Théorie Analytique des Probabilités

P.S. Laplace*

Selections†

FIRST CHAPTER
On the generating functions of one variable
pp. 7–11

2. Let y_x be any function whatever of x; if one forms the infinite series

$$y_0 + y_1 t + y_2 t^2 + y_3 t^3 + \cdots + y_x t^x + y_{x+1} t^{x+1} + \cdots + y_\infty t^\infty,$$

one can always imagine a function of t which, expanded according to the powers of t, gives this series: this function is that which I name *generating function* of y_x.

The generating function of any variable whatever y_x is therefore generally a function of t which, expanded according to the powers of t, has this variable for coefficient of t^x; and reciprocally, the corresponding variable of a generating function is the coefficient of t^x in the expansion of this function according to the powers of t; so that the exponent of the power of t indicates the rank that the variable y_x occupies in the series, which one can imagine prolonged indefinitely on the left, relatively to the negative powers of t.

It follows from these definitions that, u being the generating function of y_x, that of y_{x+r} is $\frac{u}{t^r}$; because it is clear that the coefficient of t^x in $\frac{u}{t^r}$ is equal to that of t^{x+r} in u; consequently it is equal to y_{x+r}.

The coefficient of t^x in $u\left(\frac{1}{t}-1\right)$ is therefore equal to $y_{x+1}-y_x$, or to the difference in the two consecutive quantities y_{x+1} and y_x, a difference which we will designate by $\triangle y_x$, $\triangle$ being the characteristic of the finite differences. We have therefore the generating function of the finite difference of a variable quantity, by multiplying by $\frac{1}{t}-1$ the generating function of the quantity itself. The generating function of the finite difference of $\triangle y_x$, a difference which we designate by $\triangle^2 y_x$ is thus $u\left(\frac{1}{t}-1\right)^2$; that of the finite difference of $\triangle^2 y_x$, or $\triangle^3 y_x$, is $u\left(\frac{1}{t}-1\right)^3$, whence we can generally conclude that the generating function of the finite difference $\triangle^i y_x$ is $u\left(\frac{1}{t}-1\right)^i$.

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. September 25, 2010

†These selections are those chosen by Ivo Schneider for inclusion in the *Entwicklung der Wahrscheinlichkeitsrechnung von den Anfängen bis* 1933. Einführung und Texte, Wissenschaftliche Buchgesellschaft 1988.

Similarly, the coefficient of t^x in the expansion of

$$u\left(a+\frac{b}{t}+\frac{c}{t^2}+\frac{e}{t^3}+\cdots+\frac{q}{t^n}\right)$$

is

$$ay_x+by_{x+1}+cy_{x+2}+ey_{x+3}+\cdots+qy_{x+n};$$

by naming therefore ∇y_x this quantity, its generating function will be

$$u\left(a+\frac{b}{t}+\frac{c}{t^2}+\frac{e}{t^3}+\cdots+\frac{q}{t^n}\right).$$

If we name $\nabla^2 y_x$ that which ∇y_x becomes when we change y_x into ∇y_x; if we name similarly $\nabla^3 y_x$ that which $\nabla^2 y_x$ becomes when we change ∇y_x into $\nabla^2 y_x$ and so forth, their corresponding generating functions will be

$$u\left(a+\frac{b}{t}+\frac{c}{t^2}+\cdots+\frac{q}{t^n}\right)^2,$$
$$u\left(a+\frac{b}{t}+\frac{c}{t^2}+\cdots+\frac{q}{t^n}\right)^3,$$
$$\cdots,$$

and generally the generating function of $\nabla^i y_x$ will be

$$u\left(a+\frac{b}{t}+\frac{c}{t^2}+\cdots+\frac{q}{t^n}\right)^i.$$

Whence it is easy to conclude generally that the generating function of $\triangle^i\nabla^s y_{x+r}$ is

$$\frac{u}{t^r}\left(a+\frac{b}{t}+\frac{c}{t^2}+\cdots+\frac{q}{t^n}\right)^s\left(\frac{1}{t}-1\right)^i.$$

We can generalize again these results, by supposing that ∇y_x represents any linear function whatever, finite or infinite, of $y_x,\ y_{x+1},\ y_{x+2},\ \ldots$; that $\nabla^2 y_x$ is that which ∇y_x becomes when we change y_x into ∇y_x; that $\nabla^3 y_x$ is that which $\nabla^2 y_x$ becomes when we change ∇y_x into $\nabla^2 y_x$, and so forth; u being the generating function of y_x, us^i will be the generating function of $\nabla^i y_x$, s being that which ∇y_x becomes, when we change y_x into unity, y_{x+1} into $\frac{1}{t}$, y_{x+2} into $\frac{1}{t^2}$, ... This is again true when i is a negative number or even fractional and incommensurable, by making however in this result some appropriate modifications.

Let us represent by Σ the characteristic of the finite integrals, and name z the generating function of $\Sigma^i y_x$, u being the generating function of y_x; $z\left(\frac{1}{t}-1\right)^i$ will be, by that which precedes, the generating function of y_x. But this function must, by having regard only to the positive powers of t, be reduced to u, which contains only the positive powers of t, if we extend the multiple integral $\Sigma_i y_x$ only to the positive values of x; we will have therefore then

$$z\left(\frac{1}{t}-1\right)^i=u+\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t^3}+\cdots+\frac{F}{t^i},$$

whence we draw

$$z = \frac{ut^i + At^{i-1} + Bt^{i-2} + Ct^{i-3} + \cdots + F}{(1-t)^i},$$

A, B, $C, \ldots,$ F being some arbitrary constants which correspond to the i arbitrary constants which the i successive integrations introduce from $\Sigma^i y_x$.

By setting aside these constants, the generating function of $\Sigma^i y_x$ is $u\left(\frac{1}{t}-1\right)^{-i}$; so that we obtain this generating function by changing i into $-i$ in the generating function of $\triangle^i y_x$; $\triangle^{-i} y_x$ is therefore then equal to $\Sigma^i y_x$, that is that the negative differences change themselves into integrals. But, if we have regard to the arbitrary constants, it is necessary, in passing from the positive powers of $\frac{1}{t}-1$ to its negative powers, to increase u of the series $\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t^3}+\cdots$, prolonged to where the number of its terms is equal to the exponent of these powers. We can apply some similar considerations to the generating function of $\nabla^i y_x$.

We see by that which precedes in what manner the generating functions are formed from the law of the corresponding variables. We see now how the variables are deduced from their generating functions; s being any function whatever of $\frac{1}{t}$, if we expand s^i according to the powers of $\frac{1}{t}$, and if we designate by $\frac{k}{t^n}$ any term whatever of this expansion, the coefficient of t^x in $\frac{ku}{t^n}$ will be ky_{x+n}; we will have therefore the coefficient of t^x in us^i, a coefficient which we have designated previously by $\nabla^i y_x$: 1° by substituting into s, y_x in place of $\frac{1}{t}$; 2° by expanding that which becomes then s^i according to the powers of y_x, and by transporting to the index x the exponent of the power of y_x, that is by writing y_{x+1} in place of $(y_x)^1$, y_{x+2} in place of $(y_x)^2$, etc., and by multiplying the terms independent of y_x, and which can be counted to have $(y_x)^0$ for factor, by y_x. When the characteristic ∇ is changed to $\triangle$, s is, by that which precedes, equal to $\frac{1}{t}-1$; we have therefore then

$$\nabla^i y_x = y_{x+i} - iy_{x+i-1} + \frac{i(i-1)}{1.2} y_{x+i-2} - \cdots$$

If, instead of expanding s^i according to the powers of $\frac{1}{t}$, we expand according to the powers of $\frac{1}{t}-1$, and if we designate by $k\left(\frac{1}{t}-1\right)^n$ any term whatever of this expansion, the coefficient of t^x in $ku\left(\frac{1}{t}-1\right)^n$ will be $k\nabla^n y_x$; we will have therefore $\nabla^i y_x$: 1° by substituting, into s, $\triangle y_x$ in place of $\frac{1}{t}-1$, or, that which reverts to the same, $1+\triangle y_x$ in place of $\frac{1}{t}$; 2° by expanding that which becomes then s^i according to the powers of $\triangle y_x$, and by applying to the characteristic $\triangle$ the exponents of the powers of $\triangle y_x$, that is by writing $\triangle y_x$ in place of $(\triangle y_x)^1$, $\triangle^2 y_x$ in place of $(\triangle y_x)^2$, etc., and by multiplying by $(\triangle y_x)^0$, or, that which is the same thing, by y_x the terms independent of $\triangle y_x$.

