Jean Bernoulli's 5th and 6th Problems

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> restart:
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First step, define base probability distributions. Now d1 through d6 are the probability distributions of the sum of the faces obtained through the cast of 1 to 6 dice. Each sequence starts with the outcome 1 and extends as far as 36.

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> d1 := [1/6, 1/6, 1/6, 1/6, 1/6,
  1/6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]:
> d2 := [0, 1/36, 1/18, 1/12, 1/9, 5/36, 1/6, 5/36, 1/9, 1/12, 1/18,
  1/36,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]:
> d3 := [0, 0, 1/216, 1/72, 1/36, 5/108, 5/72, 7/72, 25/216, 1/8,
  1/8,25/216, 7/72, 5/72, 5/108, 1/36, 1/72,
  1/216,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]:
> d4 := [0, 0, 0, 1/1296, 1/324, 5/648, 5/324, 35/1296, 7/162, 5/81,
  13/162, 125/1296, 35/324, 73/648, 35/324, 125/1296, 13/162, 5/81,
  7/162, 35/1296, 5/324, 5/648, 1/324,
  1/1296,0,0,0,0,0,0,0,0,0,0,0,0]:
> d5 := [0, 0, 0, 0, 1/7776, 5/7776, 5/2592, 35/7776, 35/3888,
  7/432, 205/7776, 305/7776, 35/648, 5/72, 217/2592, 245/2592,
  65/648, 65/648, 245/2592, 217/2592, 5/72, 35/648, 305/7776,
  205/7776, 7/432, 35/3888, 35/7776, 5/2592, 5/7776,
  1/7776,0,0,0,0,0,0]:
> d6 := [0, 0, 0, 0, 0, 1/46656, 1/7776, 7/15552, 7/5832, 7/2592,
  7/1296, 19/1944, 7/432, 43/1728, 833/23328, 749/15552, 119/1944,
  3431/46656, 217/2592, 469/5184, 361/3888, 469/5184, 217/2592,
  3431/46656, 119/1944, 749/15552, 833/23328, 43/1728, 7/432,
  19/1944, 7/1296, 7/2592, 7/5832, 7/15552, 1/7776, 1/46656]:
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Second. Analysis of the game.

The first roll of Player A determines the number of rolls of Player B. Hence Player B may roll equivalently 1 to 6 dice each with probability 1/6 and the distributions of the outcomes of Player B are as d1 through d6 respectively. On the other hand, Player A, given the outcome of the first roll, always has a sum distributed as d2.

Therefore, the more direct solution of the problem must lie through conditioning on the 1st roll of A.

Third part. What is the probability of a tie? We condition on the outcome of the first die.

For example, given 1st roll yields 1, Player A can obtain any value from 2+1 to 12+1 but B can only obtain values from 1 to 6. A tie occurs if B rolls k and A rolls k-1 on the remaining two tosses. Therefore, the probability of a tie is the sum of $\mathrm{d1}(k)\ \mathrm{d2}(k-1)$ for $k = 3$ to 6.

Another example: Given the 1st roll yields 3, Player A can obtain any value from 2+3 to 12+3 and B any value from 3 to 18. Given B has k, a tie occurs if Player A rolls $k-3$ on the remaining two tosses. The probability of a tie is therefore $\mathrm{d3}(k)\,\mathrm{d2}(k-3)$ for $k=5$ to 18.

In general, given that the first roll yields outcome n, we want to compute $\mathrm{dn}(k)\,\mathrm{d2}(k-n)$ for $k=2+n$ to $6\,n$.

$$> \left(\sum_{k=2+1}^{6(1)} d1_k\, d2_{k-1}\right)+\left(\sum_{k=2+2}^{6(2)} d2_k\, d2_{k-2}\right)+\left(\sum_{k=2+3}^{6(3)} d3_k\, d2_{k-3}\right)+\left(\sum_{k=2+4}^{6(4)} d4_k\, d2_{k-4}\right)+\left(\sum_{k=2+5}^{6(5)} d5_k\, d2_{k-5}\right)+\left(\sum_{k=2+6}^{6(6)} d6_k\, d2_{k-6}\right)$$

$$\frac{320573}{839808}$$

Since the outcomes of the first roll are equiprobable, the probability of a tie is 1/6 of this value.

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> Tie:=%/6;
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$$Tie := \frac{320573}{5038848}$$

Fourth part. What is the probability Player A beats Player B? Again we condition on the outcome of the first roll.

If the 1st roll yields 1, then, as before, Player A may obtain any value from 2+1 to 12+1 and Player B only values from 1 to 6. If Player B obtains, say, the outcome k, then Player A wins with any outcome from $k+1$ to 13. This requires that on the remaining two throws that Player A obtain at least k. The probability of this event is clearly $\mathrm{d1}(k)\,(\mathrm{d2}(k)+\mathrm{d2}(k+1))$ + ...).

