BOOK II
CHAPTER I

PRINCIPES GÉNÉRAUX DE CETTE THÉORIE

Pierre Simon Laplace*

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GENERAL PRINCIPLES OF THIS THEORY

Definition of probability. Its measure is the ratio of the number of favorable cases, to the one of all possible cases.

The probability of an event composed of two simple events, is the product of the probability of one of these events, by the probability that, this event having arrived the other event will take place.

The probability of a future event, deduced from an observed event, is the quotient of the division of the probability of the event composed of these two events, and determined *à priori* by the probability of the observed event, determined similarly *à priori*.

If an observed event is able to result from n different causes, their probabilities are respectively, as the probabilities of the event, deduced from their existence, and the probability of each of them, is a fraction of which the numerator is the probability of the event under the hypothesis of the existence of the cause, and of which the denominator is the sum of the similar probabilities, relative to all the causes. If these diverse causes considered *à priori* are unequally probable, it is necessary, in place of the probability of the event, resulting from each cause, to employ the product of this probability by that of the cause itself.

The probability of a future event, is the sum of the products of the probability of each cause, deduced from the observed event, by the probability that this cause existing, the future event will take place.

On the influence that the unknown difference, which is able to exist among some simple events that we suppose equally possible, must have on the results of the calculus of probabilities. This difference increases the probability of the events composed of the repetition of one same event. . . . . Nº 1.

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§1. We have seen in the Introduction, that the probability of an event, is the ratio of the number of cases which are favorable to it, to the number of all possible cases; when nothing supports belief that one of these cases must arrive rather than the others, that which renders them for us, equally possible. The just estimation of these diverse cases, is one of the most delicate points of the analysis of chances. [179]

If all the cases are not equally possible, we will determine their respective possibilities; and then the probability of the event will be the sum of the probabilities of each favorable case. In fact, let us name p the probability of the first of these cases. This probability is relative to the subdivision of all the cases, into some others equally possible. Let N be the sum of all the cases thus subdivided, and n the sum of those cases which are favorable to the first case; we will have $p = \frac{n}{N}$. We will have similarly $p' = \frac{n'}{N}$, $p'' = \frac{n''}{N}$, etc.; by marking with one stroke, with two strokes, etc., the letters p and n, relatively to the second case, to the third, etc. Now, the probability of the event of which there is concern, is, by the same definition of probability, equal to

$$\frac{n + n' + n'' + \text{etc.}}{N};$$

it is therefore equal to $p + p' + p'' + \text{etc.}$

When an event is composed of two simple events, the one independent of the other; it is clear that the number of all possible cases, is the product of the two numbers which express all the possible cases relative to each simple event; because each of the cases relative to one of these events, can be combined with all the cases relative to the other event. By the same reason, the number of cases favorable to the compound event, is the product of the two numbers which express the cases favorable to each simple event; the probability of the compound event, is therefore then the product of the probabilities of each simple event. Thus the probability to bring forth twice consecutively, one ace with one die, is one thirty-sixth, when we suppose the faces of the die perfectly equal; because the number of all possible cases in two coups[1], is thirty-six, each case of the first cast being able to be combined with the six cases of the second; and among all these cases, one alone gives two aces consecutively. [180]

In general, if p, p', p'', etc. are the respective possibilities of any number of simple events independent of one another; the product $p.p'.p''$, etc. will be the probability of an event composed of these events.

If the simple events are linked among them, in a manner that the supposition of the arrival of the first, influences the probability of the arrival of the second; we will have the probability of the compound event, by determining, 1° the probability of the first event; 2° the probability that this event having arrived, the second will take place.