Generally, if we consider s as a function of r, r being a function of $\frac{1}{t}$, such that the coefficient of t_x in ur is $\Box y_x$, we will have $\nabla^i y_x$, by substituting, into s, $\Box y_x$, in place of r; by expanding next s^i according to the powers of $\Box y_x$ and by applying to the characteristic $\Box$ the exponents of $\Box y_x$, that is by writing $\Box y_x$ in place of $(\Box y_x)$, $\Box^2 y_x$ in place of $(\Box y_x)^2$, etc., and by multiplying by y_x the terms independent of $\Box y_x$.

The expansion of $\nabla^i y_x$ by a series ordered according to the successive variables $\Box y_x$, $\Box^2 y_x$, etc. is reduced therefore to the formation of the generating function of

y_x, in the expansion of this function according to the powers of a given function; finally, to return to the generating function thus expanded, to the corresponding variable coefficients, the exponents of the powers of the expansion of the generating function becoming those of the characteristics of these coefficients. We see thus the analogy of the powers with the differences, or with each other combination of the consecutive variable coefficients. The passage of these coefficients to their generating functions, and the return of these expanded functions to the coefficients constitutes the *Calculus of generating functions*. The following applications make known the spirit and the advantages of them.

FIRST CHAPTER

Theorems on the expansion of functions and of their differences into series

pp. 37–41

10. By applying to some particular functions the general principles exposed in No. 1, we will have an infinity of theorems on the expansion of functions into series. We are going to present here the most remarkable.

We have generally

$$u\left(\frac{1}{t^i}-1\right)^n = u\left[\left(1+\frac{1}{t}-1\right)^i-1\right]^n.$$

Now it is clear that the coefficient of t^x in the first member of this equation is the nth difference of y_x, x varying with i; because this coefficient in $u\left(\frac{1}{t^i}-1\right)$ is $y_{x+i}-y_x$ or ${}'\triangle y_x$, by designating by the character ${}'\triangle$ the finite differences, when x varies with the quantity i; whence it is easy to conclude that this same coefficient, in the expansion of $u\left(\frac{1}{t^i}-1\right)^n$, is ${}'\triangle^n y_x$. Moreover, if we expand $u\left[\left(1+\frac{1}{t}-1\right)^i-1\right]^n$ according to the powers of $\frac{1}{t}-1$, the coefficients of t^x in the expansions of $u\left(\frac{1}{t^i}-1\right)$, $u\left(\frac{1}{t^i}-1\right)^2$,... are, by No. 2, $\triangle y_x$, $\triangle^2 y_x$, ...; so that this coefficient, in $u\left[\left(1+\frac{1}{t}-1\right)^i-1\right]^n$, is $[(1+\triangle y_x)^i-1]^n$, provided that in the expansion of this quantity we apply to the characteristic $\triangle$ the exponents of the powers of $\triangle y_x$, and that thus, instead of any power whatever $(\triangle y_x)^r$, we write $\triangle^r y_x$; we will have therefore with this condition

$$(1)\qquad {}'\triangle^n y_x = [(1+\triangle y_x)^i-1]^n.$$

If we designate by the characteristic ${}'\Sigma$ the finite integral, when x varies with i, ${}'\Sigma^n y_x$ will be, by No. 2, the coefficient of t^x in the expansion of the function $u\left(\frac{1}{t^i}-1\right)^{-n}$, by setting aside some arbitrary constants that the integration introduced; now we have

$$u\left(\frac{1}{t^i}-1\right)^{-n} = u\left[\left(1+\frac{1}{t}-1\right)^i-1\right]^{-n};$$

moreover, the coefficient of t^x in $u\left(\frac{1}{t^i}-1\right)^{-r}$ is $\Sigma^r y_x$, by setting aside some arbitrary constants; this coefficient in $u\left(\frac{1}{t^i}-1\right)^r$ is $\Sigma^r y_x$; we will have therefore

$$(2)\qquad {}'\Sigma^n y_x = [(1+\triangle y_x)^i-1]^{-n},$$

provided that, in the expansion of the second member of this equation, we apply to the characteristic $\triangle$ the exponents of the powers of $\triangle y_x$, that we change the negative differences into integrals and that we substitute y_x in place of $\triangle^0 y_x$, and as this expansion contains the integral $\Sigma^n y_x$, which can be counted to contain n arbitrary constants, the equation (2) is again true, by having regard to the arbitrary constants.

We can observe that this equation is deduced from equation (1), by making in that one n negative and by changing the negative differences to integrals, that is, by writing

${}'\Sigma^n y_x$ instead of ${}'\triangle^n y_x$ in the first member; and generally, in the expansion of the second member, $\Sigma^r y_x$ instead of $\triangle^{-r} y_x$.

Equations (1) and (2) would hold equally, if x, instead of varying from unity in $\triangle y_x$, would vary from any quantity whatever ϖ, provided that the variation of x in ${}'\triangle y_x$ be equal to $i\varpi$. Indeed, it is clear that, if in y_x we make $x = \frac{x'}{\varpi}$, x' will vary from ϖ when x will vary from unity; $\triangle y_x$ will be changed into $\triangle y_{x'}$, the variation of x' being ϖ, and ${}'\triangle y_x$ will be changed into ${}'\triangle y_{x'}$, the variation of x' being $i\varpi$. Now if, after having substituted these quantities into the equations (1) and (2), we suppose ϖ infinitely small and equal to dx', $\triangle y_{x'}$ will be changed into the infinitely small difference $dy_{x'}$. If moreover we make i infinity and $idx' = \alpha$, α being a finite quantity, the variation of x' in ${}'\triangle y_{x'}$ will be α; we will have therefore

$$(q) \qquad \begin{cases} {}'\triangle^n y_{x'} = [(1 + dy_{x'})^i - 1]^n \\ {}'\Sigma^n y_{x'} = \frac{1}{[(1+dy_{x'})^i - 1]^n}. \end{cases}$$

Now we have

$$\log(1 + dy_{x'})^i = i \log(1 + dy_{x'}) = idy_{x'} = \alpha \frac{dy_{x'}}{dx'},$$

this which gives

$$(1 + dy_{x'})^i = c^{\alpha \frac{dy_{x'}}{dx'}},$$

c being the number for which the hyperbolic logarithm is unity; we have therefore

$$(3) \qquad {}'\triangle^n y_{x'} = \left(c^{\alpha \frac{dy_{x'}}{dx'}} - 1\right)^n,$$

$$(4) \qquad {}'\Sigma^n y_{x'} = \frac{1}{\left(c^{\alpha \frac{dy_{x'}}{dx'}} - 1\right)^n},$$

by taking care to apply to the characteristic d the exponents of the powers of $dy_{x'}$, to change the negative differences into integrals and the quantity $d^0 y_{x'}$ to y_x.

