} \cdot \dfrac{q'^2}{p'^2} \\ \qquad\qquad \dots + \dfrac{(x+x'-1)\dots(x+1)}{1.2.3\dots(x'-1)} \cdot \dfrac{q'^{x'-1}}{p'^{x'-1}} \end{array} \right\};$$

and the second can be transformed into this one,

$$q'^{x+x'-1} \left\{ \begin{array}{l} 1 + \dfrac{(x+x'-1)}{1} \cdot \dfrac{p'}{q'} + \dfrac{(x+x'-1)(x+x'-2)}{1.2} \cdot \dfrac{p'^2}{q'^2} \\ \qquad\qquad \dots + \dfrac{(x+x'-1)\dots(x'+1)}{1.2.3\dots(x-1)} \cdot \dfrac{p'^{x-1}}{q'^{x-1}} \end{array} \right\}.$$

The sum of these expressions is the development of the binomial $(p' + q')^{x+x'-1}$, and [210] consequently it is equal to unity; because A or B needing to win each coup, the sum $p' + q'$ of their probabilities for this, is unity.

The problem which we just resolved, is the one which we name the *problem of points* in the analysis of chances. The chevalier de Meré proposed it to Pascal, with some other problems on the game of dice. Two players of whom the skills are equal, have put into the game the same sum; they must play until one of them has beat a given number of given times, his adversary; but they agree to quit the game, when there lack yet x points to the first player in order to attain this given number, and when there lack x' points to the second player. We demand in what way they must share the sum put into the game. Such is the problem that Pascal resolved by means of his arithmetic triangle. He proposed it to Fermat who gave the solution to it by way of combinations; that which occasioned between these two great geometers a discussion, after which Pascal recognized the goodness of the method of Fermat, for any number of players. Unhappily we have only one part of their correspondence, in which we see the first elements of the theory of probabilities, and their application to one of the most curious problems of this theory.[6]

The problem proposed by Pascal to Fermat, reverts to determining the respective probabilities of the players in order to win the game; because it is clear that the stake must be shared between the players, proportionally to their probabilities. These probabilities are the same as those of two players A and B, who must attain a given number of points, x being the number of those which are lacking to player A, and x' being the number of those which are lacking to player B, by imagining an urn containing two balls of which one is white and the other black, both bearing the no. 1, the white ball being for player A, and the black ball for player B. We draw successively one of these balls, and we return it into the urn after each drawing. By naming $y_{x,x'}$ the probability that player A will attain first, the given number of points, or, that which reverts to the same, that he will have x points before B has x', we will have [211]

$$y_{x,x'} = \frac{1}{2}y_{x-1,x'} + \frac{1}{2}y_{x,x'-1};$$

because if the ball that we extract is white, $y_{x,x'}$ is changed into $y_{x-1,x'}$, and if the ball extracted is black, $y_{x,x'}$ is changed into $y_{x,x'-1}$, and the probability of each of these events is $\frac{1}{2}$; we have therefore the preceding equation.

The generating function of $y_{x,x'}$ in this equation in the partial differences, is, by §20 of the first Book,

$$\frac{M}{1 - \frac{1}{2}t - \frac{1}{2}t'},$$

M being an arbitrary function of t'. In order to determine it, we will observe that $y_{0,0}$ can not take place, since the game ceases, when one or the other of the variables x and x' is null; M must therefore have t' for factor. Moreover $y_{0,x'}$ is unity, whatever be x'; the probability of player A is changing then into certitude: now the generating function of unity, is generally $\frac{t'^i}{1-t'}$, because the coefficients of the powers of t' in the development of this function, are all equal to unity; in the present case, $y_{0,x'}$ being able to hold when x' is either 1, or 2, or 3, etc., i must be equal to unity; the generating function of $y_{0,x'}$ is therefore equal to $\frac{t'}{1-t'}$; this is the coefficient of t^0 in the development of the generating function of $y_{x,x'}$ or in

$$\frac{M}{1 - \frac{1}{2}t - \frac{1}{2}t'};$$

we have therefore

$$\frac{M}{1 - \frac{1}{2}t'} = \frac{t'}{1 - t'},$$

that which gives

$$M = \frac{t'(1 - \frac{1}{2}t')}{(1 - t')};$$

[6] For this correspondence, see F.N. David, *Games, gods and gambling*, [**?**].

consequently the generating function of $y_{x,x'}$ is

$$\frac{t'(1-\frac{1}{2}t')}{(1-t')(1-\frac{1}{2}t-\frac{1}{2}t')}.$$

By developing it with respect to the powers of t, we have [212]

$$\frac{t'}{1-t'}\left(1+\frac{1}{2}\cdot\frac{t}{1-\frac{1}{2}t'}+\frac{1}{2^2}\cdot\frac{t^2}{(1-\frac{1}{2}t')^2}+\frac{1}{2^3}\cdot\frac{t^3}{(1-\frac{1}{2}t')^3}+\text{etc.}\right).$$

The coefficient of t^x in this series, is

$$\frac{1}{2^x}\cdot\frac{t'}{(1-t')(1-\frac{1}{2}t')^x};$$

$y_{x,x'}$ is therefore the coefficient of $t'^{x'}$ in this last quantity: now we have

$$\frac{t'}{(1-t')(1-\frac{1}{2}t')^x} = \frac{t'+\frac{1}{2}x\,t'^2+\frac{1}{2^2}\frac{x(x+1)}{2}\,t'^3\cdots+\frac{1}{2^{x'-1}}\frac{x(x+1)(x+2)\dots(x+x'-2)}{1.2.3\dots(x'-1)}\,t'^{x'}+\text{etc.}}{1-t'}.$$

By reducing into series the denominator of this last fraction, and multiplying the numerator by this series, we see that the coefficient of $t'^{x'}$ in this product, is that which this numerator becomes when we make $t'=1$; we have therefore

$$y_{x,x'}=\frac{1}{2^x}\left\{\begin{array}{c}1+x\cdot\frac{1}{2}+\frac{x(x+1)}{1.2}\cdot\frac{1}{2^2}+\frac{x(x+1)(x+2)}{1.2.3}\cdot\frac{1}{2^3}\\ \dots+\frac{x(x+1)\dots(x+x'-2)}{1.2.3\dots(x'-1)}\cdot\frac{1}{2^{x'-1}}\end{array}\right\};$$

a result conformed to that which precedes.

Let us imagine presently that there is in the urn a white ball bearing the no. 1, and two black balls, of which one bears the no. 1, and the other bears the no. 2, the white ball being favorable to A, and the black balls to his adversary: each ball diminishing by its value, the number of points which lack to the player to which it is favorable. $y_{x,x'}$ being always the probability that player A will attain first the given number, we will have the equation in the partial differences

$$y_{x,x'}=\frac{1}{3}\,y_{x-1,x'}+\frac{1}{3}\,y_{x,x'-1}+\frac{1}{3}\,y_{x,x'-2};$$

because in the following drawing, if the white balls exits, $y_{x,x'}$ becomes $y_{x-1,x'}$; if the black [213]
ball numbered 1 exits, $y_{x,x'}$ becomes $y_{x,x'-1}$; and if the black ball numbered 2 exits, $y_{x,x'}$ becomes $y_{x,x'-2}$, and the probability of each of these events is $\frac{1}{3}$.

The generating function of $y_{x,x'}$ is

$$\frac{M}{1 - \frac{1}{3}t - \frac{1}{3}t' - \frac{1}{3}t'^2},$$

M being an arbitrary function of t', which must, by that which precedes, have for factor t', and in the present case, be equal to

$$\frac{t'}{1-t'} \cdot \left(1 - \frac{1}{3}t' - \frac{1}{3}t'^2\right);$$

so that the generating function of $y_{x,x'}$ is

$$\frac{t'(1 - \frac{1}{3}t' - \frac{1}{3}t'^2)}{(1-t')(1 - \frac{1}{3}t - \frac{1}{3}t' - \frac{1}{3}t'^2)}:$$

The coefficient of t^x in the development of this function, is

$$\frac{1}{3^x} \cdot \frac{t'}{1-t'} \cdot \frac{1}{(1 - \frac{1}{3}t' - \frac{1}{3}t'^2)^x};$$

and there results from this that we just said, that the coefficient of $t'^{x'}$ in the development of this last quantity, is equal to

$$\frac{1}{3^x} \cdot \left\{ \begin{aligned} t' + \frac{xt'^2(1+t')}{3} + \frac{x(x+1)}{1.2} \cdot \frac{t'^3(1+t')^2}{3^2} \\ + \frac{x(x+1)(x+2)}{1.2.3} \cdot \frac{t'^4(1+t')^3}{3^3} + \text{etc.} \end{aligned} \right\};$$

by rejecting from the development in this series, all the powers of t' superior to $t'^{x'}$, and supposing in this that we conserve, $t' = 1$, this will be the expression of $y_{x,x'}$.

It is easy to translate this process into a formula. Thus by supposing x' even and equal to $2r+2$, we find

$$\begin{aligned} y_{x,x'} = & \frac{1}{3^x}\left\{1 + x \cdot \frac{2}{3} + \frac{x(x+1)}{1.2}\left(\frac{2}{3}\right)^2 \dots \frac{x(x+1)\dots(x+r-1)}{1.2.3\dots r}\left(\frac{2}{3}\right)^r\right\} \\ & + \frac{x(x+1)\dots(x+r)}{1.2.3\dots(r+1)\,3^{x+r+1}}\left\{1 + (r+1) + \frac{(r+1)r}{1.2} \dots + \frac{(r+1)r\dots 2}{1.2.3\dots r}\right\} \\ & + \frac{x(x+1)\dots(x+r+1)}{1.2.3\dots(r+2)\,3^{x+r+2}}\left\{1 + (r+2) \dots + \frac{(r+2)(r+1)\dots 4}{1.2.3\dots(r-1)}\right\} \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + \frac{x(x+1)\dots(x+2r)}{1.2.3\dots(2r+1)\,3^{x+2r+1}}. \end{aligned}$$

If we suppose x' odd and equal to $2r+1$, we will have [214]

$$
\begin{aligned}
y_{x,x'} = & \frac{1}{3^x}\left\{1 + x\cdot\frac{2}{3} + \frac{x(x+1)}{1.2}\left(\frac{2}{3}\right)^2 \cdots + \frac{x(x+1)\dots(x+r-1)}{1.2.3\dots r}\left(\frac{2}{3}\right)^r\right\} \\
& + \frac{x(x+1)\dots(x+r)}{1.2.3\dots(r+1)\,3^{x+r+1}}\left\{1 + (r+1) + \frac{(r+1)r}{1.2}\cdots + \frac{(r+1)r\dots 3}{1.2.3\dots(r-1)}\right\} \\
& + \frac{x(x+1)\dots(x+r+1)}{1.2.3\dots(r+2)\,3^{x+r+2}}\left\{1 + (r+2) + \frac{(r+2)(r+1)}{1.2}\cdots + \frac{(r+2)(r+1)\dots 5}{1.2.3\dots(r-2)}\right\} \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& + \frac{x(x+1)\dots(x+2r-1)}{1.2.3\dots 2r\,3^{x+2r}}.
\end{aligned}
$$

Thus in the case of $x = 2$ and $x' = 5$, we have

$$y_{2,5} = \frac{350}{729}.$$

Let us imagine further that there are in the urn two white balls distinguished as the two black balls, by the nos. 1 and 2; the probability of player A will be given by the equation in the partial differences

$$y_{x,x'} = \frac{1}{4}y_{x-1,x'} + \frac{1}{4}y_{x-2,x'} + \frac{1}{4}y_{x,x'-1} + \frac{1}{4}y_{x-1,x'-2}.$$

The generating function of $y_{x,x'}$ is then, by §20 of the first Book,

$$\frac{M + Nt}{1 - \frac{1}{4}t - \frac{1}{4}t' - \frac{1}{4}t^2 - \frac{1}{4}t'^2};$$

M and N being two arbitrary functions of t'. In order to determine them, we will observe that $y_{0,x'}$ is always equal to unity, and that it is necessary to exclude in M the null power of t'; we have therefore

$$M = \frac{t'}{1-t'}\left(1 - \frac{1}{4}t' - \frac{1}{4}t'^2\right).$$

In order to determine N, let us seek the generating function of $y_{1,x'}$. If we observe that $y_{0,x'}$ is equal to unity, and that player A having no more need but of one point, he wins the game, either that he brings forth the white ball numbered 1, or the white ball numbered 2; the preceding equation in the partial differences will give

$$y_{1,x'} = \frac{1}{2} + \frac{1}{4}y_{1,x'-1} + \frac{1}{4}y_{1,x'-2}.$$

Let us suppose $y_{1,x'} = 1 - y'_{x'}$; we will have [215]

$$y'_{x'} = \frac{1}{4}y'_{x'-1} + \frac{1}{4}y'_{x'-2}.$$

The generating function of this equation is

$$\frac{m + nt'}{1 - \frac{1}{4}t' - \frac{1}{4}t'^2},$$

m and n being two constants. In order to determine them, we will observe that $y_{1,0} = 0$, and that consequently $y'_0 = 1$, that which gives $m = 1$. The generating function of $y'_{x'}$ is therefore

$$\frac{1 + nt'}{1 - \frac{1}{4}t' - \frac{1}{4}t'^2}.$$

We have next evidently $y_{1,1} = \frac{1}{2}$, that which gives $y'_1 = \frac{1}{2}$; y'_1 is the coefficient of t' in the development of the preceding function, and this coefficient is $n + \frac{1}{4}$; we have therefore $n + \frac{1}{4} = \frac{1}{2}$, or $n = \frac{1}{4}$. The generating function of unity is $\frac{1}{1-t'}$, because here all the powers of t' can be admitted; we have thus

$$\frac{1}{1 - t'} - \frac{1 + \frac{1}{4}t'}{1 - \frac{1}{4}t' - \frac{1}{4}t'^2}, \quad \text{or} \quad \frac{\frac{1}{2}t'}{(1 - t')(1 - \frac{1}{4}t' - \frac{1}{4}t'^2)},$$

for the generating function of $y_{1,x'}$. This same function is the coefficient of t in the development of the generating function of $y_{x,x'}$, a function which, by that which precedes, is

$$\frac{\frac{t'}{1-t'}(1 - \frac{1}{4}t' - \frac{1}{4}t'^2) + Nt}{1 - \frac{1}{4}t - \frac{1}{4}t' - \frac{1}{4}t^2 - \frac{1}{4}t'^2};$$

this coefficient is

$$\frac{\frac{1}{4}t'}{(1 - t')(1 - \frac{1}{4}t' - \frac{1}{4}t'^2)} + \frac{N}{1 - \frac{1}{4}t' - \frac{1}{4}t'^2};$$

by equating it to

$$\frac{\frac{1}{2}t'}{(1 - t')(1 - \frac{1}{4}t' - \frac{1}{4}t'^2)};$$

we will have

$$N = \frac{\frac{1}{4}t'}{1 - t'}.$$

The generating function of $y_{x,x'}$ is thus [216]

$$\frac{t'(1 - \frac{1}{4}t' - \frac{1}{4}t'^2) + \frac{1}{4}tt'}{(1 - t')(1 - \frac{1}{4}t - \frac{1}{4}t' - \frac{1}{4}t^2 - \frac{1}{4}t'^2)}.$$

If we develop into series the function

$$\frac{t'(1 - \frac{1}{4}t' - \frac{1}{4}t'^2) + \frac{1}{4}tt'}{1 - \frac{1}{4}t - \frac{1}{4}t' - \frac{1}{4}t^2 - \frac{1}{4}t'^2} - t';$$

we will have

$$\frac{(2+t)tt'}{4}\left\{\begin{array}{l}1+\frac{1}{4}t'(1+t')+\frac{1}{4^2}t'^2(1+t')^2+\frac{1}{4^3}t'^3(1+t')^3+\text{etc.}\\ \quad+\frac{t(1+t)}{4}\left[1+\frac{2}{4}t'(1+t')+\frac{3}{4^2}t'^2(1+t')^2+\frac{4}{4^3}t'^3(1+t')^3+\text{etc.}\right]\\ \quad+\frac{t^2(1+t)^2}{4^2}\left[1+\frac{3}{4}t'(1+t')+\frac{3.4}{1.2.4^2}t'^2(1+t')^2+\frac{3.4.5}{1.2.3.4^3}t'^3(1+t')^3+\text{etc.}\right]\\ \quad+\frac{t^3(1+t)^3}{4^3}\left[1+\frac{4}{4}t'(1+t')+\frac{4.5}{1.2.4^2}t'^2(1+t')^2+\frac{4.5.6}{1.2.3.4^3}t'^3(1+t')^3+\text{etc.}\right]\\ \quad+\text{etc.}\end{array}\right\}.$$

If we reject from this series, all the powers of t other than t^x, and all the powers of t' superior to $t'^{x'}$, and if in that which remains, we make $t=1$, $t'=1$, we will have the expression of $y_{x,x'}$ when x is equal or greater than unity: when x is null, we have $y_{0,x'}=1$. It is easy to translate this process into a formula, as we have done for the preceding case.