If the 1st roll should yield a 3, Player A can obtain any value from 2+3 to 12+3 and B any value from 3 to 18. If B threw a 12, for example, then A would need at least a 10 from the remaining 2 dice. In general, if B throws k, A needs at least $k-2$ from remaining dice so the probability that A beats B must be $\mathrm{d3}(k)\,(\mathrm{d2}(k-2)+\mathrm{d2}(k-1))$ + ...)

$$> \left(\sum_{k=1}^{6(1)}\left(\sum_{j=k}^{36} d1_k\, d2_j\right)\right)+\left(\sum_{k=2}^{6(2)}\left(\sum_{j=k-1}^{36} d2_k\, d2_j\right)\right)+\left(\sum_{k=3}^{6(3)}\left(\sum_{j=k-2}^{36} d3_k\, d2_j\right)\right)+\left(\sum_{k=4}^{6(4)}\left(\sum_{j=k-3}^{36} d4_k\, d2_j\right)\right)+\left(\sum_{k=5}^{6(5)}\left(\sum_{j=k-4}^{36} d5_k\, d2_j\right)\right)+\left(\sum_{k=6}^{6(6)}\left(\sum_{j=k-5}^{36} d6_k\, d2_j\right)\right)$$

$$\frac{215555}{93312}$$

Again, since the outcomes of the first cast are equiprobable, the probability that A beats B is 1/6 of this value.

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> Beat:=%/6;
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$$Beat := \frac{215555}{559872}$$

Player A loses to Player B if his total is less than that of A.

For example, if the first toss is 1, then Player B may throw from 1 to 6. Now since Player A is able to throw from 2+1 to 12+1, B cannot win with a 1 or a 2. Moreover, if B throws a 3, he can only tie. Therefore, only $k = 4$ to 6 are potentially winning throws. In this case, on the remaining casts, Player A will lose if he throws at most $k - 2$. This occurs with probability $\mathrm{d2}(k-2) + \mathrm{d2}(k-3) + \ldots\mathrm{d2}(1)$.

If the first toss should yield a 3, Player B can produce any number between 3 and 18 and Player A from 2+3 to 12+3. Since A has minimum score 5, B cannot win with a 3 or 4 and can only tie with a 5. Thus, only outcomes 6 to 18 are of concern. if Player B casts, say, 11, Player A will lose if the remaining two throws produce at most 7. This occurs with probability $\mathrm{d2}(7) + \mathrm{d2}(6) + \ldots$. In general, if B rolls k, the probability of A losing is $\mathrm{d2}(k-4) + \mathrm{d2}(k-5) + \ldots\mathrm{d2}(1)$.

In complete generality, if the first cast is n, only the outcomes $n + 3$ to $6\,n$ need be considered. If then B rolls k, Player A can produce at most $k - n - 1$.

$$> \left(\sum_{k=4}^{6(1)}\left(\sum_{j=1}^{k-2} d1_k\, d2_j\right)\right) + \left(\sum_{k=5}^{6(2)}\left(\sum_{j=1}^{k-3} d2_k\, d2_j\right)\right) + \left(\sum_{k=6}^{6(3)}\left(\sum_{j=1}^{k-4} d3_k\, d2_j\right)\right) + \left(\sum_{k=7}^{6(4)}\left(\sum_{j=1}^{k-5} d4_k\, d2_j\right)\right) + \left(\sum_{k=8}^{6(5)}\left(\sum_{j=1}^{k-6} d5_k\, d2_j\right)\right) + \left(\sum_{k=9}^{6(6)}\left(\sum_{j=1}^{k-7} d6_k\, d2_j\right)\right)$$

$$\frac{347285}{104976}$$

Again, since the outcomes of the first cast are equiprobable, we divide by 6.

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> Lose:=%/6;
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$$Lose := \frac{347285}{629856}$$

Check.

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> Lose+Beat+Tie;
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$$1$$

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> Expect_A:=Beat+Tie/2;
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$$Expect\_A := \frac{4200563}{10077696}$$

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> Expect_B:=Lose+Tie/2;
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$$Expect\_B := \frac{5877133}{10077696}$$

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> Expect_A/Expect_B;
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$$\frac{4200563}{5877133}$$

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> evalf(%);
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$$0.7147299542$$

Problem VI.

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> Expect_A2:=Beat+Tie;
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$$Expect\_A2 := \frac{282571}{629856}$$

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> Expect_B2:=Lose;
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$$Expect\_B2 := \frac{347285}{629856}$$

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> Expect_A2/Expect_B2;
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$$\frac{282571}{347285}$$

And Bernoulli's solution agrees with this.