DE RATIONCINIIS IN LUDO ALEAE

CHRISTIAAN HUYGENS

TO THE READER[1]

When I had already taken the resolution to end these exercises,[2] there occurred to me the feeling, Dear Reader, that there would remain to me again many other amusing and remarkable subjects, which, if I had added them to these Sections[3] and if I had succeeded in this treatise worthily, they would have greatly adorned my work and perhaps facilitate and render more profitable your Studies; only the trouble that I would have had to take in order to develop them, in the same way as the work there would have cost me, should make me become too dull. This is why (similarly that, among other matters treated in the preceding Sections of my work, I have indicated how some of the best and most subtle propositions found in part by the Mathematicians of Antiquity and in part by the most Excellent Mathematicians of this century, would be able to be sought and found by means of the *Algebra*) it does not seem inopportune to me, in order to extend the applications of this Art, to add here, instead of the subjects which remained to me, that which has been invented lately by the very Noble and very Celebrated Mr. CHRISTIANUS HUGENIUS for the calculation in the Games of Chance, a Treatise which he has communicated to me with a letter that I have equally added. I presume that his Writing will please you so much more as the considerations of the author will appear to you more subtle and more extraordinary; especially since he employs the same Analysis of which I myself am served and of which I have taught to him formerly the basics, and that thus it indicates to those who have studied this art a method for analysis of similar Problems. If I have given you thus, Dear Reader, beyond the rest of my work, enough subjects in order to exercise you on this type of Study, you will understand, I trust, my good will towards you, that which it will be agreed to you the trouble that I have taken for good style and the one of the Studies. Farewell.

Date: 1656–1657.

Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, Ohio. Printed on July 18, 2009.

[1]In this preface, which one finds on pages 485–486 of the "Mathematiche Oefferningen," Professor van Schooten introduces to his readers the small treatise, which is going to follow, on the games of chance, composed by his student Christiaan Huygens.

[2]The work in question of which the title page has been reproduced in facsimile.

[3]The treatise of Huygens was added to the fifth and last Book of the work of van Schooten. The Latin version preceded the Dutch in publication.

To Mr.
FRANCISCUS van SCHOOTEN
Sir,

Knowing that by publishing the laudable fruits of your intelligence and of your zeal, you yourself propose among others to show, by the diversity of subjects treated, the grandeur of the field on which our excellent *Algebraic* Art is extended, I do not doubt that the present writing on the subject of the Calculus in the Games of chance is able to serve you to attain this target. Indeed, the more it seems difficult to determine by reason that which is uncertain or formed by chance, the more the science which attains to this result will appear admirable. As this is therefore by your demand and by consequence of your exhortations that I have begun to put this Calculus in writing,[4] and that you judge it worthy to appear together with the results of your profound researches, not only do I give to you willingly the permission to publish it in this fashion but further I value that this manner of publication will be all to my advantage. Because if some readers should be well able to imagine that I have worked on some subjects of feeble importance, they will not condemn nonetheless as completely useless and unworthy of all praise that which you well wish to adopt in this fashion as if this were your proper work, after having translated it, not without some labor, from our language into Latin.[5] However I wish to believe that in considering these things more attentively, the reader will perceive shortly that it is not a question of some simple witticism here, but that we cast there the fundamentals of a very interesting and profound speculation. The Problems belonging to this Matter will not be, it seems to me, judged more easy than those of Diophantus, but one will find them perhaps more amusing whereas they comprise something more than simple properties of numbers. It is necessary to know besides that there is already a certain time that a few of the more Celebrated Mathematicians of all France occupied themselves in this kind of Calculus,[6] so that no person attribute to me the honor of the first Invention which will not belong to me. But these scholars, although they themselves both of them put to proof by proposing to themselves many difficult questions to resolve, have however hidden[7] their methods. I have therefore needed to examine and to delve myself into all this matter by beginning with the elements, and it is impossible to me for the reason that I just mentioned to assert that we are starting from the same first principle. But for that which is the result, I have established as well the case that my solutions are not different from theirs. You will find that at the end of this Treatise I have proposed yet some questions of the same type without indicating the manner of solution of them, firstly because I saw that it will cost me too much work to expose conveniently the reasoning leading to the answers, and in second place because it seems to me useful to leave something to seek to our readers (if there are found some few), so that this serves to exercise them and to pass time.

The Hague — Your devoted servant
27 April 1657 — *CHR. HUYGENS of Zuylichem*

[4] See the letter to Roberval of 18 April 1656.

[5] Indeed, the present Treatise on the Calculus of the Games of Chance appeared for the first time in 1657 in a Latin edition which preceded by three years the Dutch edition.

[6] Refers here to the work of Pascal and Fermat.

[7] In truth, they did not publish their methods.