In order to demonstrate this principle in a general manner, let us name p the number of all the possible cases, and let us suppose that in this number, there are of them p' favorable

[1] *Translator's note*: *Two coups*, that is, two rolls of a die. The word *coup* can take on many meanings depending upon context. A statistical experiment often consists of a sequence of individual *coups* where they are typically called trials. Similarly in the case of a game among players, these are the sub-games or rounds which comprise it. In general, the word coup can be replaced by the word trial, round or attempt.

to the first event. Let us suppose next that in the number p', there are q favorable to the second event; it is clear that $\frac{q}{p}$ will be the probability of the compound event. But the probability of the first event is $\frac{p'}{p}$, the probability that this event having arrived, the second will take place, is $\frac{q}{p'}$; because then one of the cases p' needing to exist, we must consider only these cases. Now we have [181]

$$\frac{q}{p} = \frac{p'}{p} \cdot \frac{q}{p'};$$

that which is the translation into analysis, of the principle enunciated above.

In considering how a compound event, the observed event, join to a future event; the probability of this last event, deduced from the observed event, is evidently the probability that the observed event taking place, the future event will take place similarly; now, by the principle that we have just exposed, this probability multiplied by that of the observed event, determined *à priori*, or independently from that which is already arrived, is equal to that of the compound event, determined *à priori*; we have therefore this new principle, relative to the probability of future events, deduced from observed events.

The probability of a future event, deduced from an observed event, is the quotient of the division of the probability of the event composed of these two events, and determined *à priori*, by the probability of the observed event, determined similarly *à priori*.

Thence proceeds further this other principle relative to the probability of causes, deduced from observed events.

If an observed event can result from n different causes; their probabilities are respectively, as the probabilities of the event, deduced from their existence; and the probability of each of them, is a fraction of which the numerator is the probability of the event, under the hypothesis of the existence of the cause, and of which the denominator is the sum of the similar probabilities, relative to all the causes.

Let us consider, indeed, as a compound event, the observed event, resulting from one of these causes. The probability of this compound event, a probability that we will designate by E, will be, by that which precedes, equal to the product of the probability of the observed event, determined *à priori*, and that we will name F, by the probability that this event taking place, the cause of which there is concern, exists, a probability which is that of the cause, deduced from the observed event, and that we will name P.[2] will have therefore [182]

$$P = \frac{E}{F}.$$

The probability of the compound event, is the product of the probability of the cause, by the probability that this cause taking place, the event will arrive, a probability that we will designate by H.[3] All the causes being supposed *à priori*, equally possible, the probability

[2]*Translator's note*: In modern notation, $E = \Pr(\text{Event and Cause}_i)$, $F = \Pr(\text{Event})$, and $P = \Pr(\text{Cause}_i|\text{Event})$, $i = 1, 2, \ldots n$..

[3]*Translator's note*: $H = \Pr(\text{Event}|\text{Cause}_i)$, $i = 1, 2, \ldots n$.

of each of them is $\frac{1}{n}$; we have therefore

$$E = \frac{H}{n}.$$

The probability of the observed event, is the sum of all the E relative to each cause; by designating therefore by $S.\frac{H}{n}$, the sum of all the values of $\frac{H}{n}$, we will have

$$F = S.\frac{H}{n};$$

the equation $P = \frac{E}{F}$ will become therefore

$$P = \frac{H}{S.H};$$

that which is the principle enunciated above, when all the causes are *à priori* equally possible. If this is not, by naming p the probability *à priori* of the cause that we just considered; we will have $E = Hp$; and, by following the preceding reasoning, we will find

$$P = \frac{Hp}{S.Hp};$$

that which gives the probabilities of the diverse causes, when they are not all, equally possible *à priori*.

In order to apply the preceding principle to an example, let us suppose that an urn contains three balls of which each is able to be only white or black; that after having drawn [183] a ball, we restore it to the urn in order to proceed to a new drawing, and that after m drawings, we have brought forth only white balls. It is clear that we can make *à priori*, only four hypotheses; because the balls can be, either all white, or two whites and one black, or two blacks and one white, or finally all black. If we consider these hypotheses as so many causes of the observed event; the probabilities of the event, relative to these causes, will be

$$1, \quad \frac{2^m}{3^m}, \quad \frac{1}{3^m}, \quad 0.$$

The respective probabilities of these hypotheses, deduced from the observed event, will be therefore, by the third principle,

$$\frac{3^m}{3^m + 2^m + 1}, \quad \frac{2^m}{3^m + 2^m + 1}, \quad \frac{1}{3^m + 2^m + 1}, \quad 0.$$

We see, besides, that it is useless to have regard to the hypotheses which exclude the event, because the probability resulting from these hypotheses, being null, their omission changes not at all the expressions of the other probabilities.