We can give to equation (3) this singular form which will be useful to us in the following,

$${}'\triangle^n y_{x'} = \left(c^{\frac{\alpha}{2} \frac{dy_{x'} + \frac{n\alpha}{2}}{dx'}} - c^{-\frac{\alpha}{2} \frac{dy_{x'} + \frac{n\alpha}{2}}{dx'}}\right)^n.$$

Indeed, it gives

$${}'\triangle^n y_{x'} = c^{\frac{n\alpha}{2} \frac{dy_{x'}}{dx'}} \left(c^{\frac{\alpha}{2} \frac{dy_{x'}}{dx'}} - c^{-\frac{\alpha}{2} \frac{dy_{x'}}{dx'}}\right)^n.$$

We will consider any term whatever in the expansion of

$$\left(c^{\frac{\alpha}{2} \frac{dy_{x'}}{dx'}} - c^{-\frac{\alpha}{2} \frac{dy_{x'}}{dx'}}\right)^n,$$

such as $k\left(\frac{dy_{x'}}{dx'}\right)$. By multiplying it by $c^{\frac{n\alpha}{2} \frac{dy_{x'}}{dx'}}$, and expanding this last quantity, we will have

$$k \frac{d^r}{dx'^r} \left[r_{x'} + \frac{n\alpha}{2} \frac{dy_{x'}}{dx'} + \left(\frac{n\alpha}{2}\right)^2 \frac{d^2 y_{x'}}{1.2.dx'^2} + \cdots \right];$$

this quantity is equal to $k\frac{d^r y_{x'+\frac{n\alpha}{2}}}{dx'^r}$, whence it is easy to conclude

$$c^{\frac{n\alpha}{2}\frac{dy_{x'}}{dx'}}\left(c^{\frac{\alpha}{2}\frac{dy_{x'}}{dx'}} - c^{-\frac{\alpha}{2}\frac{dy_{x'}}{dx'}}\right)^n = \left(c^{\frac{\alpha}{2}\frac{dy_{x'+\frac{n\alpha}{2}}}{dx'}} - c^{-\frac{\alpha}{2}\frac{dy_{x'+\frac{n\alpha}{2}}}{dx'}}\right)^n =' \triangle^n y_{x'}.$$

If in the equations (1) and (2) we suppose again i infinitely small and equal to dx, we will have

$$'\triangle^n y_x = d^n y_x, \qquad '\Sigma^n y_x = \frac{1}{dx^n}\int^n y_x dx^n;$$

we will have moreover

$$(1+\triangle y_x)^i = c^{dx\log(1+\triangle y_x)} = 1 + dx\log(1+\triangle y_x);$$

the equations (1) and (2) will become thus

$$\frac{d^n y_x}{dx^n} = [\log(1+\triangle y_x)]^n, \tag{5}$$

$$\int^n y_x dx^n = \frac{1}{[\log(1+\triangle y_x)]^n}. \tag{6}$$

We can observe here a singular analogy between the positive powers and the differences and between the negative powers and the integrals. The equation

$$'\triangle y_x = c^{\alpha\frac{dy_x}{dx}} - 1 \tag{o}$$

is the translation of the theorem known to Taylor, when, in the expansion of its second member according to the powers of $\frac{dy_x}{dx}$, we apply to the characteristic d the exponents of these powers. By raising the two members of this equation to the power n, and applying to the characteristics $'\triangle$ and d the exponents of the powers of $'\triangle y_x$ and of dy_x, we will have equation (3), whence results equation (4) by changing the negative differences to integrals.

The preceding equation gives

$$c^{\alpha\frac{dy_x}{dx}} = 1 +' \triangle y_x.$$

By taking the logarithms of each member, we will have

$$\alpha\frac{dy_x}{dx} = 1 +' \triangle y_x. \tag{r}$$

Supposing next $\alpha = 1$, this which changes $'\triangle y_x$ into $\triangle y_x$, and raising the two members of this equation to the power n, we will have equation (5), provided that we apply the exponents of the powers to the characteristics. We will have equation (6) by making n negative and changing the negative powers into integrals.

If, in the preceding equation (r), one changes α into i, one will have

$$\frac{dy_x}{dx} = \log(1 +' \triangle y_x)^{\frac{1}{i}},$$

and if one supposes $\alpha = 1$, one will have

$$\frac{dy_x}{dx} = \log(1 + \triangle y_x).$$

The comparison of these two values of $\frac{y_x}{dx}$ gives

$$\log(1 + \triangle y_x) = \log(1 + {}'\triangle y_x)^{\frac{1}{i}},$$

whence one draws

$${}'\triangle y_x = (1 + \triangle y_x) - 1.$$

By raising each member to the power n and applying the exponents of the powers to the characteristics, one will have equation (1), whence results equation (2), by changing the negative differences into integrals. Equations (1), (2), (3), (4), (5) and (6) result therefore from the theorem of Taylor, put under the form of equation (o), by transforming this equation according to the rules of Analysis, provided that in the results one applies to the characteristics the exponents of the powers, that one changes the negative differences into integrals and that one substitutes the variable itself y_x instead of its zero differences.

This analogy of the positive powers with the differences and of the negative powers with the integrals becomes evident by the theory of generating functions. It holds, as one has seen, to this that the products of the function u, generator of y_x, by the powers $\frac{1}{t^i} - 1$ are the generating functions of the successive finite differences of y_x, x varying by any quantity i, while the quotients of u, divided by these same powers, are the generating functions of the integrals of y_x.

By considering, instead of the factor $\frac{1}{t^i - 1}$ and of its powers, the powers of any rational and entire function of $\frac{1}{t}$, one is able to conclude from the theorems analogous to the preceding, on the successive *derived* of the functions. I name *derived* of a function y_x each quantity which derives from it, such as $ay_x + by_{x+1} + ey_{x+2} + \ldots$ By regarding next this derived function as a new function that I designate by y'_x the quantity $ay'_x + b'_{x+1} + ey'_{x+2} + \ldots$ will be a second derived from the function y_x and thus consecutively. When the function $ay_x + by_{x+1} + \ldots$ becomes $-y_x + y_{x+1}$, the derived becomes a finite difference.

Now one has

$$(q) \qquad \left\{ \begin{array}{l} u\left(a + \frac{b}{t} + \frac{e}{t^2} + \frac{h}{t^3} + \cdots\right)^n \\ \quad = u\left[a + b\left(1 + \frac{1}{t^{dx}} - 1\right)^{\frac{1}{dx}} + e\left(1 + \frac{1}{t^{dx}} - 1\right)^{\frac{2}{dx}} + \cdots\right]^n; \end{array} \right.$$

one has next generally, by n° 2, by designating by ∇y_x the quantity $ay_x + by_{x+1} + ey_{x+2} + \ldots$, $\nabla^n y_x$ for the coefficient of the generating function of the first member of this equation; moreover one has

$$u\left(1 + \frac{1}{t^{dx}} - 1\right)^{\frac{r}{dx}} = u\left[1 + \frac{r}{dx}\left(\frac{1}{t^{dx}} - 1\right) + \frac{r^2}{1.2.dx^2}\left(\frac{1}{t^{dx}} - 1\right)^2 + \cdots\right].$$

The second member of this equation is the generating function of

$$y_x + r\frac{ry_x}{dx} + \frac{r^2}{1.2}\frac{d^2y_x}{dx^2} + \cdots$$

or of $c^{r\frac{dy_x}{dx}}$, by applying to the characteristic d the exponents of powers of $\frac{dy_x}{dx}$, and writing y_x instead of $\left(\frac{dy_x}{dx}\right)^0$. Thence one concludes that, under the same conditions, the second member of equation (q) is the generating function of

$$\left(a + bc^{\frac{dy_x}{dx}} + ec^{\frac{2dy_x}{dx}} + hc^{\frac{3dy_x}{dx}} + \cdots\right)^n,$$

and that thus this equation gives, by passing again from the generating functions to the coefficients,

$$\nabla^n y_x = \left[a + bc^{\frac{dy_x}{dx}} + ec^{\frac{2dy_x}{dx}} + hc^{\frac{3dy_x}{dx}} + \cdots\right]^n. \tag{7}$$