ON THE CALCULUS OF THE GAMES OF CHANCE

Although in the games of pure chance the results are uncertain, the chance that a player has to win or to lose has nevertheless a determined value. Example: if any one wagers to throw with one die six points on the first cast, it is uncertain if he will win or if he will lose; but that which is determined and calculable is by how much the chance that he has to lose his wager surpasses that which he has to win it. Similarly, if I play with another person to whom will win the first three points and that I have already won one of them, it is again uncertain which of the two will carry it off; but one is able to calculate with certitude the ratio of my chance to win to his own, and one knows consequently also how much, if we wish to interrupt the game, the division of the stake to which I have right surpasses his own.[8] We are able to calculate equally for what price I would reasonably cede my game to another who would desire to continue in my place. Many questions of this type are all able to be present in some similar cases where there are 2, 3 or more persons,[9] and as these calculations are not universally known and are able often to be useful, I will indicate briefly the method of it, after which I will consider also the game of dice.[10]

In these two matters I start from the hypothesis that in a game the chance which one has to win something has a value as much as if one possessed that value when one is able to procure the same chance by an equitable game, that is by a game which is not seen to the detriment to the person. Example: if someone hides without my knowledge three coins[11] in one hand and seven in the other, and gives me to choose between the two hands, I say that this offer has for me the same value as if I was certain to obtain five coins; indeed, when I possess five coins, I am able anew to stake myself in the case of having equal chances to obtain three or seven coins, and this in an equitable game, as this will be demonstrated below.[12]

PROPOSITION I

To have equal chances to obtain a or b is worth $\frac{a+b}{2}$ to me.

In order that not only to prove this rule but also to discover it, we call x the value of my chance. It is necessary therefore that, possessing x, I am able to procure anew the same chance by an equitable game. Suppose that this game is the following: I wager x against another person, with whom the stake is equally x; it is agreed that the one who wins will give a to the one who loses. This game is equitable, and it appears that I have thus an equal chance to have a in losing, or $2x-a$ in winning the game; because in this last case I obtain the stake $2x$, of which I must give a to the other player. If $2x-a$ was equal to b, I would have therefore an equal chance to have a or to have b. I put therefore $2x-a=b$, whence I draw the value of my chance $x=\frac{a+b}{2}$. The proof of it is easy. Indeed, possessing $\frac{a+b}{2}$, I am able to wager this sum against another player who will set equally $\frac{a+b}{2}$, and to agree with him that the winner will give a to the other. I will have in this way an equal chance to

[8]See Prop. VI.

[9]See Prop. IV, V, VII, VIII, IX.

[10]See Prop. X–XIV.

[11]The Dutch text uses "shilling," the Latin "solidus."

[12]See Prop. I which follows.

have a if I lose, or b if I win; because in this last case I obtain the stake $a + b$ and I give a of it to him.

In numbers. When I have an equal chance to have 3 or to have 7, the value of my chance is 5 according to this Proposition; and it is certain that having 5 I myself am able to procure anew the same chance. Indeed, if I wager 5 against another person with whom the wager is equally 5, under condition that the winner will give 3 to the other, it is thence an equitable game, and it is evident that I have the same chance to have 3 by losing, or to have 7 by winning; because in this case I obtain 10, of which I give 3 of them to him.

PROPOSITION II

To have equal chances to obtain a, b or c is worth $\frac{a+b+c}{3}$ to me.

In order to find this, call again x the value of my chance. It is necessary therefore that, possessing x, I am able to procure myself anew the same chances by an equitable game. Let this game be the following: I play against two other persons; each of us three set x, I agree with the first that she will give me b if she wins the game and reciprocally, with the second that she will give me c if she wins and reciprocally. It appears that this game is equitable. I will have therefore an equal chance to have b, namely if the first player wins, or c, if the second wins, or finally $3x - b - c$ if I myself win; because in this last case I obtain the stake $3x$, of which I give b to the one and c to the other. Now, if $3x - b - c$ was equal to a, I will have an equal chance to have a, b, or c. I put therefore $3x - b - c = a$, whence I take $x = \frac{a+b+c}{3}$, the value of my chance. One finds similarly that to have equal chances to obtain a, b, c or d is worth $\frac{a+b+c+d}{4}$ to me, and thus the rest.

PROPOSITION III[13]

To have p chances to obtain a and q to obtain b, the chances being equivalent, is worth $\frac{pa+qb}{p+q}$ to me.