Let us name $z_{x,x'}$ the probability of player B; the generating function of $z_{x,x'}$ will be that which the generating function of $y_{x,x'}$ becomes when we change in it t into t', and reciprocally; that which gives for this function,

$$\frac{t(1-\frac{1}{4}t-\frac{1}{4}t^2)+\frac{1}{4}tt'}{(1-t)(1-\frac{1}{4}t-\frac{1}{4}t'-\frac{1}{4}t^2-\frac{1}{4}t'^2)}.$$

By adding the two generating functions, their sum is reduced to

$$\frac{t}{1-t}+\frac{t'}{1-t'}+\frac{tt'}{(1-t)(1-t')},$$

in which the coefficient of $t^x t'^{x'}$ is unity; thus we have

$$y_{x,x'}+z_{x,x'}=1;$$

that which is clear besides, since the game must be necessarily won by one of the players. [217]

§9. Let us imagine in an urn, r balls marked with the nº 1, r balls marked with nº 2, r balls marked with nº 3, and so forth to the nº n. These balls being well mixed in the urn, we draw them successively; we require the probability that there will exit at least one of these balls, at the rank[7] indicated by its label[8], or that there will exit at least two of them, or at least three, etc.

[7]*Translator's note*: This means that a ball marked with 1 will be drawn first, a ball marked with 2 will be drawn second, and so on. In other words, balls will be drawn consecutively by number.

[8]*Translator's note*: The word here is *numéro*, number. However, this refers to the use of a number as a label. In order to distinguish it from *nombre*, number or quantity, I choose to render it as such.

Let us seek first the probability that there will exit at least one of them. For this, we will observe that each ball can exit at its rank, only in the first n drawings; we can therefore here set aside the following drawings; now the total number of balls being rn, the number of their combinations n by n, by having regard for the order that they observe among themselves, is, by that which precedes,

$$rn(rn-1)(rn-2)\dots(rn-n+1);$$

this is therefore the number of all possible cases in the first n drawings.

Let us consider one of the balls marked with the n° 1, and let us suppose that it exits at its rank, or the first. The number of combinations of the $rn-1$ other balls taken $n-1$ by $n-1$, will be

$$(rn-1)(rn-2)\dots(rn-n+1);$$

this is the number of cases relative to the assumption that we just made; and as this assumption can be applied to r balls marked with n° 1, we will have

$$r(rn-1)(rn-2)\dots(rn-n+1)$$

for the number of cases relative to the hypothesis that one of the balls marked with the n° 1 will exit at its rank. The same result holds for the hypothesis that any one of the $n-1$ other kinds of balls will exit at the rank indicated by its label: by adding therefore all the results relative to these diverse hypotheses, we will have

$$rn(rn-1)(rn-2)\dots(rn-n+1), \qquad (a)$$

for the number of cases in which one ball at least will exit at its rank, provided however [218]
that we remove from them the cases which are repeated.

In order to determine these cases, let us consider one of the balls of the n° 1, exiting first, and one of the balls of the n° 2, exiting second. This case is comprehended twice in the preceding number; for it is comprehended one time in the number of the cases relative to the assumption that one of the balls labeled[9] 1, will exit at its rank, and a second time, in the number of cases relative to the assumption that one of the balls labeled 2, will exit at its rank; and as this extends to any two balls exiting at their rank, we see that it is necessary to subtract from the number of the cases preceding, the number of all the cases in which two balls exit at their rank.

The number of combinations of two balls of different labels is $\frac{n(n-1)}{1.2}r^2$; for the number of the labels being n, their combinations two by two are in number $\frac{n(n-1)}{1.2}$, and in each of these combinations, we can combine the r balls marked with one of the labels, with the r balls marked with the other label. The number of combinations of the $rn-2$ balls remaining, taken $n-2$ by $n-2$, by having regard for the order that they observe among themselves, is

$$(rn-2)(rn-3)\dots(rn-n+1);$$

[9] *Translator's note*: The word is *numérotées*, numbered. I have chosen to render it as labeled for the same reason as above.

thus the number of cases relative to the assumption that two balls exit at their rank is

$$\frac{n(n-1)}{1.2}r^2(rn-2)(rn-3)\dots(rn-n+1);$$

by subtracting it from the number (*a*), we will have

$$\begin{aligned} &rn(rn-1)(rn-2)\dots(rn-n+1)\\ -&\frac{n(n-1)}{1.2}r^2(rn-2)(rn-3)\dots(rn-n+1); \end{aligned} \qquad (a')$$

for the number of all the cases in which one ball at least will exit at its rank, provided that we subtract again from this function, the repeated cases, and that we add to them those which are lacking.

These cases are those in which three balls exit at their rank. By naming k this number, [219] it is repeated three times in the first term of the function (a'); for it can result, in this term, from the three assumptions of each of the three balls exiting at its rank. The number k is likewise comprehended three times in the second term of the function; for it can result from each of the assumptions relative to any two of the three balls exiting at their rank; thus this second term being affected with the $-$ sign, the number k in not found in the function (a'); it is necessary therefore to add it to it in order that it contain all the cases in which one ball at least exits at its rank. The number of combinations of n labels taken three by three, is $\frac{n(n-1)(n-2)}{1.2.3}$, and as we can combine the r balls of one of these labels of each combination, with the r balls of the second label, and with the r balls of the third label, we will have the total number of combinations in which three balls exit at their rank, by multiplying $\frac{n(n-1)(n-2)}{1.2.3}r^3$ by $(rn-3)(rn-4)\dots(rn-n+1)$, a number which expresses that of the combinations of the $rn-3$ remaining balls, taken $n-3$ by $n-3$, by having regard for the order that they observe among themselves. If we add this product to the function (a'), we will have

$$\begin{aligned} &nr(rn-1)(rn-2)\dots(rn-n+1)\\ -&\frac{n(n-1)}{1.2}r^2(rn-2)(rn-3)\dots(rn-n+1)\\ +&\frac{n(n-1)(n-2)}{1.2.3}r^3(rn-3)(rn-4)\dots(rn-n+1); \end{aligned} \qquad (a'')$$

this function expresses the number of all cases in which one ball at least exits at its rank, provided that we subtract from it again the repeated cases. These cases are those in which four balls exit at their rank. By applying here the preceding reasonings, we will see that it is necessary again to subtract from the function (a'') the term

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}r^4(rn-4)(rn-5)\dots(rn-n+1).$$

By continuing thus, we will have for the expression of the cases in which one ball at least exits at its rank

$$
\begin{aligned}
&nr(rn-1)(rn-2)\ldots(rn-n+1) \\
&-\frac{n(n-1)}{1.2}r^2(rn-2)(rn-3)\ldots(rn-n+1) \\
&+\frac{n(n-1)(n-2)}{1.2.3}r^3(rn-3)(rn-4)\ldots(rn-n+1) \qquad \text{(A)} \\
&-\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}r^4(rn-4)(rn-5)\ldots(rn-n+1) \\
&+\text{etc.}
\end{aligned}
$$ [220]

the series being continued as far at it can be. In this function, each combination is not repeated; thus the combination of s balls exiting at their rank, is found here only one time; for this combination is comprehended s times in the first term of the function, since it can result from each of the s balls exiting at its rank; it is subtracted $\frac{s(s-1)}{1.2}$ times in the second term, since it can result from two by two combinations of the s balls exiting at their rank; it is added $\frac{s(s-1)(s-2)}{1.2.3}$ times in the third term, since it can result from the combinations of s letters taken three by three, and so forth; it is therefore, in the function (A), comprehended a number of times equal to

$$s-\frac{s(s-1)}{1.2}+\frac{s(s-1)(s-2)}{1.2.3}-\text{etc.};$$

and consequently equal to $1-(1-1)^s$, or to unity. By dividing the function (A) by the number $rn(rn-1)(rn-2)\ldots(rn-n+1)$ of all possible cases, we will have for the expression of the probability that one ball at least will exit at its rank,

$$
\begin{aligned}
&1-\frac{(n-1)r}{1.2(rn-1)}+\frac{(n-1)(n-2)r^2}{1.2.3(rn-1)(rn-2)} \\
&-\frac{(n-1)(n-2)(n-3)r^3}{1.2.3.4(rn-1)(rn-2)(rn-3)}+\text{etc.}
\end{aligned}
\qquad \text{(B)}
$$

Let us seek now the probability that at least s balls will exit at their rank. The number of cases in which s balls exit a their rank, is, by that which precedes,

$$\frac{n(n-1)(n-2)\ldots(n-s+1)}{1.2.3\ldots s}r^s(rn-s)(rn-s-1)\ldots(rn-n+1), \qquad (b)$$

provided that we subtract from this function, the cases which are repeated. These cases are those in which $s+1$ balls exit at their rank, for they can result in the function, from $s+1$ balls taken s by s; these cases are therefore repeated $s+1$ times in this function; consequently it is necessary to subtract them s times. Now the number of cases in which $s+1$ balls exit at their rank, is [221]

$$\frac{n(n-1)(n-2)\ldots(n-s)}{1.2.3\ldots(s+1)}r^{s+1}(rn-s-1)(rn-s-2)\ldots(rn-n+1).$$

By multiplying it by s, and subtracting it from the function (b), we will have

$$\frac{n(n-1)(n-2)\dots(n-s+1)}{1.2.3\dots s} r^s (rn-s)(rn-s-1)\dots(rn-n+1) \times \left\{1 - \frac{s(n-s)r}{(s+1)(rn-s)}\right\}. \qquad (b')$$

In this function, many cases are again repeated, namely, those in which $s+2$ balls exit at their rank; for they result in the first term, from $s+2$ balls exiting at their rank, and taken s by s; they result, in the second term, from $s+2$ balls exiting at their rank, and taken $s+1$ by $s+1$, and moreover multiplied by the factor s, by which we have multiplied the second term. They are therefore comprehended in this function, the number of times $\frac{(s+2)(s+1)}{1.2} - s(s+2)$; thus it is necessary to multiply by unity less this number of times, the number of cases in which $s+2$ balls exit at their rank. This last number is

$$\frac{n(n-1)(n-2)\dots(n-s-1)}{1.2.3\dots(s+2)} r^{s+2} (rn-s-2)(rn-s-3)\dots(rn-n+1);$$

the product in question will be therefore

$$\frac{n(n-1)\dots(n-s-1)}{1.2.3\dots(s+2)} r^{s+2} (rn-s-2)\dots(rn-n+1) \frac{s(s+1)}{1.2}.$$

By adding it to the function (b'), we will have

$$\frac{n(n-1)\dots(n-s+1)}{1.2.3\dots s} r^s (rn-s)(rn-s-1)\dots(rn-n+1) \times \left\{\begin{array}{l} 1 - \frac{s}{s+1}\cdot\frac{(n-s)r}{rn-s} \\ \quad + \frac{s}{s+2}\cdot\frac{(n-s)(n-s-1)r^2}{1.2(rn-s)(rn-s-1)} \end{array}\right\}; \qquad (b'')$$

this is the number of all possible cases in which s balls exit at their rank, provided that [222]
we subtract from it again the cases which are repeated. By continuing to reason so, and by dividing the final function by the number of all possible cases; we will have for the expression of the probability that s balls at least will exit at their rank,

$$\frac{(n-1)(n-2)\dots(n-s+1)r^{s-1}}{1.2.3\dots s(rn-1)(rn-2)\dots(rn-s+1)} \times \left\{\begin{array}{l} 1 - \frac{s}{s+1}\cdot\frac{(n-s)r}{rn-s} + \frac{s}{s+2}\cdot\frac{(n-s)(n-s-1)r^2}{1.2.(rn-s)(rn-s-1)} \\ \quad - \frac{s}{s+3}\cdot\frac{(n-s)(n-s-1)(n-s-2)r^3}{1.2.3(rn-s)(rn-s-1)(rn-s-2)} + \text{etc.} \end{array}\right\}. \qquad (C)$$

We will have the probability that none of the balls will exit at its rank, by subtracting formula (B) from unity; and we will find, for its expression,

$$\frac{[1.2.3\ldots rn] - nr[1.2.3\ldots(rn-1)] + \frac{n(n-1)}{1.2}r^2[1.2.3\ldots(rn-2)] - \text{etc.}}{1.2.3\ldots rn}.$$

We have, by §33 of the first Book, whatever be i,

$$1.2.3\ldots i = \int x^i dx c^{-x},$$

the integral being taken from x null to x infinity. The preceding expression can therefore be put under this form

$$\frac{\int x^{rn-n}dx(x-r)^n c^{-x}}{\int x^{rn}dx c^{-x}}. \qquad (o)$$

Let us suppose the number rn of balls in the urn, very great; then by applying to the preceding integrals, the method of §24 of the first Book, we will find more or less nearly, for the integral of the numerator,

$$\frac{\sqrt{2\pi}X^{rn+2}\left(1-\frac{r}{X}\right)^{n+1}c^{-X}}{\sqrt{nX^2+n(r-1)(X-r)^2}},$$

X being the value of x which renders a *maximum*, the function $x^{rn-n}(x-r)^n c^{-x}$. The equation relative to this *maximum* gives for X, the two values

$$X = \frac{rn+r}{2} \pm \frac{\sqrt{r^2(n-1)^2+4rn}}{2}.$$

We can consider here only the greatest of these values which is, to the quantities nearly, of the order $\frac{1}{rn}$, equal to $rn+\frac{n}{n-1}$; then the integral of the numerator of the function (o) becomes nearly [223]

$$\frac{\sqrt{2\pi}(rn)^{rn+\frac{1}{2}}c^{-rn}\left(1-\frac{1}{n}\right)^{n+1}\sqrt{r}}{\sqrt{(r-1)(1-\frac{1}{n})^2+1}}.$$

The integral of the denominator of the same function is, by §33, quite nearly,

$$\sqrt{2\pi}(rn)^{rn+\frac{1}{2}}c^{-rn};$$

the function (o) becomes thus

$$\frac{\left(1-\frac{1}{n}\right)^{n+1}\sqrt{r}}{\sqrt{(r-1)(1-\frac{1}{n})^2+1}}.$$

We can put it under the form

$$\frac{\left(1-\frac{1}{n}\right)^{n+1}}{\sqrt{(1-\frac{1}{n})^2+\frac{2}{rn}-\frac{1}{rn^2}}},$$

rn being supposed a very great number, this function is reduced quite nearly to this very simple form

$$\left(\frac{n-1}{n}\right)^n.$$

This is therefore the quite close expression of the probability that none of the balls of the urn will exit at its rank, when there is a great number of balls. The hyperbolic logarithm of this expression being

$$-1-\frac{1}{2n}-\frac{1}{3n^2}-\text{etc.};$$

we see that it always increases in measure as n increases; that it is null, when $n=1$, and that it becomes $\frac{1}{c}$, when n is infinity, c being always the number of which the hyperbolic [224]
logarithm is unity.

Let us imagine now a number i of urns each containing the number n of balls, all of different colors; and that we draw successively all the balls from each urn. We can, by the preceding reasonings, determine the probability that one or more balls of the same color will exit at the same rank in the i drawings. In fact, let us suppose that the ranks of the colors are settled after the complete drawing of the first urn, and let us consider first the first color: let us suppose that it exits first in the drawings of the $i-1$ other urns. The total number of combinations of the $n-1$ other colors from each urn is, by having regard for their situation among them, $1.2.3\ldots(n-1)$; thus the total number of these combinations relative to $i-1$ urns, is $[1.2.3\ldots(n-1)]^{i-1}$; this is the number of cases in which the first color is drawn the first altogether from all these urns; and as there are n colors, we will have

$$n[1.2.3\ldots(n-1)]^{i-1}$$

for the number of cases in which one color at least will arrive at its rank in the drawings from the $i-1$ urns. But there are in this number, some repeated cases: thus the cases where two colors arrive at their rank in these drawings, are comprehended twice in this number; it is necessary therefore to subtract them from it. The number of these cases is, by that which precedes,

$$\frac{n(n-1)}{1.2}[1.2.3\ldots(n-2)]^{i-1};$$

by subtracting it from the preceding number, we will have the function

$$n[1.2.3\ldots(n-1)]^{i-1}-\frac{n(n-1)}{1.2}[1.2.3\ldots(n-2)]^{i-1}.$$

But this function contains itself repeated cases. By continuing to exclude from them, as we have done above relatively to a single urn; by dividing next the final function, by the number of all possible cases, and which is here $[1.2.3\dots n]^{i-1}$; we will have, for the probability that one of the $n-1$ colors at least will exit at its rank in the $i-1$ drawings which follow the first, [225]

$$\frac{1}{n^{i-2}} - \frac{1}{1.2[n(n-1)]^{i-2}} + \frac{1}{1.2.3[n(n-1)(n-2)]^{i-2}} - \text{etc.},$$

an expression in which it is necessary to take as many terms as there are units in n. This expression is therefore the probability that at least one of the colors will exit at the same rank in the drawings from the i urns.

§10. Let us consider two players A and B, of whom the skills are p and q, and of whom the first has a tokens, and the second, b tokens. Let us suppose that at each coup, the one who loses gives a token to his adversary, and that the game ends only when one of the players will have lost all his tokens; we demand the probability that one of the players, A for example, will win the game, before or at the n^{th} coup.