One is able thus to obtain an infinity of similar results. We ourselves will be limited to the following, which will be useful to us in the following: $u\left(\frac{1}{\sqrt{t}} - \sqrt{t}\right)^n$ is the generating function of

$$y_{x+\frac{n}{2}} - ny_{x+\frac{n}{2}-1} + \frac{n(n-1)}{1.2}y_{x+\frac{n}{2}-2} - \cdots,$$

or of $\triangle^n y_{x+\frac{n}{2}}$. Moreover, one has

$$\left(\frac{1}{\sqrt{t}} - \sqrt{t}\right)^n = u\left[\left(1 + \frac{1}{t^{dx}} - 1\right)^{\frac{1}{2dx}} - \left(1 + \frac{1}{t^{dx}} - 1\right)^{-\frac{1}{2dx}}\right]^n,$$

whence one draws, by passing again by the preceding analysis from the generating functions to the coefficients

$$\triangle^n y_{x-\frac{n}{2}} = \left(c^{\frac{dy_x}{2dx}} - c^{-\frac{dy_x}{2dx}}\right)^n.$$

BOOK II CHAPTER III

On the laws of probability which result from the indefinite multiplication of events

pp. 280–284

16. In measure as the events are multiplied, their respective probabilities are developed more and more; their mean results and the profits or the losses which depend on them converge toward some limits which they bring together with the probabilities always increasing. The determination of these increases and of these limits is one of the most interesting and most delicate parts of the analysis of chances.

We will consider first the manner in which the possibilities of two simple events, of which one alone must arrive at each trial, is developed when one multiplies the number of trials. It is clear that the event of which the facility is greatest must probably arrive more often in a given number of trials, and one is carried naturally to think that by repeating the trials a great number of times, each of these events will arrive proportionally to its facility, that one will be able thus to discover by experience. We will demonstrate analytically this important theorem.

One has seen in n$^{\text{o}}$ 6 that, if p and $1-p$ are the respective probabilities of two events a and b, the probability that in $x+x'$ trials the event a will arrive x times and the event b, x' times, is equal to

$$\frac{1.2.3\ldots(x+x')}{1.2.3\ldots x.1.2.3\ldots x'}p^x(1-p)^{x'};$$

this is the $(x'+1)^{\text{st}}$ term of the binomial $[p+(1-p)]^{x+x'}$. We will consider the greatest of these terms that we will designate by k. The anterior term will be $\frac{kp}{1-p}\frac{x'}{x+1}$, and the following term will be $k\frac{1-p}{p}\frac{x}{x'+1}$. In order that k be the greatest term, it is necessary that one has

$$\frac{x}{x'+1}<\frac{p}{1-p}<\frac{x+1}{x'};$$

it is easy to conclude from it that, if one makes $x+x'=n$, one will have

$$(n+1)p-1<x<(n+1)p;$$

thus x is the greatest whole number contained within $(n+1)p$; by making therefore

$$x=(n+1)p-s,$$

this which gives

$$p=\frac{x+s}{n+1},\quad 1-p=\frac{x'+1-s}{n+1},\quad \frac{p}{1-p}=\frac{x+s}{x'+1-s},$$

s will be less than unity. If x and x' are very great numbers, one will have, very nearly,

$$\frac{p}{1-p}=\frac{x}{x'},$$

that is to say that the exponents of p and of $1-p$ in the greatest term of the binomial are quite nearly in the ratio of these quantities; so that, in all the combinations which are able to take place in a very great number n of trials, the most probable is that in which each event is repeated proportionally to its probability.

The l^{th} term, after the greatest, is

$$\frac{1.2.3\ldots n}{1.2.3\ldots(x-l).1.2.3\ldots(x'+l)}p^{x-l}(1-p)^{x'+l}.$$

One has, by n° 33 of Book I,

$$1.2.3\ldots n = n^{n+\frac{1}{2}}c^{-n}\sqrt{2\pi}\left(1+\frac{1}{12n}+\cdots\right),$$

this which gives

$$\frac{1}{1.2.3\ldots(x-l)} = (x-l)^{l-x-\frac{1}{2}}\frac{c^{x-l}}{\sqrt{2\pi}}\left[1-\frac{1}{12(x-l)}-\cdots\right],$$
$$\frac{1}{1.2.3\ldots(x'+l)} = (x'+l)^{-x'-l-\frac{1}{2}}\frac{c^{x'+l}}{\sqrt{2\pi}}\left[1-\frac{1}{12(x'+l)}-\cdots\right].$$

We develop the term $(x-l)^{l-x-\frac{1}{2}}$. Its hyperbolic logarithm is

$$\left(l-x-\frac{1}{2}\right)\left[\log x+\log\left(1-\frac{l}{x}\right)\right];$$

now one has

$$\log\left(1-\frac{l}{x}\right) = -\frac{l}{x}-\frac{l^2}{2x^2}-\frac{l^3}{3x^3}-\frac{l^4}{4x^4}-\cdots;$$

we will neglect the quantities of order $\frac{1}{n}$, and we will suppose that l^2 does not surpass at all the order n; then one will be able to neglect the terms of order $\frac{l^4}{x^3}$, because x and x' are of order n. One will have thus

$$\left(l-x-\frac{1}{2}\right)\left[\log x+\log\left(1-\frac{l}{x}\right)\right]$$
$$= \left(l-x-\frac{1}{2}\right)\log x + l + \frac{l}{2x} - \frac{l^2}{2x} - \frac{l^3}{6x^2},$$

this which gives, by passing again from the logarithms to the numbers,

$$(x-l)^{l-x-\frac{1}{2}} = c^{l-\frac{l^2}{2x}}x^{l-x-\frac{1}{2}}\left(1+\frac{l}{2x}-\frac{l^3}{6x^2}\right);$$

one will have similarly

$$(x'+l)^{-l-x'-\frac{1}{2}} = c^{-l-\frac{l^2}{2x'}}x'^{-l-x'-\frac{1}{2}}\left(1+\frac{l}{2x'}-\frac{l^3}{6x'^2}\right).$$

One has next, by that which precedes, $p=\frac{x+s}{n+1}$, s being less than unity; by making therefore $p=\frac{x-z}{n}$, z will be contained within the limits $\frac{x}{n+1}$ and $-\frac{n-x}{n+1}$, and consequently it will be, setting aside the sign, below unity. The value of p gives $1-p=\frac{x'+z}{n}$;

one will have, by the preceding analysis,

$$p^{x-l}(1-p)^{x'+l} = \frac{x^{x-l}x'^{x'+l}}{n^n}\left(1+\frac{nzl}{xx'}\right);$$

thence one draws

$$\begin{aligned}&\frac{1.2.3\ldots n}{1.2.3\ldots(x-l).1.2.3\ldots(x'+l)}p^{x-l}(1-p)^{x'+l}\\ &\quad= \frac{\sqrt{n}c^{-\frac{nl^2}{2xx'}}}{\sqrt{\pi}\sqrt{2xx'}}\left[1+\frac{nzl}{xx'}+\frac{l(x'-x)}{2xx'}-\frac{l^3}{6x^2}+\frac{l^3}{6x'^2}\right].\end{aligned}$$

One will have the term anterior to the greatest term and which is extended from it at the distance l, by making l negative in this equation; by reuniting next these two terms, their sum will be

$$\frac{2\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}}c^{-\frac{nl^2}{2xx'}}.$$

The finite integral

$$\sum\frac{2\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}}c^{-\frac{nl^2}{2xx'}},$$

taken from $l = 0$ inclusively, will express therefore the sum of all the terms of the binomial $[p+(1-p)]^n$, contained between the two terms, of which the one has p^{x+l} for factor, and the other has p^{x-l} for factor, and which are thus equidistant from the greatest term; but it is necessary to subtract from this sum the greatest term which is evidently contained twice.