If we wish to have the probability to bring forth only some black balls in the following m' drawings, we will determine *à priori*, the probabilities to bring forth first m white balls, next m' black balls. These probabilities are, relatively to the preceding hypotheses,

$$0, \quad \frac{2^m}{3^{m+m'}}, \quad \frac{2^{m'}}{3^{m+m'}}, \quad 0;$$

and as, *à priori*, the four hypotheses are equally possible, the probability of the compound event will be the quarter of the sum of the four preceding probabilities, or

$$\frac{1}{4}\,\frac{2^m + 2^{m'}}{3^{m+m'}}.$$

The probabilities of the observed event, determined *à priori*, under the preceding four hypotheses, being respectively

$$\frac{3^m}{3^m},\quad \frac{2^m}{3^m},\quad \frac{1}{3^m},\quad 0,$$ [184]

the quarter of their sum, or

$$\frac{1}{4}\left(\frac{3^m + 2^m + 1}{3^m}\right),$$

will be the probability of the observed event, determined *à priori*; by dividing therefore the probability of the compound event, by this probability, we will have by the second principle,

$$\frac{2^m + 2^{m'}}{3^{m'}(3^m + 2^m + 1)},$$

for the probability to bring forth m' black balls in the m' following drawings.

We are able further to determine this probability, by the following principle.

The probability of a future event is the sum of the products of the probability of each cause, deduced from the observed event, by the probability that this cause existing, the future event will take place.

Here the probabilities of each cause, deduced from the observed event, are, as we have seen,

$$\frac{3^m}{3^m + 2^m + 1},\quad \frac{2^m}{3^m + 2^m + 1},\quad \frac{1}{3^m + 2^m + 1},\quad 0;$$

the probabilities of the future event, relative to these causes, are respectively

$$0,\quad \frac{1}{3^{m'}},\quad \frac{2^{m'}}{3^{m'}},\quad 1;$$

the sum of their respective products, or

$$\frac{2^m + 2^{m'}}{3^{m'}(3^m + 2^m + 1)},$$

will be the probability of the future event, deduced from the observed event; that which is conformed to that which precedes.

If we suppose four balls in the urn, and that having brought forth a white ball at the first drawing, we seek the probability to bring forth only black balls in the following m' drawings; we will find, by the principles exposed above, this probability equal to [185]

$$\frac{3 + 2^{m'+1} + 3^{m'}}{10.4^{m'}}.$$

If the number of white balls equals the one of the blacks; the probability to bring forth only black balls in m' drawings, is $\frac{1}{2^{m'}}$. It surpasses the preceding, when m' is equal or less than 5; but it becomes inferior to it, when m' surpasses 5, although the white ball extracted first from the urn, indicates a superiority in the number of white balls. The explication of this paradox, holds in this that this indication excludes not at all the superiority of the number of black balls; it renders it only less probable; whereas the supposition of a perfect equality between the number of the whites and the one of the blacks, excludes this superiority; now this superiority, however small that its probability be, must render the probability to bring forth consecutively, m' black balls, greater than the case of equality of the colors, when m' is considerable.