This problem can be resolved with facility by the following process which is in some way, mechanical. Let us suppose b equal or less than a, and let us consider the development of the binomial $(p+q)^b$. The first term p^b of this development will be the probability of A to win the game at coup b. We will subtract this term, from the development, and we will subtract similarly the last term q^b, if $b=a$; because then this term expresses the probability of B to win the game at coup b. Next we will multiply the rest by $p+q$. The first term of this product will have for factor p^bq, and, as the exponent b surpasses only by $b-1$ the exponent of q, there results from it that the game cannot be won by player A, at the coup $b+1$, that which is clear besides; because if A has lost a token in the first b coups, he must, in order to win the game win this token plus the b tokens of player B, that which requires $b+2$ coups. But if $a=b+1$, we will subtract from the product, its last term which expresses the probability of the player B to win the game at the coup $b+1$.

We will multiply anew this second remainder, by $p+q$. The first term of the product will have for factor $p^{b+1}q$, and as the exponent of p surpasses by b there the one of q, this term will express the probability of A to win the game at the coup $b+2$. We will subtract similarly from the product, the last term, if the exponent of q there surpasses by a the one [226]
of p.

We will multiply anew this third remainder, by $p+q$, and we will continue these multiplications up to the number of times $n-b$, by subtracting at each multiplication, the first term, if the exponent of p there surpasses by b, the one of q, and the last term, if the exponent of q there surpasses by a, the one of p. This premised, the sum of the first terms thus subtracted, will be the probability of A to win the game, before or at coup n; and the sum of the last terms subtracted will be the similar probability relative to player B.

In order to have an analytic solution of the problem, let $y_{x,x'}$ be the probability of player A to win the game, when he has x tokens, and when he has no more than x' coups to play in order to attain the n coups. This probability becomes at the following coup, either $y_{x+1,x'-1}$, or $y_{x-1,x'-1}$, according as player A wins or loses the coup; now the respective probabilities of these two events are p and q: we have therefore the equation in the partial differences,

$$y_{x,x'} = py_{x+1,x'-1} + qy_{x-1,x'-1}.$$

In order to integrate this equation, we will consider, as previously, a function u of t and of t' generator of $y_{x,x'}$, so that $y_{x,x'}$ be the coefficient of $t^x t'^{x'}$ in the development of this function. In passing again from the coefficients, to the generating functions, the preceding equation will give

$$u = u.\left(\frac{pt'}{t} + qtt'\right);$$

whence we deduce

$$1 = \frac{pt'}{t} + qtt';$$

consequently,

$$\frac{1}{t} = \frac{1}{2pt'} \pm \frac{\sqrt{\frac{1}{t'^2} - 4pq}}{2p};$$

that which gives

$$\frac{1}{t^x} = \frac{1}{(2p)^x}\left(\frac{1}{t'} \pm \sqrt{\frac{1}{t'^2} - 4pq}\right)^x;$$

therefore [227]

$$\frac{u}{t^x t'^{x'}} = \frac{u}{(2p)^x t'^{x'}}\left(\frac{1}{t'} \pm \sqrt{\frac{1}{t'^2} - 4pq}\right)^x.$$

This equation can be put under the following form,

$$\frac{u}{t^x t'^{x'}} = \frac{u}{2(2p)^x t'^{x'}} \times \left\{ \begin{array}{l} \left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}\right)^x + \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}\right)^x \\ \pm \sqrt{\frac{1}{t'^2} - 4pq}\, \frac{\left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}\right)^x - \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}\right)^x}{\sqrt{\frac{1}{t'^2} - 4pq}}. \end{array} \right\}$$

The preceding expression of $\frac{1}{t}$ gives

$$\pm\sqrt{\frac{1}{t'^2} - 4pq} = \frac{2p}{t} - \frac{1}{t'};$$

we have therefore

$$\frac{u}{t^x t'^{x'}} = \frac{u}{2(2p)^x t'^{x'}} \left[\left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq} \right)^x + \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq} \right)^x \right] + \frac{u\left(\frac{1}{t} - \frac{1}{2pt'}\right)}{2(2p)^{x-1} t'^{x'}} \frac{\left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq} \right)^x - \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq} \right)^x}{\sqrt{\frac{1}{t'^2} - 4pq}} :$$

under this form, the ambiguity of the $\pm$ sign disappears.

Now if we pass again from the generating functions to their coefficients, and if we observe that $y_{0,x'}$ is null, because player A loses the game necessarily, when he has no more tokens; the preceding equation will give, by passing again from the generating functions to the coefficients,

$$y_{x,x'} = \frac{1}{2^x p^{x-1}} \times [X^{(x-1)} y_{1,x+x'-1} + X^{(x-3)} y_{1,x+x'-3} \cdots + X^{(x-2r-1)} y_{1,x+x'-2r-1} + \text{etc.}],$$

the series of the second member being arrested when $x-2r-1$ has a negative value. $X^{(x-1)}$, $X^{(x-3)}$, etc., are the coefficients of $\frac{1}{t'^{x-1}}$, $\frac{1}{t'^{x-3}}$, etc., in the development of the function [228]

$$\frac{\left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq} \right)^x - \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq} \right)^x}{\sqrt{\frac{1}{t'^2} - 4pq}} \qquad (i)$$

If we name u' the coefficient of t^x in the development of u, u' will be a function of t' and of x, generator of $y_{x,x'}$. If we name similarly T' the coefficient of t in the development of u, the product of $\frac{T'}{2^x p^{x-1}}$ by the function (i), will be the generating function of the second member of the preceding equation; this function is therefore equal to u'. Let us suppose $x = a + b$, then $y_{x,x'}$ becomes $y_{a+b,x'}$, and this quantity is equal to unity; because it is certain that A has won the game, when he has won all the tokens of B; u' is therefore then the generating function of unity; now x' is here zero or an even number, because the number of coups in which A can win the game, is equal to b plus an even number: indeed, A must for this win all the tokens of B, and moreover he must win again each token that he has lost, that which requires two coups. Next n expressing a number of coups in which A can win the game, it is equal to b plus an even number; x' being the number of coups which are lacking to player A in order to arrive to n, is therefore zero or an even number. Thence it follows in the case of $x = a + b$, u' becomes $\frac{1}{1-t'^2}$; we have therefore

$$\frac{T'}{2^{a+b} p^{a+b-1}} \frac{\left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq} \right)^{a+b} - \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq} \right)^{a+b}}{\sqrt{\frac{1}{t'^2} - 4pq}} = \frac{1}{1 - t'^2};$$

that which gives the value of T'. By multiplying it by the function (i) divided by $2^a p^{a-1}$, and in which we make $x = a$, we will have the generating function of $y_{a,x'}$ equal to

$$\frac{2^b p^b t'^b[(1+\sqrt{1-4pqt'^2})^a - (1-\sqrt{1-4pqt'^2})^a]}{(1-t'^2)[(1+\sqrt{1-4pqt'^2})^{a+b} - (1-\sqrt{1-4pqt'^2})^{a+b}]}. \qquad (o)$$

In the case of $a = b$, it becomes [229]

$$\frac{2^a p^a t'^a}{(1-t'^2)[(1+\sqrt{1-4pqt'^2})^a + (1-\sqrt{1-4pqt'^2})^a]}.$$

By developing the function

$$(1+\sqrt{1-4pqt'^2})^a - (1-\sqrt{1-4pqt'^2})^a, \qquad (q)$$

according to the powers of t'^2, the radical disappears, and the highest exponent of t' in this development, is equal to or smaller than a. But if we develop $(1-\sqrt{1-4pqt'^2})^a$ according to the powers of t'^2, the least exponent of t' will be $2a$; the function (q) is therefore equal to the development of $(1+\sqrt{1-4pqt'^2})^a$, by rejecting the powers of t' superior to a.

Now we have, by §3 of the first Book,

$$z^a = 1 - a\alpha + \frac{a(a-3)}{1.2}\alpha^2 - \frac{a(a-4)(a-5)}{1.2.3}\alpha^3 + \text{etc.},$$

z being one of the roots of the equation

$$z = 1 - \frac{\alpha}{z},$$

which is reduced to unity, when α is null. This root is

$$\frac{1+\sqrt{1-4\alpha}}{2};$$

by supposing therefore $\alpha = pqt'^2$, we will have

$$\left(1+\sqrt{1-4pqt'^2}\right)^a = 2^a\left\{1 - apqt'^2 + \frac{a(a-3)}{1.2}p^2q^2t'^4 - \frac{a(a-4)(a-5)}{1.2.3}p^3q^3t'^6 + \text{etc.}\right\};$$

we will have thus,

$$\frac{2^a p^a t'^a}{\left(1+\sqrt{1-4pqt'^2}\right)^a + \left(1-\sqrt{1-4pqt'^2}\right)^a} = \frac{p^a t'^a}{1 - apqt'^2 + \frac{a(a-3)}{1.2}p^2q^2t'^4 - \frac{a(a-4)(a-5)}{1.2.3}p^3q^3t'^6 + \text{etc.}},$$

the series of the denominator being continued exclusively until the powers of t' superior to a. This second member must be, by that which precedes, divided by $1-t'^2$, in order to have the generating function of $y_{a,x'}$; the quantity $y_{a,x'}$ is therefore the sum of the coefficients of the powers of t', by considering in the development of this member, with respect to the powers of t', only the powers equal or inferior to x'. Each of these coefficients will express the probability that A will win the game at the coup indicated by the exponent of the power of t'. [230]

If we name z_i the coefficient corresponding to t'^{a+2i}, we will have generally

$$0 = z_i - apqz_{i-1} + \frac{a(a-3)}{1.2}p^2q^2z_{i-2} - \text{etc.};$$

whence it is easy to conclude the values of z_1, z_2, etc., by observing that z_{-1}, z_{-2}, etc. are nulls, and that $z_0 = p^a$. The value of z_i being equal to $y_{a,a+2i} - y_{a,a+2i-2}$, we will have those of $y_{a,a}$, $y_{a,a+2}$, $y_{a,a+4}$, etc. The equation in the partial differences to which we are immediately led, is found thus restored to one equation in the ordinary differences which determines, by integrating it, the value of $y_{a,x'}$. But we can obtain this value by the following process which is applied in the general case where a and b are equal or different between them.

Let us resume the generating function of $y_{a,x'}$ found above; $y_{a,x'}$ is the coefficient of $t^{x'-b}$ in the development of the function

$$2^bp^b\frac{P}{Q(1-t'^2)},$$

by supposing

$$P = \frac{\left(1+\sqrt{1-4pqt'^2}\right)^a - \left(1-\sqrt{1-4pqt'^2}\right)^a}{\sqrt{1-4pqt'^2}}$$

$$Q = \frac{\left(1+\sqrt{1-4pqt'^2}\right)^{a+b} - \left(1-\sqrt{1-4pqt'^2}\right)^{a+b}}{\sqrt{1-4pqt'^2}}.$$

It results from §5 of the first Book, that if we consider the two terms

$$\frac{P}{2t'^{2i}Q}, \qquad -\frac{P}{(1-t'^2)t'^{2i+1}\frac{dQ}{dt'}};$$

that we make next successively $t' = 1$ and $t' = -1$ in the first term, and t' equal successively to all the roots of the equation $Q = 0$ in the second term; the sum of all the terms which we obtain in this manner, will be the coefficient of t'^{2i} in the development of the fraction [231]

$$\frac{P}{Q(1-t'^2)}.$$

That which the first term produces in this sum is

$$\frac{p^a - q^a}{2^b(p^{a+b} - q^{a+b})}.$$

In order to have the roots of the equation $Q = 0$, we make

$$t' = \frac{1}{2\sqrt{pq}\cos\varpi};$$

that which gives

$$Q = \frac{(\cos\varpi + \sqrt{-1}\sin\varpi)^{a+b} - (\cos\varpi - \sqrt{-1}\sin\varpi)^{a+b}}{\sqrt{-1}\sin\varpi\,(\cos\varpi)^{a+b-1}},$$

or

$$Q = \frac{2\sin(a+b)\varpi}{\sin\varpi\,(\cos\varpi)^{a+b-1}}.$$

The roots of the equation $Q = 0$ are therefore represented by

$$\varpi = \frac{(r+1)\pi}{a+b},$$

r being a positive whole number which be extended from $r = 0$ to $r = a + b - 2$. When $a+b$ is an even number, $\frac{1}{2}\pi$ is one of the values of ϖ; it is necessary to exclude it, because, $\cos\varpi$ becoming null then, this value of ϖ does not render Q null. In this case, the equation $Q = 0$ has only $a + b - 2$ roots; but, as the term depending on the value $\varpi = \frac{1}{2}\pi$, is multiplied in the expression of $y_{a,x'}$, by a positive power of $\cos\frac{(r+1)\pi}{a+b}$, we can conserve the value of r which gives $\varpi = \frac{1}{2}\pi$, since the term which corresponds to it in the expression of $y_{a,x'}$ disappears.

Now we have

$$\frac{dQ}{dt'} = \left(\frac{dQ}{d\varpi}\right)\cdot\frac{d\varpi}{dt'};$$

whence we deduce, by virtue of the equation $\sin(a+b)\varpi = 0$, [232]

$$\frac{dQ}{dt'} = \frac{4(a+b)\sqrt{pq}\cos(r+1)\pi}{\sin^2\varpi\,(\cos\varpi)^{a+b-3}} = \frac{4(a+b)\sqrt{pq}\,(-1)^{r+1}}{\sin^2\varpi\,(\cos\varpi)^{a+b-3}},$$

the term

$$\frac{-P}{(1-t'^2)t'^{2i+1}\,\frac{dQ}{dt'}}$$

becomes thus, by observing that

$$P = \frac{2\sin a\varpi}{\sin\varpi\,(\cos\varpi)^{a-1}},$$

$$\frac{(-1)^{r+1}2^{2i+2}(pq)^{i+1}\sin\frac{(r+1)\pi}{a+b}\sin\frac{(r+1)a\pi}{a+b}\left(\cos\frac{(r+1)\pi}{a+b}\right)^{b+2i+1}}{(a+b)\left(p^2-2pq\cos\frac{2(r+1)\pi}{a+b}+q^2\right)}; \qquad (h)$$

the sum of all the terms which we obtain, by giving to r all the whole and positive values, from $r=0$ to $r=a+b-2$, will be that which produces the function

$$\frac{-P}{(1-t'^2)t'^{2i+1}\frac{dQ}{dt'}}:$$

we will designate this sum by the characteristic S placed before the function (h).

If we make $r'+1=a+b-(r+1)$, we will have

$$\begin{aligned}
\sin\frac{(r'+1)\pi}{a+b} &= \sin\frac{(r+1)\pi}{a+b},\\
\cos\frac{(r'+1)\pi}{a+b} &= -\cos\frac{(r+1)\pi}{a+b},\\
\cos\frac{2(r'+1)\pi}{a+b} &= \cos\frac{2(r+1)\pi}{a+b},\\
\sin\frac{(r'+1)a\pi}{a+b} &= (-1)^{a+1}\sin\frac{(r+1)a\pi}{a+b}.
\end{aligned}$$

Thence it is easy to conclude that in the function (h), the term relative to $r+1$ is the same as the term relative to $r'+1$; we can therefore double this term, and extend then the characteristic S only to the values of r comprehended from $r=0$ to $r=\frac{a+b-2}{2}$, if $a+b$ is [233]
even, or $r=\frac{a+b-1}{2}$, if $a+b$ is odd. This premised, by observing that

$$\sin\frac{(r+1)a\pi}{a+b}=(-1)^r\sin\frac{(r+1)b\pi}{a+b},$$

we will have

$$\begin{aligned}
y_{a,b+2i} = \frac{p^b(p^a-q^a)}{p^{a+b}-q^{a+b}} &- \frac{2^{b+2i+2}p^b(pq)^{i+1}}{a+b}\\
\times S&\left\{\frac{\sin\frac{2(r+1)\pi}{a+b}\sin\frac{(r+1)b\pi}{a+b}\left(\cos\frac{(r+1)\pi}{a+b}\right)^{b+2i}}{p^2-2pq\cos\frac{2(r+1)\pi}{a+b}+q^2}\right\}.
\end{aligned} \qquad \text{(H)}$$

By changing a into b, p into q, and reciprocally, we will have the probability that player B will win the game before the coup $a+2i$, or at this coup.

Let us suppose $a=b$; $\sin\frac{(r+1)a\pi}{a+b}$ will become $\sin\frac{1}{2}(r+1)\pi$. This sine is null, when $r+1$ is even; therefore it suffices then to consider in the expression of $y_{a,a+2i}$, the odd values of $r+1$. By expressing them as $2s+1$, and observing that $\sin\frac{(2s+1)\pi}{2}=(-1)^s$, we will have

$$y_{a,a+2i} = \frac{p^a}{p^a+q^a} - \frac{2^{a+2i+2}p^a(pq)^{i+1}}{a}$$
$$\times S\left\{\frac{(-1)^s \sin\frac{(2s+1)\pi}{a}\left(\cos\frac{(2s+1)\pi}{2a}\right)^{a+2i}}{p^2-2pq\cos\frac{(2s+1)\pi}{a}+q^2}\right\},$$

$2s+1$ needing to comprehend all the odd values contained in $a-1$.