Now, in order to have this finite integral, we will observe that one has, by n° 10 of Book I, y being function of l,

$$\sum y = \frac{1}{c^{\frac{dy}{dl}}-1} = \left(\frac{dy}{dl}\right)^{-1} - \frac{1}{2}\left(\frac{dy}{dl}\right)^0 + \frac{1}{12}\frac{dy}{dl} + \cdots,$$

whence one draws, by the preceding number,

$$\sum y = \int y\,dl - \frac{1}{2}y + \frac{1}{12}\frac{dy}{dl} + \cdots + \text{ const.};$$

y being here equal to $\frac{2\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}}c^{-\frac{nl^2}{2xx'}}$, the successive differentials of y acquire for factor $\frac{nl}{2xx'}$, and its powers. Thus, l not being supposed to be able to be more than order $\sqrt{n}$, this factor is of order $\frac{1}{\sqrt{n}}$, and consequently its differentials, divided by the respective powers of dl, decrease more and more; by neglecting therefore, as one has done previously, the terms of order $\frac{1}{n}$, one will have, by starting with l the two finite and infinitely small integrals, and designating by Y the greatest term of the binomial,

$$\sum y = \int ydl - \frac{1}{2}y + \frac{1}{2}Y.$$

The sum of all the terms of the binomial $[p + (1 - p)]^n$ contained between the two terms equidistant from the greatest term by the number l being equal to $\sum y - \frac{1}{2}Y$, it will be

$$\int ydl - \frac{1}{2}y,$$

and if one adds there the sum of these extreme terms, one will have, for the sum of all these terms,

$$\int ydl + \frac{1}{2}y.$$

If one makes

$$t = \frac{l\sqrt{n}}{\sqrt{2xx'}},$$

this sum becomes

$$\text{(o)} \qquad \frac{2}{\sqrt{\pi}} \int dt\, c^{-t^2} + \frac{\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}} c^{-t^2}.$$

The terms that one has neglected being of the order $\frac{1}{n}$, this expression is so much more exact as n is greater; it is rigorous when n is infinity. It would be easy, by the preceding analysis, to have regard to the terms of order $\frac{1}{n}$ and of the superior orders.

One has, by that which precedes, $x = np + z$, z being a number smaller than unity; one has therefore

$$\frac{x + l}{n} - p = \frac{l + z}{n} = \frac{t\sqrt{2xx'}}{n\sqrt{n}} + \frac{z}{n};$$

thus formula (o) expresses the probability that the difference between the ratio of the number of times that the event a must arrive to the total number of trials, and the facility p of this event, is contained within the limits

$$(l) \qquad \pm \frac{t\sqrt{2xx'}}{n\sqrt{n}} + \frac{z}{n}.$$

$\sqrt{2xx'}$ being equal to

$$n\sqrt{2p(1 - p) + \frac{2z}{n}(1 - 2p) - \frac{2z^2}{n^2}},$$

one sees that the interval contained between the preceding limits is of order $\frac{1}{\sqrt{n}}$.

If the limit of t, that we will designate by T, is supposed invariable, the probability determined by the function (o) remains very nearly the same; but the interval comprehended between the limits (l) diminishes without ceasing in measure as the trials are repeated, and it becomes null, when their number is infinite.

This interval being supposed invariable, when the events are multiplied, T increases without ceasing, and quite nearly as the square root of the number of trials. But, when

T is considerable, formula (o) becomes, by n° 27 of Book I,

$$1-\frac{c^{-T^2}}{2T\sqrt{\pi}}\cfrac{1}{1+\cfrac{q}{1+\cfrac{2q}{1+\cfrac{3q}{1+\cdots}}}}+\frac{c^{-T^2}}{\sqrt{2\pi n\left[p(1-p)+\frac{z}{n}(1-2p)-\frac{z^2}{n^2}\right]}},$$

q being equal to $\frac{1}{2T^2}$. When one makes T increase, c^{-T^2} diminishes with an extreme rapidity, and the preceding probability approaches rapidly to unity, to which it becomes equal, when the number of trials is infinite.

There are here two sorts of approximations: the one of them is relative to the limits taken on both sides of the facility of the event a; the other approximation is related to the probability that the ratio of the arrivals of this event to the total number of trials will be contained within these limits. The indefinite repetition of the trials increases more and more this probability, the limits remaining the same; it narrows more and more the interval of these limits, the probability remaining the same. Into infinity, this interval becomes null, and the probability is changed into certitude.

The preceding analysis reunites to the advantage to demonstrate this theorem the one to assign the probability that, in a great number n of trials, the ratio of the arrivals of each event will be comprehended within some given limits. We suppose, for example, that the facilities of the births of boys and of girls are in the ratio of 18 to 17, and that there are born in one year 14000 infants; one demands the probability that the number of boys will not surpass 7363, and will not be less than 7037.

In this case, one has

$$p=\frac{18}{35},\quad x=7200,\quad x'=6800,\quad n=14000,\quad l=163;$$

formula (o) gives quite nearly 0.994303 for the sought probability.

If one knows the number of times that out of n trials the event a is arrived, formula (o) will give the probability that its facility p, supposed unknown, will be comprehended within the given limits. In fact, if one names i this number of times, one will have, by that which precedes, the probability that the difference $\frac{i}{n}-p$ will be comprehended within the limits $\pm\frac{T\sqrt{2xx'}}{n\sqrt{n}}+\frac{z}{n}$; consequently, one will have the probability that p will be comprehended within the limits

$$\frac{i}{n}\mp\frac{T\sqrt{2xx'}}{n\sqrt{n}}-\frac{z}{n}.$$

The function $\frac{T\sqrt{2xx'}}{n\sqrt{n}}$ being of the order $\frac{1}{\sqrt{n}}$, one is able, by neglecting the quantities of order $\frac{1}{n}$, to substitute there i instead of x and $n-i$ instead of x'; the preceding limits become thus, by neglecting the terms of order $\frac{1}{n}$,

$$\frac{i}{n}\mp\frac{T\sqrt{2i(n-i)}}{n\sqrt{n}},$$

and the probability that the facility of the event a is comprehended within these limits is equal to

$$(o') \qquad \frac{2}{\sqrt{\pi}} \int dt\, c^{-t^2} + \frac{\sqrt{n} c^{-T^2}}{\sqrt{\pi}\sqrt{2i(n-i)}}.$$

One sees thus that, in measure as the events are multiplied, the interval of the limits is narrowed more and more, and the probability that the value of p falls within these limits approaches more and more unity or certitude. It is thus that the events, in being developed, make known their respective probabilities.

One arrives directly to these results, by considering p as a variable which is able to be extended from zero to unity, and by determining, after the observed events, the probability of its diverse values, as we will see it when we will treat the probability of causes deduced from observed events.

If one has three or a greater number of events a, b, c, ..., of which one alone must arrive at each trial, one will have, by that which precedes, the probability that, in a very great number n of trials, the ratio of the number x of times that one of these events, a for example, will arrive, to the number n, will be comprehended within the limits $p \pm \alpha$, α being a very small fraction, and one sees that, in the extreme case of the number n infinite, the interval 2α of these limits is able to be supposed null, and the probability is able to be supposed equal to certitude, so that the numbers of arrivals at each event will be proportional to their respective facilities.

Sometimes the events, instead of making known directly the limits of the value of p, give those of a function of this value; then one concludes from it the limits of p, by the resolution of equations. In order to give a quite simple example of it, we will consider two players A and B, of whom the respective skills are p and $1-p$, and playing together on this condition, that the game is won by the one of the two players who, out of three trials, will have vanquished twice his adversary, the third trial being not played, as useless, when one of the players is vanquished in the first two trials.