The inequality which is able to exist among some things that we suppose perfectly similar, is able to have on the results of the calculus of probabilities, a sensible influence which merits a particular attention. Let us consider the game of *heads* and *tails*, and let us suppose that it is equally easy to bring forth *heads* as *tails*; then the probability to bring forth *heads* at the first coup, is $\frac{1}{2}$, and that to bring it forth two times consecutively, is $\frac{1}{4}$. But if there exists in the coin an inequality which makes one of the faces appear rather than the other, without us knowing the face that this inequality favors; the probability to bring forth *heads* at the first coup, will remain always $\frac{1}{2}$; because, in the ignorance in which one is, of the face that this inequality favors; as much as the probability of the simple event is increased, if this inequality is favorable to it, so much is it diminished, if this inequality is contrary to it. But the probability to bring forth *heads* two times consecutively, is increased, notwithstanding this ignorance; because this probability is equal to that to bring forth *heads* at the first coup, multiplied by the probability that having brought it forth at the first coup, [186]
we will bring it forth at the second; now its arrival at the first coup, is a motive to believe that the inequality of the coin, favors it; it increases therefore the probability to bring it forth at the second; thus the product of the two probabilities is increased by this inequality. In order to submit this object to calculation, let us suppose that the inequality of the coin increases by the quantity α, the probability of the simple event that it favors. If this event is *heads*, the probability will be $\frac{1}{2} + \alpha$, and the probability to bring it forth two times consecutively will be $\left(\frac{1}{2} + \alpha\right)^2$. If the event favored is *tails*, the probability of *heads* will be $\frac{1}{2} - \alpha$, and the probability to bring it forth two times consecutively will be $\left(\frac{1}{2} - \alpha\right)^2$. As we have in advance, no reason to believe that the inequality favors the one of the simple events rather than the other, it is clear that in order to have the probability of the compound event *heads-heads*, it is necessary to add the two preceding probabilities, and to take the half of their sum, that which gives $\frac{1}{4} + \alpha^2$ for this probability: it is also the probability of *tails-tails*. We will find by the same reasoning, that the probability of the compound event *heads-tails* or *tails-heads* is $\frac{1}{4} - \alpha^2$; consequently, it is less than that of the repetition of the same simple event.

The preceding considerations can be extended to any events whatsoever. p representing the probability of a simple event, and $1 - p$ that of the other event; if we designate by P, the probability of a result relative to these events, and if we suppose that p is really $p \pm \alpha$, α being an unknown quantity, as well as the sign which affects it; the probability P of the result will be

$$P + \frac{1}{1.2}\alpha^2.\frac{ddP}{dp^2} + \frac{1}{1.2.3.4}\alpha^4.\frac{d^4P}{dp^4} + \text{ etc.}$$

By making $P = p^n$, that is by supposing that the result relative to the events, be n times the repetition of the first; the probability P will become

$$p^n + \frac{n(n-1)}{1.2}\alpha^2 p^{n-2} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\alpha^4 p^{n-4} + \text{etc.}$$

Thus the unknown error that we are able to suppose in the probability of the simple events, [187] increases always the probability of the events composed of the repetition of the same event.

§2. The probability of events serves to determine the expectation[4] and the fear of the persons interested in their existence. The word *espérance* has diverse meanings; it expresses generally the advantage of the one who awaits any good, under a supposition that is only likely. In the theory of chances, this advantage is the product of the expected sum, by the probability to obtain it; it is the partial sum which must return, when we no longer wish to incur the risks of the event, by supposing that the apportionment of the entire sum is made proportional to the probabilities. This manner to apportion it, is alone equitable, when we set aside all strange circumstance, because with an equal degree of probability, we have an equal right with respect to the expected sum. We will name this advantage *mathematical expectation*, in order to distinguish it from moral expectation which depends, as it does, on the expected good and on the probability to obtain it, but which is regulated further on a thousand variable circumstances that it is nearly always impossible to define, and yet more, to subject to the calculus. These circumstances, it is true, making only to increase or to decrease the value of the expected good, we can consider the moral expectation itself as the product of this value, by the probability to obtain it; but we must then distinguish in the expected good, its relative value, from its absolute value: the latter is independent of the motives which make it desired, whereas the first increases with these motives.

We are not able to give a general rule in order to estimate this relative value; however it is natural to suppose the value relative to an infinitely small sum, in direct ratio to its absolute value, in inverse ratio of the total good of the interested person. In fact, it is clear that a franc has very little value for the one who possesses a great number of them, and that the most natural manner to estimate its relative value, is to suppose it in inverse ratio to this number.

Such are the general principals of the analysis of probabilities. We will now apply them [188] to the most delicate and the most difficult questions of this analysis. But in order to put in order in this matter, we will treat first the questions in which the probabilities of the simple events, are given; we will consider next those in which these possibilities are unknown, and must be determined by the observed events.

[4] espérance