If we change in this expression, p into q, and reciprocally, we will have the probability of player B to win the game in $a+2i$ coups. The sum of these two probabilities will be the probability that the game will end after this number of coups; this last probability is therefore

$$1-\frac{2^{a+2i+1}}{a}(p^a+q^a)(pq)^{i+1}S\left\{\frac{(-1)^s \sin\frac{(2s+1)\pi}{a}\left(\cos\frac{(2s+1)\pi}{2a}\right)^{a+2i}}{p^2-2pq\cos\frac{(2s+1)\pi}{a}+q^2}\right\}.$$

If the skills p and q are equal, this expression becomes [234]

$$1-\frac{2}{a}S\left\{\frac{(-1)^s\left(\cos\frac{(2s+1)\pi}{2a}\right)^{a+2i+1}}{\sin\frac{(2s+1)\pi}{2a}}\right\}.$$

When $a+2i$ is a large number, we can conclude from it in a manner quite near, the number of coups necessary in order that the probability that the game will end in this number of coups, be equal to a given fraction $\frac{1}{k}$. We will have then

$$\frac{2}{a}S\left\{\frac{(-1)^s\left(\cos\frac{(2s+1)\pi}{2a}\right)^{a+2i+1}}{\sin\frac{(2s+1)\pi}{2a}}\right\} = \frac{k-1}{k},$$

$a+2i$ being supposed a very great number quite superior to the number a, it suffices to consider the term of the first member which corresponds to s null, and then we have

$$a+2i+1 = \frac{\log\left(\frac{a(k-1)}{2k}\sin\frac{\pi}{2a}\right)}{\log(\cos\frac{\pi}{2a})},$$

these logarithms can be at will hyperbolic or tabular.

If in the preceding formulas, we suppose a infinite, b remaining a finite number; we will have the case in which player A plays against player B who has originally the number b of tokens, until he has won all the tokens of B, without that ever the latter is able to beat

A, whatever be the number of tokens that he has won from him. In this case, the generating function (o) of $y_{a,x'}$ is reduced to

$$\frac{2^b p^b t'^b}{(1-t'^2)\left(1+\sqrt{1-4pqt'^2}\right)^b};$$

because then $\left(1-\sqrt{1-4pqt'^2}\right)^a$ and $\left(1-\sqrt{1-4pqt'^2}\right)^{a+b}$ developed, contain only infinite powers of t', powers which we must neglect, when we consider only a finite number of coups. We have by that which precedes [235]

$$\begin{aligned}&\left(1+\sqrt{1-4pqt'^2}\right)^{-b}\\ &=\frac{1}{2^b}\left\{\begin{array}{c}1+bpqt'^2+\dfrac{b(b+3)}{1.2}p^2q^2t'^4+\dfrac{b(b+4)(b+5)}{1.2.3}p^3q^3t'^6\\ \cdots+\dfrac{b(b+i+1)(b+i+2)\ldots(b+2i-1)p^iq^it'^{2i}}{1.2.3\ldots i}+\text{etc.}\end{array}\right\}.\end{aligned}$$

By multiplying this second member by $\frac{2^b p^b t'^b}{1-t'^2}$, the coefficient of t'^{b+2i} will be

$$p^b\left\{1+bpq+\frac{b(b+3)}{1.2}p^2q^2\cdots+\frac{b(b+i+1)(b+i+2)\ldots(b+2i-1)p^iq^i}{1.2.3\ldots i}\right\};$$

this is the value of $y_{a,b+2i}$, or the probability that A will win the game before or at the coup $b+2i$.

This value will be very painful to reduce into numbers, if b and $2i$ were large numbers; it will be especially very difficult to obtain by its means, the number of coups in which A can wager one against one to win the game; but we can attain it easily in this manner.

Let us resume formula (H) found above. In the case of a infinite, and p being supposed equal or greater than q, if we suppose $\frac{(r+1)}{a}\pi=\phi$, and $\frac{\pi}{a}=d\phi$, it becomes

$$y_{a,b+2i}=1-\frac{2^{b+2i+2}p^b(pq)^{i+1}}{\pi}\int\frac{d\phi\sin 2\phi\sin b\phi\,(\cos\phi)^{b+2i}}{p^2-2pq\cos 2\phi\,+q^2},$$

the integral needing to be taken from $\phi=0$ to $\phi=\frac{1}{2}\pi$. In the case of p less than q, the same expression holds, provided that we change the first term 1, into $\frac{p^b}{q^b}$.

If $p=q$, this expression becomes

$$1-\frac{2}{\pi}\int\frac{d\phi\sin b\phi\,(\cos\phi)^{b+2i+1}}{\sin\phi},$$

the integral being taken from ϕ null to $\phi=\frac{1}{2}\pi$. Let us suppose now that b and i are great numbers. The *maximum* of the function

$$\frac{\phi\,(\cos\phi)^{b+2i+1}}{\sin\phi}$$

corresponds to $\phi = 0$, that which gives 1 for this *maximum*. The function decreases next with an extreme rapidity, and in the interval where it has a sensible value, we can suppose [236]

$$\begin{aligned}\log\sin\phi &= \log\phi + \log(1 - \tfrac{1}{6}\phi^2) = \log\phi - \tfrac{1}{6}\phi^2,\\ \log(\cos\phi)^{b+2i+1} &= (b+2i+1)\log(1 - \tfrac{1}{2}\phi^2 + \tfrac{1}{24}\phi^4)\\ &= -\frac{(b+2i+1)}{2}\phi^2 - \frac{(b+2i+1)}{12}\phi^4,\end{aligned}$$

that which gives, by neglecting the sixth powers of ϕ, and its fourth powers which are not multiplied by $b + 2i + 1$,

$$\log\left(\frac{(\cos\phi)^{b+2i+1}}{\sin\phi}\right) = -\log\phi - \frac{(b+2i+\frac{2}{3})}{2}\phi^2 - \frac{(b+2i+\frac{2}{3})}{12}\phi^4;$$

by making therefore

$$a^2 = \frac{b+2i+\frac{2}{3}}{2};$$

we will have

$$\frac{(\cos\phi)^{b+2i+1}}{\sin\phi} = \frac{(1 - \frac{a^2}{6}\phi^4)}{\phi}c^{-a^2\phi^2};$$

hence,

$$\int\frac{d\phi\sin b\phi\,(\cos\phi)^{b+2i+1}}{\sin\phi} = \int\frac{d\phi\left(1 - \frac{a^2}{6}\phi^4\right)}{\phi}\sin b\phi\,c^{-a^2\phi^2}.$$

This last integral can be taken from $\phi = 0$ to ϕ infinity; because it must be taken from $\phi = 0$ to $\phi = \frac{1}{2}\pi$; now a^2 being a considerable number, $c^{-a^2\phi^2}$ becomes excessively small, when we make $\phi = \frac{1}{2}\pi$, so that we can suppose it null, seeing the extreme rapidity with which this exponential diminishes, when ϕ increases. Now we have

$$\frac{d}{db}\int\frac{d\phi\left(1 - \frac{a^2}{6}\phi^4\right)}{\phi}\sin b\phi\,c^{-a^2\phi^2} = \int d\phi\left(1 - \frac{a^2}{6}\phi^4\right)\cos b\phi\,c^{-a^2\phi^2};$$

we have besides, by §25 of the first Book,

$$\int d\phi\cos b\phi\,c^{-a^2\phi^2} = \frac{\sqrt{\pi}}{2a}c^{-\frac{b^2}{4a^2}},$$ [237]

$$\begin{aligned}\int\phi^4\,d\phi\cos b\phi\,c^{-a^2\phi^2} &= \frac{\sqrt{\pi}}{2a}\frac{d^4c^{-\frac{b^2}{4a^2}}}{db^4},\\ &= \frac{3\sqrt{\pi}}{8a^5}c^{-\frac{b^2}{4a^2}}\left(1 - \frac{b^2}{a^2} + \frac{b^4}{12.a^4}\right);\end{aligned}$$

whence we deduce, by supposing

$$\frac{b^2}{4a^2} = t^2,$$

$$\int \frac{d\phi \sin b\phi\,(\cos\phi)^{b+2i+1}}{\sin\phi} = \sqrt{\pi}\left\{\int dt\, c^{-t^2} - \frac{tc^{-t^2}}{8a^2}\left(1 - \tfrac{2}{3}t^2\right)\right\}.$$

Thus the probability that A will win the game in the number $b + 2i$ coups, is

$$1 - \frac{2}{\sqrt{\pi}}\left[\int dt\, c^{-t^2} - \frac{Tc^{-T^2}}{8a^2}\left(1 - \tfrac{2}{3}T^2\right)\right];$$

the integral being taken from t null to $t = T$, T^2 being equal to $\frac{b^2}{4a^2}$.

If we seek the number of coups in which we can wager one against one that this will take place, we will make this probability equal to $\frac{1}{2}$, that which gives

$$\int dt\, c^{-t^2} = \frac{\sqrt{\pi}}{4} + \frac{Tc^{-T^2}}{8a^2}\left(1 - \tfrac{2}{3}T^2\right).$$

Let us name T' the value of t, which corresponds to

$$\int dt\, c^{-t^2} = \frac{\sqrt{\pi}}{4};$$

and let us suppose

$$T = T' + q,$$

q being of order $\frac{1}{a^2}$. The integral $\int dt\, c^{-t^2}$ will be increased very nearly by $qc^{-T'^2}$; that [238]
which gives

$$qc^{-T'^2} = \frac{T'c^{-T'^2}}{8a^2}\left(1 - \tfrac{2}{3}T'^2\right);$$

we will have therefore

$$T^2 = T'^2 + \frac{T'^2}{4a^2}\left(1 - \tfrac{2}{3}T'^2\right).$$

Having therefore T^2 to the quantities near the order $\frac{1}{a^4}$, the equation

$$2a^2 = b + 2i + \tfrac{2}{3} = \frac{b^2}{2T^2}$$

will give, to the quantities near the order $\frac{1}{a^2}$,

$$b + 2i = \frac{b^2}{2T'^2} - \frac{7}{6} + \frac{1}{3}T'^2.$$

In order to determine the value of T'^2, we will observe that here T' is smaller than $\frac{1}{2}$; thus the transcendent and integral equation

$$\int dt\, c^{-t^2} = \frac{\sqrt{\pi}}{4},$$

can be transformed into the following,

$$T' - \frac{1}{3}T'^3 + \frac{1}{1.2}\cdot\frac{1}{5}T'^5 - \frac{1}{1.2.3}\cdot\frac{1}{7}T'^7 + \text{etc.} = \frac{\sqrt{\pi}}{4}.$$

By resolving this equation, we find

$$T'^2 = 0.2102497.$$

By supposing $b = 100$, we will have

$$b + 2i = 23780, 14.$$

There is therefore then disadvantage to wager one against one, that A will win the game in 23780 coups, but there is advantage to wager that he will win it in 23781 coups.

§11. A number $n+1$ of players play together with the following conditions. Two among them play first, and the one who loses is retired after having put a franc into the game, in order to return only after all the other players have played; that which holds generally for [239]
all the players who lose, and who thence become the last. The one of the first two players who has won, plays with the third, and, if he wins it, he continues to play with the fourth, and so forth until he loses, or until he has beat successively all the players. In this last case, the game is ended. But if the player winning at the first coup, is vanquished by one of the other players, the vanquisher plays with the following player, and continues to play until he is vanquished, or until he has beat consecutively all the players; the game continues thus until there is one player who beats consecutively all the others, that which ends the game, and then the player who wins it, takes away all that which has been set into the game. This premised,

Let us determine first the probability that the game will end precisely at coup x; let us name z_x this probability. In order that the game finish at coup x, if is necessary that the player who enters into the game at coup $x - n + 1$, wins this coup and the $n - 1$ coups following; now he is able to enter against a player who has won only a single coup: by naming P the probability of this event, $\frac{P}{2^n}$ will be the corresponding probability that the game will end at coup x. But the probability z_{x-1} that the game will end at coup $x - 1$, is evidently $\frac{P}{2^{n-1}}$. Because it is necessary for this that there is a player who has won a coup, at coup $x - n + 1$, and who playing at this coup, wins it and the following $n - 2$ coups; and the probability of each of these events being P and $\frac{1}{2^{n-1}}$, the probability of the composite event will be $\frac{P}{2^{n-1}}$; we will have therefore $z_{x-1} = \frac{P}{2^{n-1}}$, and consequently,

$$\frac{P}{2^n} = \frac{1}{2}z_{x-1};$$

$\frac{1}{2}z_{x-1}$ is therefore the probability that the game will end at coup x, relative to this case.

If the player who enters into the game at coup $x-n+1$, plays at this coup against a player who has already won two coups; by naming P' the probability of this case, $\frac{P'}{2^n}$ will be the probability relative to this case, that the game will end at coup x. But we have [240]

$$\frac{P'}{2^{n-2}} = z_{x-2};$$

because in order that the game end at coup $x-2$, it is necessary that at coup $x-n+1$, one of the players has already won two coups, and that he wins this coup and the $n-3$ following coups. We have therefore

$$\frac{P'}{2^n} = \frac{1}{2^2}z_{x-2};$$

$\frac{1}{2^2}z_{x-2}$ is therefore the probability that the game will end at coup x, relative to this case; and so forth.

By reassembling all these partial probabilities, we will have

$$z_x = \frac{1}{2}z_{x-1} + \frac{1}{2^2}z_{x-2} + \frac{1}{2^3}z_{x-3}\cdots + \frac{1}{2^{n-1}}z_{x-n+1}.$$

The generating function of z_x is, by the first Book,

$$\frac{\psi(t)}{1 - \frac{1}{2}t - \frac{1}{2^2}t^2 \cdots - \frac{1}{2^{n-1}}t^{n-1}}$$

or

$$\frac{\frac{1}{2}\psi(t)(2-t)}{1 - t + \frac{1}{2^n}t^n}.$$

In order to determine $\psi(t)$, we will observe that the game can end no earlier than at coup n, and that the probability for this is $\frac{1}{2^{n-1}}$; because it is necessary that the vanquisher at the first coup, wins the $n-1$ following coups; $\psi(t)$ must therefore contain only the power n of t, and $\frac{1}{2^{n-1}}$ must be the coefficient of this power; that which gives $\psi(t) = \frac{t^n}{2^{n-1}}$: thus the generating function of z_x is

$$\frac{\frac{1}{2^n}t^n(2-t)}{1 - t + \frac{1}{2^n}t^n}.$$

The sum of the coefficients of the powers of t to infinity, in the development of this function, is the probability that the game must end after an infinity of coups; now we have this sum [241] by making $t = 1$ in the function, that which reduces it to unity; it is therefore certain that the game must end.

We will have the probability that the game will be ended at coup x or before this coup, by determining the coefficient of t^x in the development of the preceding function, divided by $1-t$; the generating function of this probability is therefore

$$\frac{\frac{1}{2^n}t^n(2-t)}{(1-t)\left(1 - t + \frac{1}{2^n}t^n\right)}.$$

Let us give to the generating function of z_x, this form

$$\frac{1}{2^n} \cdot \frac{t^n(2-t)}{1-t} \left(1 - \frac{1}{2^n} \cdot \frac{t^n}{1-t} + \frac{1}{2^{2n}} \cdot \frac{t^{2n}}{(1-t)^2} - \text{etc.}\right);$$

the coefficient of t^x in $\frac{t^{rn}(2-t)}{2^{rn}(1-t)^r}$ is

$$\frac{1}{2^{rn}} \cdot \frac{(x-rn+1)(x-rn+2)\dots(x-rn+r-2)}{1.2.3\dots(r-1)}(x-rn+2r-2);$$

we have therefore

$$\begin{aligned} z_x = \frac{1}{2^n} - \frac{(x-2n+2)}{2^{2n}} &+ \frac{(x-3n+1)}{1.2.2^{3n}}(x-3n+4) \\ &- \frac{(x-4n+1)(x-4n+2)}{1.2.3.2^{4n}}(x-4n+6) + \text{etc.}, \end{aligned}$$

an expression which is relative only to x greater than n, and in which it is necessary to take only as many terms as there are integral units in the quotient $\frac{x}{n}$: When $x = n$, we have $z_x = \frac{1}{2^{n-1}}$.

By developing in the same manner the generating function of the probability that the game will end before or at coup x, we will find for the expression of this probability,

$$\begin{aligned} \frac{x-n+2}{2^n} &- \frac{(x-2n+1)}{1.2.2^{2n}}(x-2n+4) \\ &+ \frac{(x-3n+1)(x-3n+2)}{1.2.3.2^{3n}}(x-3n+6) - \text{etc.}, \end{aligned}$$

this expression holding even in the case $x = n$.