The probability of A to win the game is the sum of the first two terms of the binomial $[p + (1-p)]^3$; it is consequently equal to $p^3 + 3p^2(1-p)$. Let P be this function; by raising the binomial $P + (1-P)$ to the power n, one will have, by the preceding analysis, the probability that, out of the number n of games, the number of games won by A will be comprehended within the given limits. It suffices for that to change p into P in formula (o).

If one names i the number of games won by A, formula (o') will give the probability that P will be comprehended within the limits

$$\frac{i}{n} \pm \frac{T\sqrt{2i(n-i)}}{n\sqrt{n}}.$$

Let therefore p' be the real and positive root of the equation

$$p^3 + 3p^2(1-p) = \frac{i}{n};$$

by designating by $p' \mp \delta p$ the limits of p, the corresponding limits of P will be very nearly $3p'^2 - 2p'^3 \mp 6p'(1-p')\delta p$; by equating these limits to the preceding, one will have

$$\delta p = \frac{T\sqrt{2i(n-i)}}{6p'(1-p')n\sqrt{n}};$$

thus formula (o') will give the probability that p will be comprehended within the limits

$$p' \mp \frac{T\sqrt{2i(n-i)}}{6p'(1-p')n\sqrt{n}}.$$

The number n of games does not determine the number of trials, since one is able to have some games of two trials, and others of three trials. One will have the probability that the number of games of two trials will be comprehended within the given limits, by observing that the probability of a game with two trials is $p^2 + (1-p)^2$; we designate this function by P'. By elevating the binomial $P' + (1-P')$ to the power n, formula (o) will give the probability that the number of games of two trials will be comprehended within the limits $nP' \pm l$; now the number of games of two trials being $nP' \pm l$, the number of games with three trials will be $n(1-P') \mp l$; the total number of trials will be therefore $3n - nP' \mp l$; formula (o) will give therefore the probability that the number of trials will be comprehended within the limits

$$2n(1 + p - p^2) \mp T\sqrt{2nP'(1-P')}.$$

On the probability of the errors of the mean results of a great number of observations and of the most advantageous mean results

pp. 309–324

18. We consider now the mean results of a great number of observations of which one knows the law of the facility of errors. We suppose first that, for each observation, the errors are able to be equally

$$-n,\ -n+1,\ -n+2,\ \dots,\ -1,\ 0,\ 1,\ 2,\ \dots,\ n-2,\ n-1,\ n.$$

The probability of each error will be $\frac{1}{2n+1}$. If one names s the number of observations, the coefficient of $c^{l\varpi\sqrt{-1}}$ in the development of the polynomial

$$(c^{-n\varpi\sqrt{-1}}+c^{-(n-1)\varpi\sqrt{-1}}+c^{-(n-2)\varpi\sqrt{-1}}+\cdots+c^{-\varpi\sqrt{-1}}+1+c^{\varpi\sqrt{-1}}+\cdots+c^{n\varpi\sqrt{-1}})^s$$

will be the number of combinations in which the sum of the errors is l. This coefficient is the term independent of $c^{\varpi\sqrt{-1}}$ and of its powers in the development of the same polynomial multiplied by $c^{-l\varpi\sqrt{-1}}$, and it is clearly equal to the term independent of ϖ in the same development multiplied by $\frac{c^{l\varpi\sqrt{-1}}+c^{-l\varpi\sqrt{-1}}}{2}$ or by $\cos\, l\varpi$'; one will have therefore, for the expression of this coefficient,

$$\frac{1}{\pi}\int d\varpi\cos l\varpi(1+2\cos\varpi+2\cos 2\varpi+\cdots+2\cos n\varpi)^s,$$

the integral being taken from $\varpi=0$ to $\varpi=\pi$.

One has seen, in n$^{\text{o}}$ 36 of Book I, that this integral is

$$\frac{(2n+1)^s\sqrt{3}}{\sqrt{n(n+1)2s\pi}}c^{-\frac{\frac{3}{2}l^2}{n(n+1)s}};$$

the total number of combinations of the errors is $(2n+1)^s$; by dividing the preceding quantity by that here, one will have

$$\frac{\sqrt{3}}{\sqrt{n(n+1)2s\pi}}c^{-\frac{\frac{3}{2}l^2}{n(n+1)s}};$$

for the probability that the sum of the errors of the s observations will be l.

If one makes

$$l=2t\sqrt{\frac{n(n+1)s}{6}},$$

the probability that the sum of the errors will be contained within the limits $+2T\sqrt{\frac{n(n+1)s}{6}}$ and $-2T\sqrt{\frac{n(n+1)s}{6}}$ will be equal to

$$\frac{2}{\pi}\int dtc^{-t^2},$$

the integral being taken from $t = 0$ to $t = T$. This expression holds further in the case of n infinite. Then, by naming $2a$ the interval contained between the limits of the errors of each observation, one will have $n = a$, and the preceding limits would become $\pm\frac{2Ta\sqrt{s}}{\sqrt{6}}$: thus the probability that the sum of the errors will be contained within the limits $\pm ar\sqrt{s}$ is

$$2\sqrt{\frac{3}{2\pi}}\int drc^{-\frac{3}{2}r^2};$$

it is also the probability that the mean error will be contained within the limits $\pm\frac{ar}{\sqrt{s}}$; because one has the mean error by dividing by s the sum of the errors.

The probability that the sum of the inclination of the orbits of s comets will be contained within some given limits, by supposing all the inclinations equally possible, from zero to a right angle, is evidently the same as the preceding probability; the interval $2a$ of the limits of the errors of each observation is, in this case, the interval $\frac{\pi}{2}$ of the limits of the possible inclinations: then the probability that the sum of the inclinations must be contained within the limits $\pm\frac{\pi r\sqrt{s}}{4}$ is $2\sqrt{\frac{3}{2\pi}}\int drc^{-\frac{3}{2}r^2}$, this which accords with that which one has found in n° 13.

We suppose generally that the probability of each positive or negative error is expressed by $\phi\left(\frac{x}{n}\right)$, x and n being of infinite numbers. Then, in the function

$$1 + 2\cos\varpi + 2\cos 2\varpi + 2\cos 3\varpi + \cdots + 2\cos n\varpi,$$

each term, such as $2\cos x\varpi$, must be multiplied by $\phi\left(\frac{x}{n}\right)$; now one has

$$2\phi\left(\frac{x}{n}\right)\cos x\varpi = 2\phi\left(\frac{x}{n}\right) - \frac{x^2}{n^2}\phi\left(\frac{x}{n}\right)n^2\varpi^2 + \cdots$$

By making therefore

$$x' = \frac{x}{n}, \qquad dx' = \frac{1}{n},$$

the function

$$\phi\left(\frac{0}{n}\right) + 2\phi\left(\frac{1}{n}\right)\cos\varpi + 2\phi\left(\frac{2}{n}\right)\cos 2\varpi + \cdots + 2\phi\left(\frac{n}{n}\right)\cos n\varpi$$

becomes

$$2n\int dx'\phi(x') - n^2\varpi^2\int x'^2 dx'\phi(x') + \cdots$$

the integrals must be extended from $x' = 0$ to $x' = 1$. Let then

$$k = 2\int dx'\phi(x'), \quad k'' = \int x'^2 dx'\phi(x'), \quad \ldots$$

The preceding series becomes

$$nk\left(1 - \frac{k''}{k}n^2\varpi^2 + \cdots\right).$$

Now the probability that the sum of the errors of the s observations will be contained within the limits $\pm l$ is, as it easy to be assured of it by the preceding reasonings,

$$\frac{2}{\pi}\iint d\varpi\, dl \cos l\varpi \left\{ \begin{array}{r} \phi\left(\frac{0}{n}\right) + 2\phi\left(\frac{1}{n}\right)\cos\varpi + 2\phi\left(\frac{2}{n}\right)\cos 2\varpi + \cdots \\ +2\phi\left(\frac{n}{n}\right)\cos n\varpi \end{array} \right\}^s,$$

the integral being taken from ϖ null to $\varpi = \pi$; this probability is therefore

$$(u) \qquad 2\frac{(nk)^S}{\pi}\iint d\varpi\, dl \cos l\varpi \left(1 - \frac{k''}{k}n^2\varpi^2 + \cdots\right)^s.$$