Let us determine now the respective probabilities of the players in order to win the game at coup x. Let $y_{0,x,}$ be that of the player who has won the first coup. Let $y_{1,x}$, $y_{2,x}$, $\dots y_{n-1,x}$ be those of the following players, and $y_{n,x}$ that of the player who has lost at the first coup, and who thence became the last. Let us designate the players by (0), (1), (2), $\dots(n-1)$, (n). This premised, the probability $y_{r,x}$ of player (r) becomes $y_{r-1,x-1}$, if at the second coup player (0) is vanquished by player (1); because it is clear that (r) is found then, with respect to the vanquisher (1), in the same position where $(r-1)$ was with respect to the vanquisher (0); only, there is one coup less to play in order to arrive at coup x, that which changes x into $x-1$. Presently the probability that player (0) will be vanquished by (1) is $\frac{1}{2}$; thus $\frac{1}{2}y_{r-1,\,x-1}$ is the probability of player (r) to win the game at coup x, relative to the case where (0) is vanquished by (1). If (0) is vanquished only by (2), $y_{r,x}$ becomes $y_{r-2,\,x-2}$, and the probability of this event being $\frac{1}{4}$, we have $\frac{1}{4}y_{r-2,\,x-2}$ for the probability of player (r), to win the game at coup x, relative to this case. If player (0) is vanquished only by player (r), $y_{r,x}$ becomes $y_{0,x-r}$, and the probability of this event is $\frac{1}{2^r}$; thus $\frac{1}{2^r}y_{0,x-r}$ is the probability of player (r) to win the game at coup x, relative to this case. If player (0) is vanquished only by player $(r+1)$, $y_{r,x}$ is changed into $y_{n-1,\,x-r-1}$; because then player [242]

(r) is found, with respect to the vanquisher, in the original position of player ($n-1$) with respect to player (0): only there remains only $x-r-1$ coups to play in order to arrive at coup x. Now the probability that (0) will be vanquished only by player ($r+1$), is $\frac{1}{2^{r+1}}$; $\frac{1}{2^{r+1}}y_{n-1,x-r-1}$ is therefore the probability of (r) to win the game at coup x, relative to this case. By continuing thus, and reassembling all these partial probabilities, we will have the entire probability $y_{r,x}$ of player (r) to win the game; that which gives the following equation:

$$y_{r,x} = \frac{1}{2}y_{r-1,x-1} + \frac{1}{2^2}y_{r-2,\,x-2}\cdots + \frac{1}{2^r}y_{0,x-r} + \frac{1}{2^{r+1}}y_{n-1,\,x-r-1} + \frac{1}{2^{r+2}}y_{n-2,\,x-r-2}\cdots + \frac{1}{2^{n-1}}y_{r+1,\,x-n+1}.$$

This expression holds from $r=1$ to $r=n-2$. It gives [243]

$$\frac{1}{2}y_{r-1,x-1} = \frac{1}{2^2}y_{r-1,x-2} + \frac{1}{2^3}y_{r-3,\,x-3}\cdots + \frac{1}{2^n}y_{r,x-n}.$$

By subtracting this equation, from the preceding; we will have that here in the partial differences,

$$y_{r,x} - y_{r-1,x-1} + \frac{1}{2^n}y_{r,x-n} = 0; \qquad (1)$$

this equation is extended from $r=2$ to $r=n-2$.

We have, by the preceding reasoning, the following equation,

$$y_{n-1,x} = \frac{1}{2}y_{n-2,\,x-1} + \frac{1}{2^2}y_{n-3,\,x-2}\cdots + \frac{1}{2^{n-1}}y_{0,x-n+1}.$$

But the preceding expression of $y_{r,x}$ gives

$$\frac{1}{2}y_{n-2,x-1} = \frac{1}{2^2}y_{n-3,x-2}\cdots + \frac{1}{2^{n-1}}y_{0,\,x-n+1} + \frac{1}{2^n}y_{n-1,x-n}.$$

By subtracting this equation from the preceding, we will have

$$y_{n-1,x} - y_{n-2,x-1} + \frac{1}{2^n}y_{n-1,x-n} = 0:$$

thus equation (1) subsists in the case of $r=n-1$.

The preceding reasoning leads further to this equation

$$y_{n,x} = \frac{1}{2}y_{n-1,x-1} + \frac{1}{2^2}y_{n-2,\,x-2}\cdots + \frac{1}{2^{n-1}}y_{1,\,x-n+1},$$

that which gives

$$\frac{1}{2}y_{n,\,x-1} = \frac{1}{2^2}y_{n-1,\,x-2}\cdots + \frac{1}{2^n}y_{1,x-n}.$$

By subtracting this equation, from that here which gives the general expression of $y_{r,x}$,

$$y_{1,x} = \frac{1}{2}y_{0,x-1} + \frac{1}{2^2}y_{n-1,\,x-2} \cdots + \frac{1}{2^{n+1}}y_{2,\,x-n+1};$$

and making $\frac{1}{2}(y_{0,x} + y_{n,x}) = \bar{y}_{0,x}$; we will have [244]

$$y_{1,x} - \bar{y}_{0,x-1} + \frac{1}{2^n}y_{1,x-n} = 0.$$

Equation (1) subsists therefore yet even in the case of $r = 1$, provided that we change $y_{0,x}$ into $\bar{y}_{0,x}$. We must observe that $\bar{y}_{0,x}$ is the probability to win the game at coup x, of each of the first two players, at the moment where the game commences; because this probability becomes, after the first coup, $y_{0,x}$ or $y_{n,x}$, according as the player wins or loses, and the probability of each of these events is $\frac{1}{2}$.

Now, the generating function of equation (1) is, by §20 of the first Book,

$$\frac{\phi(t)}{1 - tt' + \frac{1}{2^n}t^n}, \qquad (a)$$

t being relative to the variable x, and t' being relative to the variable r, so that $y_{r,x}$ is the coefficient of $t'^r t^x$ in the development of this function; $\phi(t)$ is a function of t that there is concern to determine.

For this, we will make

$$T = \frac{1}{1 + \frac{1}{2^n}t^n};$$

the generating function of $y_{r,x}$ will be the coefficient of t'^r in the development of the function (a); it will be therefore

$$\phi(t)t^r T^{r+1};$$

the probability that the game will end precisely at coup x, is evidently the sum of the probabilities of each player to win at this coup; it is therefore

$$2\bar{y}_{0,x} + y_{1,x} + y_{2,x} \cdots + y_{n-1,x};$$

consequently the generating function of this probability is

$$T\phi(t)(2 + tT + t^2T^2 \cdots + t^{n-1}T^{n-1}),$$

or [245]

$$T\phi(t)\frac{(2 - tT - t^nT^n)}{1 - tT}.$$

By equating it to the generating function of this probability, that we have found above, and which is

$$\frac{\frac{1}{2^n}t^n(2 - t)}{1 - t + \frac{1}{2^n}t^n};$$

we will have

$$\phi(t) = \frac{\frac{1}{2^n}t^n(2-t)(1-tT)}{T(2-tT-t^nT^n)\left(1-t+\frac{1}{2^n}t^n\right)};$$

Thus the generating function of equation (1) in the partial differences, is

$$\frac{\frac{1}{2^n}t^n(2-t)(1-tT)}{T(2-tT-t^nT^n)\left(1-t+\frac{1}{2^n}t^n\right)\left(1-tt'+\frac{1}{2^n}t^n\right)};$$

the generating function of $y_{r,x}$ is therefore

$$\frac{\frac{1}{2^n}t^{n+r}(2-t)(1-tT)T^r}{(2-tT-t^nT^n)\left(1-t+\frac{1}{2^n}t^n\right)}.$$

The coefficient of t^x in the development of this function, is the probability of player (r) to win the game at coup x. We will thus be able to determine this probability through this development. The sum of all these coefficients to x infinity, is the probability of player (r) to win the game; now we have this sum, by making $t = 1$ in the preceding function, that which gives $T = \frac{2^n}{1+2^n}$; let us name p this last quantity, and let us designate by y_r the probability of (r) to win the game, we will have

$$y_r = \frac{(1-p)p^r}{2-p-p^n}.$$

This expression is extended from $r = 0$ to $r = n-1$, provided that we change y_0 into $\bar{y}_0$, [246]
$\bar{y}_0$ expressing the probability to win the game, of the first two players at the moment where they enter the game.

Now, each losing player depositing a franc into the game, let us determine the advantage of the different players. It is clear that after x coups, there were x tokens in the game; the advantage of player (r) relative to these x tokens, is the product of these tokens by the probability $y_{r,x}$ to win the game at coup x; this advantage is therefore $xy_{r,x}$. The value of $xy_{r,x}$ is the coefficient of $t^{x-1}dt$ in the differential of the generating function $y_{r,x}$; by dividing therefore this differential by dt, and by supposing next $t = 1$, we will have the sum of all the values of $xy_{r,x}$ to x infinity; this is the advantage of player (r). But it is necessary to subtract the tokens that he put into the game at each coup that he loses; now $y_{r,x}$ being his probability to win the game at coup x, $2^n y_{r,x-n+1}$ will be his probability to enter into the game, at coup $x-n+1$, since this last probability, multiplied by the probability $\frac{1}{2^n}$, that he will win this coup, and the $n-1$ following coups is his probability to win the game at coup x. By supposing therefore that he loses as many times as he enters into the game, the sum of all the values of $2^n y_{r,x-n+1}$ to x infinity, would be the disadvantage of player (r); and as the sum of all the values of $y_{r,x-n+1}$ is equal to the sum of all the values of $y_{r,x}$, or to y_r, we would have $2^n y_r$, or $\frac{2^n(1-p)p^r}{2-p-p^n}$ for the disadvantage of player (r). But he does not lose each time that he enters into the game, because he is able to enter into the game and win the game; it is necessary therefore to take off from $2^n y_r$, the sum of all the values of y_x

or y_r, and then the disadvantage of (r) is $\frac{(2^n-1)(1-p)p^r}{2-p-p^n}$. In order to have the entire advantage of (r), it is necessary to subtract this last quantity, from the sum of the values of $xy_{r,x}$; by designating therefore by S this sum, the advantage of player (r) will be

$$S - \frac{(2^n - 1)(1-p)p^r}{2 - p - p^n},$$

S being, as we have just seen, the differential of the generating function of $y_{r,x}$, divided by dt, and in which we suppose next $t = 1$. Under this supposition, we have [247]

$$T = p; \quad \frac{dT}{dt} = -np(1-p).$$

Let us designate by Y_r the advantage of (r), we will find

$$Y_r = \frac{np + 1 - n}{2 - p - p^n} p^r \left\{ (1-p)r + \frac{p^{n+1} + n(1-p)p^n - p}{2 - p - p^n} \right\}.$$

This equation will serve from $r = 0$ to $r = n - 1$, provided that we change Y_0 into $\bar{Y}_0$, $\bar{Y}_0$ being the advantage of the first two players, at the moment where they enter into the game.

If at the commencement of the game, each of the players deposits into the game a sum a; the advantage of player (r) will be increased from it by $(n + 1)a$, multiplied by the probability y_r, that this player will win the game; but it is necessary to take off from it the stake a from this player; it is necessary therefore, in order to have then his advantage, to increase the preceding expression of Y_r, by the quantity

$$\frac{(n+1)a(1-p)p^r}{2 - p - p^n} - a.$$

When the advantage of (r) becomes negative, it is changed into disadvantage.

§12. Let q be the probability of a simple event, at each coup; we demand the probability to bring it forth i times consecutively, in the number x coups.

Let us name z_x the probability that this composite event will take place precisely at coup x. For this, it is necessary that the simple event not arrive at coup $x - i$, and that it arrives in the i coups following, the composite event being not at all arrived previously. Let then P be the probability that the simple event will not arrive at all at coup $x - i - 1$. The corresponding probability that it will not arrive at all at coup $x-i$, will be $(1-q)P$; and the corresponding probability that the composed event will take place precisely at coup x, will be $(1 - q)Pq^i$. This will be the part of z_x corresponding to this case. But the probability [248]
that the composed event will arrive at coup $x - 1$, is evidently Pq^i; we have therefore

$$P = \frac{z_{x-1}}{q^i};$$

thus the partial value of z_x, relative to this case, is $(1 - q)z_{x-1}$.

Let us consider now the cases where the simple event will arrive at coup $x-i-1$. Let us name P' the probability that it will not arrive at coup $x-i-2$; the probability that it will arrive in this case at coup $x-i-1$, will be qP', and the probability that it will not arrive at coup $x-i$, will be $(1-q)qP'$; the partial value of z_x relative to this case, will be therefore $(1-q)qP'q^i$. But the probability that the composite event will arrive precisely at coup $x-2$, is $P'q^i$: this is the value of z_{x-2}; that which gives

$$P' = \frac{z_{x-2}}{q^i};$$

$(1-q)qz_{x-2}$ is therefore the partial value of z_x, relative to the case where the simple event will arrive at coup $x-i-1$, without arriving at coup $x-i-2$.

We will find in the same manner that $(1-q)q^2z_{x-3}$ is the partial value of z_x, relative to the case where the simple event will arrive at coups $x-i-1$ and $x-i-2$, without arriving at coup $x-i-3$; and so forth.

By uniting all these partial values of z_x, we will have

$$z_x = (1-q)(z_{x-1} + qz_{x-2} + q^2z_{x-3} \cdots + q^{i-1}z_{x-i}).$$

It is easy to conclude from it that the generating function of z_x is

$$\frac{q^i(1-qt)t^i}{1-t+(1-q)q^it^{i+1}};$$

because this generating function is

$$\frac{\phi(t)}{1-(1-q)(t+qt^2\cdots+q^{i-1}t^i)},$$

or

$$\frac{\phi(t)(1-qt)}{1-t+(1-q)q^it^{i+1}},$$

The function $\phi(t)$ must be determined by the condition that it must contain only the power i of t, since the composed event is able to commence to be possible only at coup i; moreover, the coefficient of this power is the probability q^i, that this event will take place precisely at this coup. [249]

By dividing the preceding generating function, by $1-t$, we will have

$$\frac{q^i(1-qt)t^i}{(1-t)^2\left(1+\frac{(1-q)q^it^{i+1}}{1-t}\right)}$$

for the generating function of the probability that the composite event will take place before or at coup x.

By developing this function, we will have for the coefficient of t^{x+i}, the series

$$q^i[(1-q)x+1] - (1-q)q^{2i}\frac{(x-i)}{1.2}[(1-q)(x-i-1)+2]$$
$$+ (1-q)^2q^{3i}\frac{(x-2i)(x-2i-1)}{1.2.3}[(1-q)(x-2i-2)+3]$$
$$- (1-q)^3q^{4i}\frac{(x-3i)(x-3i-1)(x-3i-2)}{1.2.3.4}[(1-q)(x-3i-3)+4]$$
$$+ \text{etc.},$$

the series being continued until we arrive to some negative factors. This is the expression of the probability that the composed event will take place at coup $x+i$ or before this coup.

Let us suppose further that two players A and B, of whom the respective skills to win a coup, are q and $1-q$, play with this condition, that the one of the two who will have first vanquished i times consecutively his adversary, will win the game; we demand the respective probabilities of the two players to win the game precisely at coup x.

Let y_x be the probability of A, and y'_x that of B. Player A is able to win the game at coup x, only as long as he commences or recommences to beat B at coup $x-i+1$, and that he continues to beat him the following $i-1$ coups. Now, before commencing coup $x-i+1$, B will have already beat A, either one time, or two times, ... or $i-1$ times. In [250] the first case, if we name P the probability of this case, $P(1-q)^{i-1}$ will be the probability y'_{x-1} of B to win the game at coup $x-1$, that which gives

$$P = \frac{y'_{x-1}}{(1-q)^{i-1}}.$$

But if B loses at coup $x-i+1$ and at the $i-1$ following coups, A will win the game at coup x, and the probability of this is Pq^i; $\frac{q^iy'_{x-1}}{(1-q)^{i-1}}$ is therefore the part of y_x, relative to the first case.

In the second case, if we name P' its probability, $P'(1-q)^{i-2}$ will be the probability y'_{x-2} of B to win the game at coup $x-2$. The probability of A to win the game at coup x, relative to this case, is $P'q^i$; we have therefore $\frac{q^iy'_{x-2}}{(1-q)^{i-2}}$ for this probability.

By continuing thus, we will have

$$y_x = \frac{q^i}{(1-q)^i}[(1-q)y'_{x-1} + (1-q)^2y'_{x-2} \cdots + (1-q)^{i-1}y'_{x-i+1}].$$

If we change q into $1-q$, y_x into y'_x and reciprocally, we will have

$$y'_x = \frac{(1-q)^i}{q^i}(qy_{x-1} + q^2y_{x-2} \cdots + q^{i-1}y_{x-i+1}).$$

Now, u being the generating function of y_x, that of y'_x will be, by all that which precedes,

$$kq.ut.(1+qt+qt^2 \cdots + q^{i-2}t^{i-2}),$$

k being equal to $\frac{(1-q)^i}{q^i}$. But the preceding expression of y'_x commencing to hold only when $x = i+1$, because for the smaller values of x, y_{x-1}, y_{x-2}, etc. are nulls; it is necessary, in order to complete the preceding expression of the generating function of y'_x, to add to it a rational and integral function of t, of order i, and of which the coefficients of the powers of t are the values of y'_x, when x is equal or smaller than i. Now y'_x is null, when x is less than i; and when it is equal to i, y'_x is $(1-q)^i$, because it expresses then the probability of B to [251]
win the game after i coups; the function to add is therefore $(1-q)^i t^i$; thus the generating function of y'_x is

$$kq.ut.(1 + qt + qt^2 \cdots + q^{i-2}t^{i-2}) + (1-q)^i t^i.$$

If we name u' this function, the expression of y_x in y'_{x-1}, y'_{x-2}, etc., will give for the generating function of y_x, by changing in that of y'_x, k into $\frac{1}{k}$, q into $1-q$,

$$\frac{1}{k}(1-q).u'.t[1 + (1-q)t \cdots + (1-q)^{i-2}t^{i-2}] + q^i t^i.$$

This quantity is therefore equal to u; whence we deduce, by substituting in it for u its preceding value,

$$u = \frac{q^i t^i (1-qt)[1-(1-q)^i t^i]}{1 - t + q(1-q)^i t^{i+1} + (1-q)q^i t^{i+1} - q^i (1-q)^i t^{2i}}.$$

By changing q into $1-q$, we will have the generating function u' of y'_x. If we divide these functions by $1-t$, we will have the generating functions of the respective probabilities of A and of B, to win the game before or at coup x.

If we suppose $t = 1$ in u, we will have the probability that A will win the game; because it is clear that by developing u according to the powers of t, and by supposing next $t = 1$, the sum of all the terms of this development will be that of all the values of y_x. We find thus the probability of A to win the game equal to

$$\frac{[1-(1-q)^i]q^{i-1}}{(1-q)^{i-1} + q^{i-1} - q^{i-1}(1-q)^{i-1}};$$

the probability of B is therefore

$$\frac{(1-q)^{i-1}[1-q^i]}{(1-q)^{i-1} + q^{i-1} - q^{i-1}(1-q)^{i-1}}.$$

Let us suppose now that the players, at each coup that they lose, deposit a franc into the game, and let us determine their respective lot. It is clear that the gain of player A will be x, if he wins the game at coup x, since there will be x francs deposited into the game; thus [252]
the probability of this event being y_x by that which precedes. Sxy_x will be the expression of the advantage of A, the sign S extending to all the possible values of x. The generating function of y_x being u or $\frac{T'}{T}$, T' being the numerator of the preceding expression of u, and

T being its denominator; it is easy to see that we will have Sxy_x by differentiating $\frac{T'}{T}$, and by supposing next $t = 1$ in this differential, that which gives with this condition,

$$Sxy_x = \frac{dT'}{Tdt} - \frac{T'dT}{T^2dt}.$$

In order to have the disadvantage of A, we will observe that at each coup that he plays, the probability that he will lose, and consequently that he will deposit a franc into the game, is $1-q$; his loss is therefore the product of $1-q$, by the probability that the coup will be played; now the probability that coup x will be played, is $1-Sy_{x-1}-Sy'_{x-1}$; the generating function of unity, is here $\frac{t}{1-t}$, and that of $Sy_{x+1}+Sy'_{x+1}$ is $\frac{T't+T''t}{T(1-t)}$; T'' being that which T' becomes when we change q into $1-q$ and reciprocally; thus the generating function of the disadvantage of A is

$$\frac{(1-q)t(T-T'-T'')}{(1-t)T}.$$

The numerator and the denominator of this function are divisible by $1-t$; moreover, we will have the sum of all the disadvantages of A, or his total disadvantage, by making $t = 1$ in this generating function; the total disadvantage is therefore by the known methods, and by observing that $T'+T'' = T$, when $t = 1$,

$$-\frac{(1-q)(dT-dT'-dT'')}{Tdt},$$

t being supposed equal to unity, after the differentiations. If we subtract this expression, from that of the total advantage of A, we will have, for the expression of the lot of this player,

$$\frac{qdT'+(1-q)(dT-dT'')}{Tdt} - \frac{T'dT}{T^2dt}.$$

The lot of B will be [253]

$$\frac{(1-q)dT''+q(dT-dT')}{Tdt} - \frac{T''dT}{T^2dt},$$

t being supposed unity after the differentiations; that which gives

$$\begin{aligned}
T &= q(1-q)[q^{i-1}+(1-q)^{i-1}-q^{i-1}(1-q)^{i-1}];\\
\frac{dT}{dt} &= (i+1)q(1-q)[q^{i-1}+(1-q)^{i-1}]-2^iq^i(1-q)^i-1;\\
T' &= (1-q)q^i[1-(1-q)^i];\\
\frac{dT'}{dt} &= i(1-q)q^i[1-2(1-q)^i]-qq^i[1-(1-q)^i].
\end{aligned}$$

we will have T'' and $\frac{dT''}{dt}$ by changing in these last two expressions, q into $1-q$.

§13. An urn being supposed to contain $n+1$ balls, distinguished by the numbers 0, 1, 2, 3, $\dots n$, we draw from it a ball which we replace into the urn after the drawing. We demand the probability that after i drawings, the sum of the numbers brought forth will be equal to s.

Let t_1, t_2, t_3, $\dots t_i$ be the numbers brought forth at the first drawing, at the second, at the third, etc.; we must have

$$t_1 + t_2 + t_3 \cdots + t_i = s. \tag{1}$$

$t_2, t_3, \dots t_i$ being supposed not to vary, this equation is susceptible only of one combination. But if we make vary at the same time t_1 and t_2, and if we suppose that these variables can be extended indefinitely from zero, then the number of combinations which give the preceding equation will be

$$s + 1 - t_3 - t_4 \cdots - t_i;$$

because t_1 can be extended from zero, that which gives

$$t_2 = s - t_3 - t_4 \cdots - t_i,$$

to $s - t_3 - t_4 \cdots - t_i$, that which gives $t_2 = 0$, the negative values of the variables t_1, t_2 needing to be excluded.

Now, the number $s + 1 - t_3 - t_4 \cdots - t_i$ is susceptible of many values, by virtue of [254] the variations of t_3, t_4, etc. Let us suppose first t_4, t_5, etc. invariables, and that t_3 can be extended indefinitely from zero; then we make

$$s + 1 - t_3 - t_4 \cdots - t_i = x,$$

by integrating this variable of which the finite difference is unity, we will have $\frac{x(x-1)}{1.2}$ for its integral; but, in order to have the sum of all the values of x, it is necessary, as we know, to add x to this integral; this sum is therefore $\frac{x(x+1)}{1.2}$. It is necessary to make x equal to its greatest value, which we obtain by making t_3 null in the function $s + 1 - t_3 - t_4 \cdots - t_i$: thus the total number of combinations relative to the variations of t_1, t_2 and t_3, is

$$\frac{(s + 2 - t_4 - t_5 \cdots - t_i)(s + 1 - t_4 - t_5 \cdots - t_i)}{1.2}.$$

By making next in this function

$$s + 2 - t_4 - t_5 \cdots - t_i = x,$$

it becomes $\frac{x(x-1)}{1.2}$; by integrating it from $x = 0$, and by adding the function itself, to this integral, we will have $\frac{(x+1)x(x-1)}{1.2.3}$; the value of x null corresponds to $t_4 = s + 2 - t_5 \cdots - t_i$, and its greatest value corresponds to t_4 null, and consequently it is equal to $s + 2 - t_5 \cdots - t_i$; by substituting therefore for x, this value into the preceding integral, we will have

$$\frac{(s + 3 - t_5 - t_6 - \cdots - t_i)(s + 2 - t_5 - t_6 - \cdots - t_i)(s + 1 - t_5 - t_6 - \cdots - t_i)}{1.2.3}$$

for the sum of all the combinations relative to the variations of t_1, t_2, t_3, t_4. By continuing thus, we will find generally that the total number of the combinations which give equation (1), under the supposition where the variables $t_1, t_2, \dots t_i$ can be extended indefinitely from zero, is

$$\frac{(s+i-1)(s+i-2)(s+i-3)\dots(s+1)}{1.2.3\dots(i-1)} \qquad (a)$$

but in the present question, these variables can not be extended beyond n. In order to express this condition, we will observe that the urn containing $n+1$ balls, the probability to extract any one of them, is $\frac{1}{n+1}$; thus the probability of each of the values of t_1, from zero to n, is $\frac{1}{n+1}$. The probability of the values of t_1 equal or superior to $n+1$, is null; we can therefore represent it by $\frac{1-l^{n+1}}{n+1}$, provided that we make $l = 1$ in the result of the calculation; then the probability of any value of t_1 can be generally expressed by $\frac{1-l^{n+1}}{n+1}$, provided that we make l to begin, only when t_1 will have attained $n+1$, and that we suppose it at the end, equal to unity: it is likewise of the probabilities of the other variables. Now, the probability of equation (1) is the product of the probabilities of the values of t_1, t_2, t_3, etc.; this probability is therefore $\left(\frac{1-l^{n+1}}{n+1}\right)^i$; the number of combinations which give this equation, multiplied by their respective probabilities, is thus the product of the fraction (a) by $\left(\frac{1-l^{n+1}}{n+1}\right)^i$, or [255]

$$\frac{(s+1)(s+2)\dots(s+i-1)}{1.2.3\dots(i-1)}\left(\frac{1-l^{n+1}}{n+1}\right)^i; \qquad (b)$$

but it is necessary, in the development of this function, to apply l^{n+1} only to the combinations in which one of the variables begins to surpass n: it is necessary to apply l^{2n+2} only to the combinations in which two of the variables begin to surpass n, and thus of the rest. If in equation (1) we suppose that one of the variables, t_1, for example, surpasses n; by making $t_1 = n+1+t'$, this equation becomes

$$s - n - 1 = t'_1 + t_2 + t_3 + \text{etc.},$$

the variable t'_1 being able to be extended indefinitely. If two of the variables such as t_1 and t_2 surpass n; by making

$$t_1 = n+1+t'_1, \qquad t_2 = n+1+t'_2;$$

the equation becomes [256]

$$s - 2n - 2 = t'_1 + t'_2 + t_3 + \text{etc.},$$

and so forth. We must therefore, in the function (a) which we have derived from equation (1), diminish s by $n+1$, relatively to the system of variables t'_1, t_2, t_3, etc. We must diminish it by $2n+2$, relatively to the variables t'_1, t'_2, t_3, etc.; and thus of the rest. It is necessary consequently, in the development of the function (b) with respect to the powers of l, to diminish in each term, s from the exponent of the power of l; by making next $l = 1$, this function becomes

$$\begin{aligned}&\frac{(s+1)(s+2)\dots(s+i-1)}{1.2.3\dots(i-1)(n+1)^i} - \frac{i(s-n)(s-n+1)\dots(s+i-n-2)}{1.2.3\dots(i-1)(n+1)^i}\\ &\quad + \frac{i(i-1)}{1.2}\cdot\frac{(s-2n-1)(s-2n)\dots(s+i-2n-3)}{1.2.3\dots(i-1)(n+1)^i} - \text{etc.};\end{aligned} \qquad (c)$$

the series must be continued until one of the factors $s-n$, $s-2n-1$, $s-3n-2$, etc. becomes null or negative.

This formula gives the probability to bring forth a given number s, by projecting i dice with a number $n+1$ faces on each, the smallest number marked on the faces being 1. It is clear that this reverts to supposing in the preceding urn, all the numbers of the balls, increased by unity; and then the probability to bring forth the number $s+i$ in i drawings, is the same as that of bringing forth the number s in the case that we just considered; now, by making $s+i = s'$, we have $s = s' - i$; formula (c) will give therefore for the probability to bring forth the number s' by projecting the i dice,

$$\begin{aligned}&\frac{(s'-1)(s'-2)\dots(s'-i+1)}{1.2.3\dots(i-1)(n+1)^i} - \frac{i(s'-n-2)(s'-n-3)\dots(s'-i-n)}{1.2.3\dots(i-1)(n+1)^i}\\ &\quad + \frac{i(i-1)}{1.2}\cdot\frac{(s'-2n-3)(s'-2n-4)\dots(s'-i-2n-1)}{1.2.3\dots(i-1)(n+1)^i} - \text{etc.}\end{aligned}$$

Formula (c) applied to the case where s and n are infinite numbers, is transformed into the following

$$\frac{1}{1.2.3\dots(i-1)n}\left\{\left(\frac{s}{n}\right)^{i-1} - i\left(\frac{s}{n}-1\right)^{i-1} + \frac{i(i-1)}{1.2}\left(\frac{s}{n}-2\right)^{i-1} - \text{etc.}\right\}.$$

This expression can serve to determine the probability that the sum of the inclinations to the ecliptic, of a number i of orbits, will be comprehended within some given limits, by supposing that for each orbit, all the inclinations from zero to the right angle, are equally possible. In fact, if we imagine that the right angle $\frac{1}{2}\pi$, is divided into an infinite number n of equal parts, and if s contains an infinite number of these parts; by naming ϕ the sum of the inclinations of the orbits, we will have [257]

$$\frac{s}{n} = \frac{\phi}{\frac{1}{2}\pi}.$$

By multiplying therefore the preceding expression by ds or by $\frac{n\,d\phi}{\frac{1}{2}\pi}$, and by integrating it from $\phi - \epsilon$ to $\phi + \epsilon$, we will have

$$\frac{1}{1.2.3\dots i}\left\{\begin{aligned}&\left(\frac{\phi+\epsilon}{\frac{1}{2}\pi}\right)^i - i\left(\frac{\phi+\epsilon}{\frac{1}{2}\pi}-1\right)^i + \frac{i(i-1)}{1.2}\left(\frac{\phi+\epsilon}{\frac{1}{2}\pi}-2\right)^i - \text{etc.}\\ &-\left(\frac{\phi-\epsilon}{\frac{1}{2}\pi}\right)^i + i\left(\frac{\phi-\epsilon}{\frac{1}{2}\pi}-1\right)^i - \frac{i(i-1)}{1.2}\left(\frac{\phi-\epsilon}{\frac{1}{2}\pi}-2\right)^i + \text{etc.}\end{aligned}\right\}; \qquad (o)$$

this is the expression of the probability that the sum of the inclinations of the orbits will be comprehended within the limits $\phi - \epsilon$ to $\phi + \epsilon$.

Let us apply this formula to the orbits of the planets. The sum of the inclinations of the orbits of the planets to that of the Earth, was $91.4187°$ at the beginning of 1801: there are ten orbits, without including the ecliptic; we have therefore here $i = 10$. We make next

$$\phi - \epsilon = 0,$$
$$\phi + \epsilon = 91.4187°.$$

The preceding formula becomes thus, by observing that $\frac{1}{2}\pi$, or the quarter of the circumference is $100°$,[10]

$$\frac{1}{1.2.3\ldots 10}(0.914187)^{10}.$$

This is the expression of the probability that the sum of the inclinations of the orbits will be comprehended within the limits zero and $91,4187°$, if all the inclinations were equally possible. This probability is therefore $0,00000011235$. It is already very small; but it is necessary next to combine it with the probability of a very remarkable circumstance in the [258]
system of the world, and which consists in this that all the planets are moved in the same sense as the Earth. If the direct and retrograde movements are supposed equally possible, this last probability is $\left(\frac{1}{2}\right)^{10}$; it is necessary therefore to multiply $0,00000011235$ by $\left(\frac{1}{2}\right)^{10}$, in order to have the probability that all the movements of the planets and of the Earth will be directed in the same sense, and that the sum of their inclinations to the orbit of the earth, will be comprehended within the limits zero and $91,4187°$; we will have thus $\frac{1,0972}{(10)^{10}}$ for this probability; that which gives $1 - \frac{1,0972}{(10)^{10}}$ for the probability that this had not ought to take place; if all the inclinations, in the same way the direct and retrograde movements, have been equally facile. This probability approaches so to certainty, that the observed result becomes unlikely under this hypothesis; this result indicates therefore with a very great probability, the existence of an original cause which has determined the movements of the planets to bring themselves together to the plane of the ecliptic, or more naturally, to the plane of the solar equator, and to be moved in the sense of the rotation of the sun. If we consider next that the eighteen satellites observed until now, make their revolution in the same sense, and that the observed rotations in the number of thirteen in the planets, the satellites and the ring of Saturn, are yet directed in the same sense; finally, if we consider that the mean of the inclinations of the orbits of these stars, and of their equators to the solar equator, is quite removed from reaching a half right angle; we will see that the existence of a common cause, which has directed all these movements in the sense of the rotation of the sun, and onto some planes slightly inclined to the one of its equator, is indicated with a probability quite superior to the one of the greatest number of the historical facts on which we permit no doubt.