We suppose

$$\left(1 - \frac{k''}{k}n^2\varpi^2 + \cdots\right)^s = c^{-t^2};$$

by taking the hyperbolic logarithms, one will have, very nearly, when s is a great number,

$$s\frac{k''}{k}n^2\varpi^2 = t^2,$$

this which gives

$$\varpi = \frac{t}{n}\sqrt{\frac{k}{k''s}}.$$

If one observes next that, nk or $2\int dx\, \phi\left(\frac{x}{n}\right)$ expressing the probability that the error of an observation is contained within the limits $\pm n$, this quantity must be equal to unity, the function (u) will become

$$\frac{2}{n\pi}\sqrt{\frac{k}{k''s}}\iint dl\, dt\, c^{-t^2}\cos\left(\frac{lt}{n}\sqrt{\frac{k}{k''s}}\right),$$

the integral relative to t must be taken from t null to $t = \pi n\sqrt{\frac{k''s}{k}}$, or to $t = \infty$, n being supposed infinite. Now one has, by nº 25 of Book I,

$$\int dt \cos\left(\frac{lt}{n}\sqrt{\frac{k}{k''s}}\right)c^{-t^2} = \frac{\sqrt{\pi}}{2}c^{-\frac{l^2}{4n^2}\frac{k}{k''s}},$$

by making therefore

$$\frac{l}{n} = 2t'\sqrt{\frac{k''s}{k}},$$

the function (u) becomes

$$\frac{2}{\sqrt{\pi}}\int dt'\, c^{-t'^2}$$

Thus, by naming, as above, $2a$ the interval contained between the limits of the errors of each observation, the probability that the sum of the errors of the s observations will be contained within the limits $\pm ar\sqrt{s}$ is

$$\sqrt{\frac{k}{k''s}}\int dr\, c^{-\frac{kr^2}{4k''}}.$$

If $\phi\left(\frac{x}{n}\right)$ is constant, then $\frac{k}{k''} = 6$, and this probability becomes

$$2\sqrt{\frac{3}{2\pi}}\int drc^{-\frac{3}{2}r^2},$$

this which is conformed to that which we have found above.

If $\phi\left(\frac{x}{n}\right)$ or $\phi(x')$ is a rational and entire function of x', one will have, by the method of n° 15, the probability that the sum of the errors will be contained within the limits $\pm ar\sqrt{s}$, expressed by a sequence of powers s, $2s$, $\ldots$ of quantities of the form $s - \mu \pm r\sqrt{s}$, in which μ increases by arithmetic progression, these quantities being continued until they become negatives. By comparing this sequence to the preceding expression of the same probability, one will obtain in a manner very near the value of the sequence, and one will arrive thus with respect to this kind of sequence to some theorems analogous to those that we have given in n° 42 of Book I, on the finite differences of the powers of a variable.

If the law of facility of the errors is expressed by a negative exponential which is able to be extended to infinity, and generally if the errors are able to be extended to infinity, then a becomes infinite, and the application of the preceding method is able to offer some difficulties. In all these cases, one will make

$$\frac{x}{h} = x', \qquad \frac{1}{h} = dx',$$

h being any finite quantity whatsoever, and by following exactly the preceding analysis, one will find, for the probability that the sum of the errors of the s observations is contained within the limits $\pm hr\sqrt{s}$,

$$\sqrt{\frac{k}{k''s}}\int dr\, c^{-\frac{kr^2}{4k''}}.$$

an expression in which one must observe that $\phi\left(\frac{x}{h}\right)$ or $\phi(x')$ expresses the probability of the error $\pm x$, and that one has

$$k = 2\int dx'\phi(x'), \quad k'' = \int x'^2 dx'\phi(x'),$$

the integrals being taken from $x' = 0$ to $x' = \infty$.

24. One has seen previously that, in all the manners to combine the equations of condition in order to form some final linear equations, necessary to the determination of the elements, the most advantageous is that which results from the method of least squares of errors of the observations, at least when the observations are in great number. If, instead of considering the minimum of the squares of the errors, one considered the minimum of other powers of the errors, or even of each other function of the errors, the final equations would cease to be linear, and their resolution would become impractical, if the observations were in great number. However there is a case which merits a particular attention, in this that it gives the system in which the greatest error, setting aside the sign, is less than in every other system. This case is the one of the minimum of the infinite and even powers of the errors. We consider here only the correction of a single element, and, z expressing this correction, we represent, as previously, the equations of condition by the following,

$$\epsilon^{(i)} = p^{(i)}z - \alpha^{(i)},$$

i being able to vary from zero to $s-1$, s being the number of observations. The sum of the powers $2n$ of the errors will be $\mathrm{S}(\alpha^{(i)} - p^{(i)}z)^{2n}$, the sign S extending to all the values of i. One is able to suppose in this sum all the values of $p^{(i)}$ positive; because, if one of them was negative, it would become positive by changing, as one is able to do it, the signs of the two terms of the binomial raised to the power $2n$, to which it corresponds. We will suppose therefore the quantities $\alpha - pz$, $\alpha^{(1)} - p^{(1)}z$, $\alpha^{(2)} - p^{(2)}z$, ... disposed in a manner that the quantities p, $p^{(1)}$, $p^{(2)}$, ... are positive and increasing. This put, if $2n$ is infinite, it is clear that the greatest term of the sum $\mathrm{S}(\alpha^{(i)} - p^{(i)}z)^{2n}$ will be the entire sum, unless there was one or many other terms which were equal to it, and this is that which must take place in the case of the minimum of the sum. In fact, if there was only a single greatest quantity, setting aside the sign, such as $\alpha^{(i)} - p^{(i)}z$, one would be able to diminish it by making z vary conveniently, and then the sum $\mathrm{S}(\alpha^{(i)} - p^{(i)}z)^{2n}$ would diminish and would not be a minimum. It is necessary moreover that, if $\alpha^{(i)} - p^{(i)}z$ and $\alpha^{(i')} - p^{(i')}z$ are, setting aside the sign, the two greatest quantities and equal between them, they are of contrary sign. In fact, the sum

$$(\alpha^{(i)} - p^{(i)}z)^{2n} + (\alpha^{(i')} - p^{(i')}z)^{2n}$$

must be then a minimum, its differential

$$-2ndz[p^{(i)}(\alpha^{(i)} - p^{(i)}z)^{2n-1} + p^{(i')}(\alpha^{(i')} - p^{(i')}z)^{2n-1}]$$

must be null, this which is able to be, when n is infinite, only in the case where $\alpha^{(i)} - p^{(i)}z$ and $\alpha^{(i')} - p^{(i')}z$ are infinitely little different and of contrary sign. If there are three greatest quantities, and equals among them, setting aside the sign, one will see in the same manner that their signs are not able to be the same.