Let us see now if this cause has influence on the movement of the comets. The number of these which we have observed until the end of 1811, by counting for the same the diverse

[10]*Translator's note*: These are decimal degrees. That is, $\pi/4 = 100°$.

apparitions of the one of 1759, is raised to one hundred, of which fifty-three are direct, and forty-seven are retrograde. The sum of the inclinations of the orbits of the first is $2657,993°$, [259]
and that of the inclinations of the other orbits, is $2515,684°$: the mean inclination of all these orbits is therefore $51,73677°$; consequently the sum of all the inclinations is $\frac{i.\pi}{4} + i.1,73677°$, i being here equal to 100. We see already that the mean inclination surpassing the half right angle, the comets, far from participating in the tendency of the bodies of planetary system, in order to be moved in some planes slightly inclined to the ecliptic, appear to have a contrary tendency. But the probability of this tendency is very small. In fact, if we suppose, in formula (o),

$$\phi = \frac{i.\pi}{4}, \qquad \epsilon = i.1,73677°,$$

it becomes

$$\frac{1}{1.2.3\ldots i.2^i}\left\{\begin{array}{l} \left(i+\frac{4i.1,73677°}{\pi}\right)^i - i\left(i+\frac{4i.1,73677°}{\pi}-2\right)^i \\ \quad + \frac{i(i-1)}{1.2}\left(i+\frac{4i.1,73677°}{\pi}-4\right)^i - \text{etc.} \\ \quad - \left(i-\frac{4i.1,73677°}{\pi}\right)^i + i\left(i-\frac{4i.1,73677°}{\pi}-2\right)^i \\ \quad - \frac{i(i-1)}{1.2}\left(i-\frac{4i.1,73677°}{\pi}-4\right)^i \text{etc.} \end{array}\right\}; \qquad (p)$$

π being $200°$. This is the expression of the probability that the sum of the inclinations of the orbits of the i comets, must be comprehended within the limits $\pm i.1,73677°$. The number of terms of this formula, and the precision with which it would be necessary to have each of them, renders the calculation of it impractical; it is necessary to recur to the methods of approximation developed in the second part of the first Book. We have, by §42 of the same Book,

$$\frac{(i+r\sqrt{i})^i - i(i+r\sqrt{i}-2)^i + \frac{i(i-1)}{1.2}(i+r\sqrt{i}-4)^i - \text{etc.}}{1.2.3\ldots i.2^i}$$
$$= \frac{1}{2} + \sqrt{\frac{3}{2\pi}}\int dr\, c^{-\frac{3}{2}r^2} - \frac{3}{20.i}\sqrt{\frac{3}{2\pi}}\, r(1-r^2)c^{-\frac{3}{2}r^2},$$

the powers of the negative quantities being here excluded, as they are in the preceding [260]
formula; by making therefore

$$r\sqrt{i} = \frac{4i.1,73677°}{200°},$$

formula (p) becomes

$$2\sqrt{\frac{3}{2\pi}}\int dr\, c^{-\frac{3}{2}r^2} - \frac{3}{10.i}\sqrt{\frac{3}{2\pi}}\, r(1-r^2)c^{-\frac{3}{2}r^2}$$

the integral being taken from r null. We find thus $0,474$ for the probability that the inclination of the 100 orbits must fall within the limits $50° \pm 1,17377°$; the probability that the mean inclination must be inferior to the observed inclination, is therefore $0,737$. This probability is not great enough in order that the observed result makes rejection of the hypothesis of an equal facility of the inclinations of the orbits, and in order to indicate the existence of an original cause which has influence on these inclinations, a cause which we cannot forbid to admit in the inclinations of the orbits of the planetary system.

The same thing holds with respect to the sense of the movement. The probability that out of 100 comets, 47 moreover will be retrogrades, is the sum of the 48 first terms of the binomial $(p+q)^{100}$, by making in the result of the calculation $p = q = \frac{1}{2}$. But the sum of the 50 first terms, plus the half of the 51st or the middle term, is the half of the entire binomial, or of $\left(\frac{1}{2}+\frac{1}{2}\right)^{100}$, that is $\frac{1}{2}$; the sought probability is therefore

$$\frac{1}{2} - \frac{100.99\ldots 51}{1.2.3\ldots 50.2^{100}}\left(\frac{1}{2}+\frac{50}{51}+\frac{50.49}{51.52}\right)$$

or

$$= \frac{1}{2} - \frac{1.2.3\ldots 100.1594}{(1.2.3\ldots 50)^2.2^{100}.663}.$$

By virtue of the theorem

$$1.2.3\ldots s = s^{s+\frac{1}{2}}c^{-s}\left(1+\frac{1}{12s}+\text{etc.}\right)\sqrt{2\pi},$$

we have, very nearly,

$$1.2.3\ldots 100 = 100^{100+\frac{1}{2}}c^{-100}\left(1+\frac{1}{1200}\right)\sqrt{2\pi},$$
$$2^{100}(1.2.3\ldots 50)^2 = 100^{100+1}c^{-100}\left(1+\frac{1}{300}\right)\pi.$$

The preceding probability becomes thus, [261]

$$\frac{1}{2} - \frac{1}{\sqrt{50\pi}}\frac{1197.1594}{1200.663} = 0.3046.$$

This probability is much too great to indicate a cause which has favored, at the origin, the direct movements. Thus the cause which has determined the sense of the movements of the revolution and of the rotation of the planets and of the satellites, seems to have no influence on the movement of the comets.

§14. The method of the preceding section has the advantage to be extended to the case where the number of balls of the urn, which bear the same label, is not equal to unity, but varies according to any law whatsoever. Let us imagine, for example, that there is only one ball bearing the n° 0, only one ball bearing the n° 1, and so forth until n° r inclusively. Let us suppose moreover that there are two balls bearing the n° $r+1$, two balls bearing the n° $r+2$, and so forth until n° n inclusively. The total number of balls in the urn will be $2n-r+1$, the probability to extract from it one of the labels inferior to $r+1$, will be therefore $\frac{1}{2n-r+1}$; and the probability to extract from it the n° $r+1$ or one of the superior labels, will be $\frac{2}{2n-r+1}$: we will represent it by $\frac{1+l^{r+1}}{2n-r+1}$; but we will make $l = 1$ in the result of the calculation. Although there are no labels beyond n° n, we will be able however to consider in the urn some labels superior to n, to infinity, provided that we will give to their extraction, a null probability; we will be able therefore to represent this probability by $\frac{1+l^{r+1}-2l^{n+1}}{2n-r+1}$, by making $l = 1$ in the result of the calculation. By this artifice, we will be able to represent generally the probability of any label whatsoever, by the preceding expression; provided that we will make l^{r+1} commence only when the labels will commence to surpass r, and that we will make l^{n+1} commence only when one of the labels will commence to surpass n. This premised, we will find, by applying here the reasonings of the previous section, that the probability to bring forth the number s, in i drawings, is equal to [262]

$$\frac{(s+i-1)(s+i-2)(s+i-3)\dots(s+1)}{1.2.3\dots(i-1)(2n-r+1)^i}(1+l^{r+1}-2l^{n+1})^i,$$

provided that in the development of this function, according to the powers of l, we diminish in each term, s from the exponent of the power of l, that we suppose next $l = 1$, and that we arrest the series when we arrive to some negative factors.

§15. Let us apply now this method to the investigation of the mean result that any number of observations of which the laws of facility of the errors are known must give. For this, we will resolve the following problem:

Let i variable and positive quantities be t, t_1, t_2, $\dots t_{i-1}$, of which the sum is s, and of which the law of possibility is known; we propose to find the sum of the products of each value that a given function $\psi(t, t_1, t_2, \text{etc.})$ of these variables is able to receive, multiplied by the probability corresponding to this value.

Let us suppose for more generality, that the functions which express the possibilities of the variables t, t_1, etc. are discontinuous, and let us represent by $\phi(t)$ the possibility of t, from $t = 0$ to $t = q$; by $\phi'(t) + \phi(t)$, its possibility from $t = q$ to $t = q'$; by $\phi''(t) + \phi'(t) + \phi(t)$, its possibility from $t = q'$ to $t = q''$, and so forth to infinity. Let us designate next the same quantities relative to the variables t_1, t_2, etc. by the same letters, by writing respectively at the base, the numbers 1, 2, 3, etc.; so that q_1, q_1', etc.; $\phi_1(t_1)$, $\phi_1'(t_1)$, etc. correspond, relatively to t_1, to that which q, q', etc., $\phi(t)$, $\phi'(t)$, etc. are respectively to t, and so forth. In this manner of representing the possibilities of the variables, it is clear that the function $\phi(t)$ holds from $t = 0$ to t infinity; that the function $\phi'(t)$ holds from $t = q$ to t infinity, and so forth. In order to recognize the values of t, t_1, t_2, etc. when these diverse functions begin to hold, we will multiply conformably to the method exposed in the preceding sections, $\phi(t)$ by l^0 or unity, $\phi'(t)$ by l^q,$\phi''(t)$ by $l^{q'}$, etc.; we will multiply similarly $\phi_1(t_1)$ by unity, $\phi'_1(t_1)$ by l^{q_1}, and so forth: the exponents of the powers of l will indicate then these values. It will suffice next to make $l = 1$ in the last result of the [263] calculation. By means of these very simple artifices, we can easily resolve the proposed problem.

The probability of the function $\psi(t, t_1, t_2, \text{etc.})$ is evidently equal to the product of the probabilities of t, t_1, t_2, etc., so that if we substitute for t its values $s - t_1 - t_2 - \text{etc.}$ that the equation gives

$$t + t_1 + t_2 \cdots + t_{i-1} = s,$$

the product of the proposed function by its probability, will be

$$\begin{aligned}
&\psi(s - t_1 - t_2 - \text{etc.}, t_1, t_2, \text{etc.}) \\
&\times [\phi(s - t_1 - t_2 - \text{etc.}) + l^q \phi'(s - t_1 - t_2 - \text{etc.}) \\
&\qquad\qquad + l^{q'} \phi''(s - t_1 - t_2 - \text{etc.}) + \text{etc.}] \\
&\times [\phi_1(t_1) + l^{q_1}\phi'_1(t_1) + l^{q'_1}\phi''_1(t_1) + \text{etc.}] \\
&\times [\phi_2(t_2) + l^{q_2}\phi'_2(t_2) + l^{q'_2}\phi''_2(t_2) + \text{etc.}] \\
&\times \text{etc.}
\end{aligned} \tag{A}$$

we will have therefore the sum of all these products, 1° by multiplying the preceding quantity by dt_1, and by integrating for all the values of which t_1 is susceptible; 2° by multiplying this integral by dt_2, and by integrating for all the values of which t_2 is susceptible, and so forth to the last variable t_{i-1}; but these successive integrations require some particular attentions.

Let us consider any term whatsoever of the quantity (A), such as

$$\begin{aligned}
l^{q+q_1+q_2+\text{etc.}} &\psi(s - t_1 - t_2 - \text{etc.}, t_1, t_2, \text{etc.}) \\
&\times \phi'(s - t_1 - t_2 - \text{etc.})\phi'_1(t_1)\phi''_2(t_2).\text{etc.};
\end{aligned}$$

by multiplying it by dt_1, it is necessary to integrate for all the possible values of t_1; now the function $\phi'(s-t_1-t_2-\text{etc.})$ holds only when t, of which the value is $s-t_1-t_2-\text{etc.}$, equals or surpasses q; the greatest value that t_1 is able to receive, is therefore $s - q - t_2 - t_3 - \text{etc.}$ Moreover, $\phi'_1(t_1)$ holding only when t_1 is equal or greater than q_1, this quantity is the smallest value that t_1 is able to receive; it is necessary therefore to take the integral of which there is concern, from $t_1 = q_1$ to

$$t_1 = s - q - q_1 - t_2 - t_3 - \text{etc.};$$

or, that which reverts to the same, from $t_1 - q_1 = 0$ to

$$t_1 - q_1 = s - q - q_1 - t_2 - t_3 - \text{etc.}$$

We will find in the same manner that by multiplying this new integral by dt_2, it will be [264] necessary to integrate it from $t_2 - q'_2 = 0$ to

$$t_2 - q'_2 = s - q - q_1 - q'_2 - t_3 - \text{etc.}$$

By continuing to operate thus, we will arrive to a function of $s - q - q_1 - q'_2 -$ etc., in which there will remain none of the variables t, t_1, t_2, etc. This function must be rejected, if $s - q - q_1 - q'_2 -$ etc. is null or negative; because it is clear that in this case, the system of functions $\phi'(t)$, $\phi'_1(t_1)$, $\phi''_2(t_2)$, etc. can not be employed. In fact, the smallest values of t_1, t_2, etc. being by the nature of these functions, equals to q_1, q'_2, etc.; the greatest value that t can receive is $s - q_1 - q'_2 -$ etc.; thus the greatest value of $t - q$ is

$$s - q - q_1 - q'_2 - \text{etc.};$$

now the function $\phi'(t)$ can be employed only as long as $t - q$ is positive.

Thence results a very simple solution of the proposed problem. Let us substitute, 1° $q + t$ instead of t, into $\phi'(t)$; $q' + t$ instead of t, into $\phi''(t)$; $q'' + t$ instead of t, into $\phi'''(t)$, and so forth; 2° $q_1 + t_1$ instead of t_1, into $\phi'_1(t_1)$; $q'_1 + t_1$ instead of t_1, into $\phi''_1(t_1)$; etc.; 3° q_2+t_2 instead of t_2, into $\phi'_2(t_2)$; q'_2+t_2 instead of t_2, into $\phi''_2(t_2)$, etc.; and so forth; 4° finally, $k + t$ instead of t, $k_1 + t_1$ instead of t_1, and thus of the remainder, into $\psi(t, t_1, t_2, \text{etc.})$; the function (A) will become

$$\begin{aligned}&\psi(k + s - t_1 - t_2 - t_3 - \text{etc.},\ k_1+t_1, k_2 +\ t_2, \text{etc.})\\ &\quad\times[\phi(s - t_1 - t_2 - t_3 - \text{etc.})+l^q\phi'(s + q - t_1 - t_2 - \text{etc.})\\ &\qquad\qquad +l^{q'}\phi''(s + q' - t_2 - t_3 - \text{etc.})]\\ &\quad\times[\phi_1(t_1) + l^{q_1}\phi'_1(q_1 + t_1)+l^{q'_1}\phi''_1(q'_1 + t_1) + \text{etc.}]\\ &\quad\times[\phi_2(t_2) + l^{q_2}\phi'_2(q_2 + t_2)+\text{etc.}].\end{aligned} \tag{A$'$}$$

by multiplying this function by dt_1, we will integrate it from t_1 null to $t_1 = s-t_2-t_3-$etc. We will multiply next this first integral by dt_2, and we will integrate it from t_2 null to $t_2 = s - t_3 - t_4 -$ etc.. By continuing thus, we will arrive to a last integral which will be a function of s, and which we will designate by $\Pi(s)$; and this function will be the sought [265] sum of all the values of $\psi(t, t_1, t_2, \text{etc.})$ multiplied by their respective probabilities. But for this, it is necessary to take care to change in any term whatsoever, multiplied by a power of l, such as $l^{q+q_1+q_2+\text{etc.}}$, k in the part of the exponent of the power relative to the variable t, and which in this case is q; and if this part is lacking, it is necessary to suppose k equal to zero. It is similarly necessary to change k_1 in the part of the exponent relative to the variable t_1, and so forth; it is necessary to diminish s from the entire exponent of the power of l, and to write thus, in the present case, $s - q - q_1 - q'_2 -$ etc., instead of s, and to reject the term, if s, thus diminished, becomes negative. Finally it is necessary to suppose $l = 1$.

If $\psi(t, t_1, t_2, \text{etc.})$, $\phi(t)$, $\phi'(t)$, etc.; $\phi_1(t_1)$, etc. are some rational and integral functions of the variables t, t_1, t_2, etc.; of their exponentials, and of sines and cosines; all the successive integrations will be possible, because it is of the nature of these functions, to reproduce themselves by the integrations. In the other cases, the integrations would not be able to be possible; but the preceding analysis reduces then the problem to quadratures. The case of the rational and integral functions, offer some simplifications that we will expose.

Let us suppose that we have

$$\begin{aligned}
&\phi(t) + l^q\phi'(q+t) + l^{q'}\phi''(q'+t) + \text{etc.} &&= A + Bt + Ct^2 + \text{etc.},\\
&\phi_1(t_1) + l^{q_1}\phi_1'(q_1+t_1) + l^{q_1'}\phi_1''(q_1'+t_1) + \text{etc.} &&= A_1 + B_1t_1 + C_1t_1^2 + \text{etc.},\\
&\phi_2(t_2) + l^{q_2}\phi_2'(q_2+t_2) + l^{q_2'}\phi_2''(q_2'+t_2) + \text{etc.} &&= A_2 + B_2t_2 + C_2t_2^2 + \text{etc.},\\
& &&\text{etc.}
\end{aligned}$$

and let us designate by $H.t^n.t_1^{n_1}.t_2^{n_2}$.etc. any term whatsoever of $\psi(k+t, k_1+t_1, k_2+t_2, \text{etc.})$; it is easy to be assured that the part of $\Pi(s)$ corresponding to this term, is

$$\begin{aligned}
&1.2.3\ldots n.1.2.3\ldots n_1.1.2.3\ldots n_2.\text{etc.}.Hs^{i+n+n_1+n_2+\text{etc.}-1}\\
&\quad\times[A + (n+1)Bs + (n+1)(n+2)Cs^2 + \text{etc.}]\\
&\quad\times[A_1 + (n_1+1)B_1s + (n_1+1)(n_1+2)C_1s^2 + \text{etc.}];\\
&\quad\times[A_2 + (n_2+1)B_2s + (n_2+1)(n_2+2)C_2s^2 + \text{etc.}]\\
&\quad\times\text{etc.},
\end{aligned}\tag{B}$$

provided that in the development of this quantity, instead of any power whatsoever a of s, [266]
we write $\frac{s^a}{1.2.3\ldots a}$. We will have next the corresponding part of the entire sum of the values of $\psi(t, t_1, t_2, \text{etc.})$, multiplied by their respective probabilities, by changing any term of this development, such as $H\lambda t^\mu s^a$ into $H\lambda(s-\mu)^a$, and by substituting into H, instead of k, the part of the exponent μ, which is relative to the variable t; instead of k_1, the part relative to t_1, and thus of the remainder.