Now, we consider the sequence

$$(o)\quad \left\{\begin{array}{l} \alpha^{(s-1)} - p^{(s-1)}z,\ \alpha^{(s-2)} - p^{(s-2)}z,\ \alpha^{(s-3)} - p^{(s-3)}z,\ \ldots,\ \alpha - pz, \\ -\alpha + pz,\ \ldots,\ -\alpha^{(s-3)} + p^{(s-3)}z,\ -\alpha^{(s-2)} + p^{(s-2)}z, -\alpha^{(s-1)} + p^{(s-1)}z. \end{array}\right.$$

If one supposes $x = -\infty$, the first term of the sequence surpasses the following, and continues to surpass them by making z increase, to the moment where it becomes equal to one of them. The one here, by the increase of z, becomes greatest of all, and in measure as one makes z increase, it continues always to surpass those which precede it. In order to determine this term, one will form the sequence of quotients

$$\frac{\alpha^{(s-1)} - \alpha^{(s-2)}}{p^{(s-1)} - p^{(s-2)}}, \frac{\alpha^{(s-1)} - \alpha^{(s-3)}}{p^{(s-1)} - p^{(s-3)}}, \ldots, \frac{\alpha^{(s-1)} - \alpha}{p^{(s-1)} - p}, \frac{\alpha^{(s-1)} + \alpha}{p^{(s-1)} + p}, \ldots, \frac{\alpha^{(s-1)} + \alpha^{(s-1)}}{p^{(s-1)} + p^{(s-1)}}.$$

We suppose that $\frac{\alpha^{(s-1)} - \alpha^{(r)}}{p^{(s-1)} - p^{(r)}}$ is the smallest of these quotients by having regard to the sign, that is to say by regarding a greater negative quantity as smaller than another lesser negative quantity. If there are many least and equal quotients, we will consider the one which relates to the most distant term of the first in the sequence (o); this term will be the greatest of all, to the moment where, by the increase of z, it becomes equal to one of the following, which begins then to be the greatest. In order to determine this new term, one will form a new sequence of quotients

$$\frac{\alpha^{(r)} - \alpha^{(r-1)}}{p^{(r)} - p^{(r-1)}}, \frac{\alpha^{(r)} - \alpha^{(r-2)}}{p^{(r)} - p^{(r-2)}}, \ldots, \frac{\alpha^{(r)} - \alpha}{p^{(r)} - p}, \frac{p^{(r)} + \alpha}{p^{(r)} + p}, \ldots,$$

the term of the sequence (o) to which the least of these quotients correspond will be the new term. One will continue thus to that which one of the two terms which become equal and the greatest is in the first half of the sequence (o), and the other in the second half. Let $\alpha^{(i)} - p^{(i)}z$ and $-\alpha^{(i')} + p^{(i')}z$ be these two terms; then the value of z which corresponds to the system of the minimum of the greatest of the errors, setting aside the sign, is

$$\frac{\alpha^{(i)} + \alpha^{(i')}}{p^{(i)} - p^{(i')}}.$$

If there are many elements to correct, the equations of condition which determine their corrections contain many unknowns, and the investigation of the system of correction, in which the greatest error is, setting aside the sign, smaller than in every other system, becomes more complicated. I have considered this case in a general manner in Book III of the *Mécanique céleste*. I will observe only here that then the sum of the $2n$ powers of the errors of the observations is, as in the case of a single unknown, a minimum when $2n$ is infinite; whence it is easy to conclude that, in the system of which there is concern, it must have as many errors, plus one, equal, and greatest, setting aside the sign, as there are elements to correct. One imagines that the results corresponding to $2n$ equal to a great number must differ little from those which $2n$ infinite gives. It is not necessarily the same for this if the $2n$ power is quite elevated, and I have recognized through many examples that, in the same case where this power does not surpass the square, the results differ little from those that the system of the minimum of the greatest squares gives, this which is a new advantage of the method of least squares of the errors of observations.

For a long time, geometers take an arithmetic mean among their observations, and, in order to determine the elements that they wish to know, they choose the most favorable circumstances for this object, namely, those in which the errors of the observations alter the least that it is possible the value of these elements. But Cotes is, if I do not deceive myself, the first who has given a general rule in order to make many observations agree in the determination of an element, proportionally to their influence. By considering each observation as a function of the element and regarding the error of the observation as an infinitely small differential, it will be equal to the differential of the function, taken with respect to that element. The more the coefficient of the differential of the element will be considerable, the less it will be necessary to make the element vary, in order that the product of its variation by this coefficient is equal to the error of the observation; this coefficient will express therefore the influence of the observation on the value of the element. This put, Cotes represents all the values of the element, given by each observation, by the parts of an indefinite straight line, all these parts having a common origin. He imagines next, at their other extremities, some weights proportional to the influences respective of the observations. The distance from the common origin of the parts to the common center of gravity of all these weights is the value that he chose for the element.

We take the equation of condition of n° 20,

$$\epsilon^{(i)} = p^{(i)}z - \alpha^{(i)},$$

$\epsilon^{(i)}$ being the error of the $(i+1)^{\text{st}}$ observation, and z being the correction of the element already known quite nearly; $p^{(i)}$, that one is able always to suppose positive, will express the influence of the corresponding observation. $\frac{\alpha^{(i)}}{p^{(i)}}$ being the value of z resulting from the observation, the rule of Cotes reverts to multiplying this value by $p^{(i)}$, to make a sum of all the products relative to the diverse values, and to divide it by the sum of all the $p^{(i)}$, this which gives

$$z = \frac{\mathrm{S}\alpha^{(i)}}{\mathrm{S}p^{(i)}}.$$

This was in fact the correction adopted by the observers, having the usage of the method of least squares of the errors of the observations.

However, one does not see that, since this excellent geometer, one has employed his rule, to Euler, who in his first piece on Jupiter and Saturn, appears to me to be served himself the first of the equations of the condition in order to determine the elements of the elliptic movement of these two planets. Near the same time, Tobie Mayer made use of it in this good researches on the libration of the Moon, and next in order to form his lunar Tables. Since, the best astronomers have followed this method, and the success of the Tables which they have constructed by his means has verified the advantage of it.

When one has only one element to determine, this method leaves no embarrassment; but, when one must correct at the same time many elements, it is necessary to have as many final equations formed by the reunion of many equations of condition, and by means of which one determines by elimination the corrections of the elements. But what is the most advantageous manner to combine the equations of condition, in order to form the final equations? It is here that the observers abandoned themselves to some arbitrary gropings, which must have led them to some different results, although deduced from the same observations. In order to avoid these gropings, Mr. Legendre had the simple idea to consider the sum of the squares of the errors of the observations, and to render it a minimum, this which furnishes directly as many final equations, as there are elements to correct. This scholarly geometer is the first who has published this method; but one owes to Mr. Gauss the justice to observe that he had had, many years before this publication, the same idea of which he made a habitual usage, and that he had communicated to many astronomers. Mr. Gauss, in his *Theory of elliptic movement*, has sought to connect this method to the Theory of Probabilities, by showing that the same law of errors of the observations, which give generally the rule of the arithmetic mean among many observations, admitted by the observers, gives similarly the rule of the least squares of the errors of the observations, and it is this which one has seen in n^{o} 23. But, as nothing proves that the first of these rules gives the most advantageous result, the same uncertainty exists with respect to the second. The research on the most advantageous manner to form the final equations is without doubt one of the most useful of the Theory of Probabilities: its importance in Physics and Astronomy carries me to occupy myself with it. For this, I will consider that all the ways to combine the equations of condition, in order to form a final linear equation, returns to multiply them respectively by some factors which were null relatively to the equations that one employed not at all, and to make a sum of all these products, this which gives a first final equation. A second system of factors give a second final equation, and thus consecutively, to this that one has as many final equations as elements to correct. Now it is clear that it is necessary to choose the system of factors, such that the mean error to fear to plus or to less respecting each element is a minimum; the mean error being the sum of the products of each error by its probability. When the observations are in small number, the choice of these systems depends on the law of errors of each observation. But, if one considers a great number of observations, this which holds most often in the astronomical researches, this choice becomes independent of this law, and one has seen, in that which precedes, that Analysis leads then directly to the results of the method of least squares of the errors of the observations. Thus this method which offered first only the advantage to furnish, without groping, the final equations necessary to the correction of the elements, gives at the same time the most precise corrections, at least when one wishes to employ only final equations which are linear, an indispensable condition, when one considers at the same time a great number of observations; otherwise, the elimination of the unknowns and their determination would be impractical.