If in formula (B) we suppose $H = 1$, and n, n_1, n_2, etc. nulls; we will have the sum of the values of unity, multiplied by their respective probability; now it is clear that this sum is nothing other than the sum of all the combinations in which the equation

$$t + t_1 + t_2 \cdots + t_{i-1} = s$$

holds, multiplied by their probability; it expresses consequently the probability of this equation. If under the preceding hypotheses, we suppose moreover that the law of probability is the same for the first r variables t, t_1, t_2,..., t_{r-1}, and if for the last $i-r$, it is again the same, but different than for the first; we will have

$$\begin{aligned}
A &= A_1 = A_2 = \cdots = A_{r-1},\\
B &= B_1 = B_2 = \cdots = B_{r-1},\\
&\text{etc.}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
A_r &= A_{r+1} \ldots \cdots = A_{i-1},\\
B_r &= B_{r-1} \ldots \cdots = B_{i-1},\\
&\text{etc.},
\end{aligned}$$

and formula (B) will be changed into the following,

$$s^{i-1}(A + Bs + 2Cs^2 + \text{etc.})^r(A_r + B_rs + 2C_rs^2 + \text{etc.})^{i-r}. \tag{C}$$

This formula will serve to determine the probability that the sum of the errors of any number of observations whatsoever of which the law of facility of errors is known, will be comprehended within some given limits.

Let us suppose, for example, that we have $i-1$ observations of which the errors for each observation are able to be extended from $-h$ to $+g$, and that by naming z the error of the first of these observations, the law of facility of this error is expressed by $a+bz+cz^2$. Let us suppose next that this law is the same for the errors $z_1, z_2, \dots, z_{i-2}$ of the other observations, and let us seek the probability that the sum of these errors, will be comprehended within the limits p and $p+e$. [267]

If we make

$$z = t - h, \quad z_1 = t_1 - h, \quad z_2 = t_2 - h, \quad \text{etc.};$$

it is clear that t, t_1, t_2, etc. will be positive and will be able to be extended from zero, to $h+g$; moreover, we will have

$$z + z_1 + z_2 \cdots + z_{i-2} = t + t_1 + t_2 \cdots + t_{i-2} - (i-1)h;$$

therefore the greatest value of the sum $z + z_1 + z_2 \cdots + z_{i-2}$ being by assumption, equal to $p+e$, and the smallest being equal to p; the greatest value of $t + t_1 + t_2 \cdots + t_{i-2}$ will be $(i-1)h + p + e$, and the smallest will be $(i-1)h + p$; by making thus

$$(i-1)h + p + e = s$$

and

$$t + t_1 + t_2 \cdots + t_{i-2} = s - t_{i-1},$$

t_{i-1} will always be positive, and will be able to be extended from zero to e. This premised, if we apply in this case, formula (C); we will have $q = h + g$. Besides the law of facility of errors z being $a + bz + cz^2$, we will conclude from it the law of facility of t, by changing z into $t - h$; let

$$a' = a - bh + ch^2, \quad b' = b - 2ch;$$

we will have $a' + b't + ct^2$ for this law; this will be therefore the function $\phi(t)$. But as from $t = h + g$ to t infinity, the facility of the values of t is null by hypothesis; we will have

$$\phi'(t) + \phi(t) = 0,$$

that which gives

$$\phi'(t) = -(a' + b't + ct^2);$$

therefore if we make [268]

$$a'' = a' + b'(h+g) + c(h+g)^2,$$
$$b'' = b' + 2c(h+g),$$

we will have

$$\phi(t) + l^q \phi'(q+t) = a' + b't + ct^2 - l^{h+g}(a'' + b''t + ct^2);$$

and this equation will hold further, by changing t into t_1, t_2, etc.; since the law of facility of the errors is supposed the same for all the observations.

As for the variable t_{i-1}, we will observe that the probability of the equation

$$z + z_1 \cdots + z_{i-2} = \mu$$

being, whatever be μ, equal to the product of the probabilities of z, z_1, z_2, etc.; the probability of the equation

$$t + t_1 + t_2 \cdots + t_{i-2} = s - t_{i-1},$$

will be equal to the product of the probabilities of t, t_1, t_2, etc.; the law of probability of t_{i-1} is therefore constant and equal to unity; and, as this variable must be extended only from $t_{i-1} = 0$ to $t_{i-1} = e$; we will have

$$q_{i-1} = e, \quad \phi_{i-1}(t_{i-1}) = 1, \quad \phi'_{i-1}(t_{i-1}) + \phi_{i-1}(t_{i-1}) = 0;$$

and consequently

$$\phi'_{i-1}(t_{i-1}) = -1$$

that which gives

$$\phi_{i-1}(t_{i-1}) + l^{q_{i-1}}\phi'_{i-1}(q_{i-1} + t_{i-1}) = 1 - l^e;$$

formula (C) will become therefore

$$s^{i-1}[a' + b's + 2cs^2 - l^{h+g}(a'' + b''s + 2cs^2)]^{i-1}(1 - l^e). \qquad \text{(C}'\text{)}$$

Let

$$\begin{aligned} (a' + b's + 2cs^2)^{i-1} &= a^{(1)} + b^{(1)}s + c^{(1)}s^2 + f^{(1)}s^3 + \text{etc.}, \\ (a' + b's + 2cs^2)^{i-2}(a'' + b''s + 2cs^2) &= a^{(2)} + b^{(2)}s^2 + c^{(2)}s + \text{etc.}, \\ (a' + b's + 2cs^2)^{i-3}(a'' + b''s + 2cs^2) &= a^{(3)} + b^{(3)}s + c^{(3)}s^2 + \text{etc.}, \\ &\text{etc.} \end{aligned}$$

The preceding formula (C′) will give, by changing any term whatsoever such as $\lambda l^\mu s^a$, into [269]

$$\frac{\lambda(s-\mu)^a}{1.2.3\ldots a};$$

$$\frac{1}{1.2.3\ldots(i-1)}\left\{\begin{array}{l}
\quad a^{(1)}\left[s^{i-1}-(s-e)^{i-1}\right]\\
+\dfrac{b^{(1)}}{i}\left[s^{i}-(s-e)^{i}\right]\\
+\dfrac{c^{(1)}}{i(i+1)}\left[s^{i+1}-(s-e)^{i+1}\right]\\
+\text{etc.}\\
-(i-1)\left\{\begin{array}{l}
\quad a^{(2)}\left[(s-h-g)^{i-1}-(s-h-g-e)^{i-1}\right]\\
+\dfrac{b^{(2)}}{i}\left[(s-h-g)^{i}-(s-h-g-e)^{i}\right]\\
+\dfrac{c^{(2)}}{i(i+1)}\left[(s-h-g)^{i+1}-(s-h-g-e)^{i+1}\right]\\
+\text{etc.}
\end{array}\right.\\
+\dfrac{(i-1)(i-2)}{1.2}\left\{\begin{array}{l}
\quad a^{(3)}\left[(s-2h-2g)^{i-1}-(s-2h-2g-e)^{i-1}\right]\\
+\dfrac{b^{(3)}}{i}\left[(s-2h-2g)^{i}-(s-2h-2g-e)^{i}\right]\\
+\dfrac{c^{(3)}}{i(i+1)}\left[(s-2h-2g)^{i+1}-(s-2h-2g-e)^{i+1}\right]\\
+\text{etc.}
\end{array}\right.\\
-\text{etc.}
\end{array}\right.$$

It is necessary to reject from this expression, the terms in which the quantity raised under the sign of the powers, is negative.

Let us suppose now that z, z_1, z_2, etc., representing always the errors of $i-1$ observations, the law of facility, so much of the error z as of the negative error $-z$, be $\beta(h-z)$, and that h and $-h$ are the limits of these errors. Let us suppose moreover that this law is the same for all the observations, and let us seek the probability that the sum of the errors will be comprehended within the limits p and $p+e$.

If we make $z=t-h$, $z_1=t_1-h$, etc.; it is clear that t, t_1, etc. will be always positive, and will be able to be extended from zero to $2h$; but here the law of facility is discontinuous at two points. From $t=0$ to $t=h$, it is expressed by βt. From $t=h$ to $t=2h$, it is
expressed by $\beta(2h-t)$; finally, it is null from $t=2h$ to t infinity. We have therefore [270]

$$q=h,\quad q'=2h;$$

we have next

$$\begin{aligned}\phi(t)&=\beta t,\\ \phi'(t)+\phi(t)&=(2h-t)\beta,\\ \phi''(t)+\phi'(t)+\phi(t)&=0,\end{aligned}$$

that which gives

$$\phi'(t)=(2h-2t)\beta,\quad \phi''(t)=(t-2h)\beta,$$

thus we have in this case,

$$\phi(t) + l^q\phi'(q+t) + l^{q'}\phi''(q'+t) = \beta t(1-l^h)^2;$$

an equation which holds further by changing t into t_1, t_2, etc. Presently we have

$$z + z_1 + z_2 \cdots + z_{i-2} = t + t_1 + t_2 \cdots + t_{i-2} - (i-1)h;$$

therefore the sum of the errors z, z_1, etc. needing to be by hypothesis, contained within the limits p and $p+e$, the sum of the values of t, t_1, $\dots t_{i-2}$ will be comprehended within the limits $(i-1)h+p$ and $(i-1)h+p+e$; so that if we make

$$t + t_1 + t_2 \cdots + t_{i-2} = s - t_{i-1},$$

s being supposed equal to $(t-1)h+p+e$; t_{i-1} will be able to be extended from zero to e; and we will see, as in the preceding example, that its facility must be supposed equal to unity in this interval, and that it must be supposed null beyond this interval; thus we have $q_{i-1} = e$ and

$$\phi_{i-1}(t_{i-1}) + l^{q_{i-1}}\phi'_{i-1}(t_{i-1}) = 1 - l^e.$$

This premised, if we observe that $2\beta\int dz(h-z)$ being the probability that the error of an observation is comprehended within the limits $-h$ and $+h$, that which is certain, we have $\beta = \frac{1}{h^2}$; formula (C) will give for the expression of the sought probability,

[271]

$$\frac{1}{1.2.3\dots(2i-2)h^{2i-2}}\begin{cases} s^{2i-2} - (s-e)^{2i-2} \\ \qquad - (2i-2)\left[(s-h)^{2i-2} - (s-h-e)^{2i-2}\right] \\ \qquad + \dfrac{(2i-2)(2i-3)}{1.2}\left[(s-2h)^{2i-2} - (s-2h-e)^{2i-2}\right] \\ \qquad - \text{etc.}\end{cases};$$

by taking care to reject all the terms in which the quantity elevated to the power $2i-2$, is negative.

We are going to apply next this analysis to the following problem. If we imagine a number i of points ranked in a straight line, and on these points, ordinates of which the first is at least equal to the second, the latter at least equal to the third, and so forth; and that the sum of these i ordinates are constantly equal to s. By supposing s partitioned into an infinity of parts, we can satisfy the preceding conditions, in an infinity of ways. We propose to determine the value of each of the ordinates, a mean among all the values that it can receive.

Let z be the smallest ordinate, or the i^{th} ordinate; let $z+z_1$ be the $(i-1)^{\text{st}}$ ordinate; let $z+z_1+z_2$, the $(i-2)^{\text{nd}}$ ordinate, and so forth to the first ordinate which will be $z+z_1 \cdots + z_{i-1}$. The quantities z, z_1, z_2, etc. will be either nulls or positives, and their sum $iz + (i-1)z_1 + (i-2)z_2 \cdots + z_{i-1}$ will be, by the conditions of the problem, equal to s. Let

$$iz = t, \quad (i-1)z_1 = t_1, \quad (i-2)z_2 = t_2, \quad \dots, \quad z_{i-1} = t_{i-1};$$

we will have

$$t + t_1 + t_2 \cdots + t_{i-1} = s;$$

the variables t, t_1, t_2, etc. will be able to be extended to s. The r^{th} ordinate will be

$$\frac{t}{i} + \frac{t_1}{i-1} \cdots + \frac{t_{i-r}}{r}.$$

It is necessary to determine the sum of all the variations that this quantity is able to receive, and to divide it by the total number of these variations, in order have the mean ordinate. Formula (B) gives very easily this sum, by observing that here

$$\psi(t, t_1, t_2, \text{etc.}) = \frac{t}{i} + \frac{t_1}{i-1} \cdots + \frac{t_{i-r}}{r};$$

and we find it equal to [272]

$$\frac{s^i}{1.2.3\ldots i}\left(\frac{1}{i} + \frac{1}{i-1} \cdots + \frac{1}{r}\right).$$

By dividing this quantity by the total number of combinations, which can be only a function of i and of s, and which we will designate by N, we will have, for the mean value of the r^{th} ordinate,

$$\frac{s^i}{1.2.3\ldots iN}\left(\frac{1}{i} + \frac{1}{i-1} \cdots + \frac{1}{r}\right).$$

In order to determine N, we will observe that all the mean values must together equal s; that which gives

$$N = \frac{s^{i-1}}{1.2.3\ldots(i-1)};$$

the mean value of the r^{th} ordinate is therefore

$$\frac{s}{i}\left(\frac{1}{i} + \frac{1}{i-1} \cdots + \frac{1}{r}\right). \qquad (\epsilon)$$

Let us suppose that an observed effect has been able to be produced only by one of the i causes A, B, C, etc.; and that a person, after having estimated their respective probabilities, writes on a ballot, the letters which indicate these causes, in the order of the probabilities that he attributes to them, by writing first, the letter indicating the cause which seems to him most probable. It is clear that we will have by the preceding formula, the mean value of the probabilities that he is able to suppose to each of them, by observing that here the quantity s that we must apportion on each of the causes, is certitude or unity, since the person is assured that the effect must result from one of them. The mean value of the probability that he attributes to the cause that he has placed on his ballot at the r^{th} rank, is therefore

$$\frac{1}{i}\left(\frac{1}{i} + \frac{1}{i-1} \cdots + \frac{1}{r}\right).$$

Thence it follows that if a tribunal is summoned to decide on this object, and if each member expresses his opinion by a ballot similar to the preceding; then, by writing on each ballot, beside the letters which indicate the causes, the mean values which correspond to the rank [273] that they have on the ballot; by making next a sum of all the values which correspond to each cause, on the diverse ballots; the cause to which will correspond the greatest sum, will be that which the tribunal will judge most probable.

This rule is not at all applicable to the choice of the electoral assemblies, because the electors are not at all obliged, as the judges, to apportion one same sum taken for unity, on the diverse parts among which they must be determined: they can suppose to each candidate, all the nuances of merit comprehended between the null merit and the *maximum* of merit, which we will designate by a; the order of the names on each ballot, does only to indicate that the elector prefers the first to the second, the second to the third, etc. We will determine thus the numbers that it is necessary to write on the ballot, beside the names of the candidates.

Let t_1, t_2, t_3, $\dots t_i$ be the respective merits of the i candidates, in the opinion of the elector, t_1 being the merit that he supposes to the one of the candidates who he has set at the first rank, t_2 being the merit that he supposes at the second, and so forth. The integral $\int t_r dt_1 dt_2 \dots dt_i$ will express the sum of the merits that the elector can attribute to candidate r, provided that we integrate first with respect to t_i, from $t_i = 0$ to $t_i = t_{i-1}$; next with respect to t_{i-1}, from t_{i-1} to t_{i-2}, and so forth, to the integral relative to t_1, which we will take from t_1 null to $t_1 = a$. Because it is clear that then t_i never surpasses t_{i-1}, t_{i-1} never surpasses t_{i-2}, etc. By dividing the preceding integral by this here $\int dt_1 dt_2 \dots dt_i$ which expresses the total sum of the combinations in which the preceding condition is fulfilled, we will have the mean expression of the merit which the elector can attribute to the r^{th} candidate. In executing the integrations, we find $\frac{i-r+1}{i+1}a$ for this expression.

Thence it follows that we can write on the ballot of each elector i beside the first name, $i-1$ beside the second, $i-2$ beside the third, etc. By uniting next all the numbers relative to each candidate, on the diverse ballots; the one of the candidates who will have the greatest sum, must be presumed the candidate who, in the eyes of the electoral assembly, has the [274] greatest merit, and must consequently be chosen.

This mode of election would be without doubt the better, if some strange considerations in the merit did not influence at all often with respect to the choice of the electors, even the most honest, and did not determine them at all to place in the last ranks the most formidable candidates to the one who they prefer; that which gives a great advantage to the candidates of a mediocre merit. Also experience has caused abandoning it in the establishments which have adopted it.

Let us suppose that the errors of an observation are able to be extended within the limits $+a$ and $-a$; but that ignoring the law of probability of these errors, we subject it only to the condition to give to them a probability so much smaller, as they are greater; the probability of the positive errors being supposed the same as that of the corresponding negative errors, all things that it is natural to admit. Formula (ϵ) will give again the mean law of the errors. For this we will imagine the interval a partitioned into an infinite number

i of parts represented by dx, so that $i = \frac{a}{dx}$; we will make next $r = \frac{x}{dx}$; formula (ϵ) becomes thus

$$\frac{s\,dx}{a}\int\frac{dx}{x},$$

the integral being taken from $x = x$ to $x = a$. In the present question $s = \frac{1}{2}$; because the error must fall within the limits $-a$ and $+a$, the probability that it will fall within the limits zero and a is $\frac{1}{2}$; it is the quantity s that it is necessary to apportion on all the points of the interval a; formula (ϵ) becomes then

$$\frac{dx}{2a}\log\frac{a}{x}.$$

Thus the mean law of the probabilities of the positive errors x, or negatives $-x$, is

$$\frac{1}{2a}\log\frac{a}{x}.$$