In order to discover the rule, call anew x the value of my chance. It is necessary therefore that, possessing x, I am able to return into my first state by an equitable game. To this purpose I take a number of players such that with me there are $p + q$ of them in total, of whom each wagers x, so that the total stake will be $px + qx$; each plays for his proper amount with an equal chance to win. Suppose further that with q players, that is with each of them in particular, I adopt this agreement that if one of them wins the game, he will give me the sum b, and that if I myself win, I will give to him the same sum. Suppose finally that with the $p - 1$ players who remain, or rather with each of them in particular, I adopt the agreement that if one of them wins the game, he will give to me the sum a, and that I will give to him equally the sum a if it is I who wins the game. It is evident that under these conditions the game is equitable, seeing that the interests of each player are not found wronged. One sees moreover that I have now q chances to obtain b, $p - 1$ chances to obtain a and one chance (in case where it is I who wins) to have $px + qx - bq - ap + a$; indeed, in this last case I receive the stake $px + qx$ of which I must cede b to each of the q players and a to each of the $p - 1$ players, that which makes altogether $qb + pa - a$. Now, if $px + qx - bq - ap + a$ was equal to a, I would have p chances to have a (because I have already $p - 1$ chances to have this sum) and q chances to have b; I would be therefore returned to my first chances. I put therefore $px + qx - bq - ap + a = a$, and I find $x = \frac{ap+bq}{p+q}$ for the value of my chance, conforming to the statement.

[13] This proposition was communicated by Huygens to Carcavi in a letter of 6 July 1656 (Letter No. 308) adding to it that it be of use "in all those questions of divisions of games."

In numbers. If I have 3 chances to win 13, and 2 chances to win 8, I possess so to speak 11, according to this rule. And it is easy to show that being in possession of 11, I myself am able to procure anew these same chances. Indeed, I am able to play with 4 other persons and each of us five able to wager 11; I will agree then with two of these persons that if one of them wins the game he will give me 8 and that, if it is I who wins, I will give to each of them the same sum. Similarly I agree with the two others that that of the two who win the game will give me 13 and that, if I myself win, I will give to each of them 13 equally. This game will be equitable. And one sees that I have thus 2 chances to have 8, namely to the case where one of the two players who have promised me this sum carries off the stake, and 3 chances to have 13, namely if one of the two others who must give me this sum wins the game, or if I myself win it. Indeed, in this last case I receive the stake which is 55, of which I must give 13 to each of two players and 8 to each of two other players, so that there remain to me equally 13 of it.

PROPOSITION IV

Suppose that I play against another person to whom first three points will have won, and that I have already won two points and he one. I wish to know what share of the stake is due me in the case where we would wish to interrupt the game and divide equitably the stakes.

It is necessary to begin with the simplest case in order to arrive to the solution of the question proposed in first place on the subject of the division of the stake among many players to whom the chances are unequal.

It is necessary to remark first that it suffices to take account of the points which one part or the other lacks. Because it is certain that if we play until one will have won 20 points first, and that I have won 19 points and my adversary 18, I will have the same advantage as in the stated case; where for three points I have won two and he only one, and that because in the two cases there is only lacking to me one point whereas he lacks two of them.

Next, in order to calculate the share which reverts to each of us, it is necessary to pay attention to that which would happen if we will continue the game. It is certain that if I would win the first point, I would have ended the game, and that thus I would obtain the stake in totality which I will call a. But if the other player won the first point our chances would be made equal, seeing that we will lack one point each; we will have therefore each a right to $\frac{1}{2}a$. Now, it is evident that I have as much chance to win the first point as to lose it. I have therefore equal chances to have a or $\frac{1}{2}a$, this which, according to the first Proposition, is equivalent to the sum of the halves, that is to $\frac{3}{4}a$, so that there remains $\frac{1}{4}a$ for my adversary. I will moreover to be able to make for him also a direct calculation, by following the same method. There results from it that the one who would wish to continue the game in my place, would be able to offer me $\frac{3}{4}a$, and that one is able always to wager 3 against 1 in accepting to win a game before another player wins two of them.

PROPOSITION V

Suppose that one point is lacking to me and three to my adversary. The question is to divide the stake under this hypothesis.

Consider anew the state where we would be if I won the first point or else if my adversary won it. If I won it, I would have the stake a, but if it was he who won it, there would lack to him yet 2 against the one which would be lacking to me; we would find ourselves therefore in the state considered in the preceding proposition, and there would revert $\frac{3}{4}a$

to me, as we have seen in that place. I have therefore equal chances to obtain a or $\frac{3}{4}a$, that which, according to the first Proposition, is worth $\frac{7}{8}a$ to me. There remains $\frac{1}{8}a$ for the other player. My chance is therefore to his own as 7 to 1.

Just as that calculation depends on the preceding calculation, similarly the result obtained here is necessary to the calculation which is stated in the following case, where one point is lacking to me, while four of them are lacking to my adversary. We find in the same way that there reverts to me in that case $\frac{15}{16}$ of the stake, and $\frac{1}{16}$ to him.

PROPOSITION VI

Suppose that two points are lacking to me and that three of them are lacking to my adversary.

The first point will now have for effect, either truly one point will be lacking to me yet and three to the other player (in that case there will revert to me $\frac{7}{8}a$ according to that which precedes), or else that two points will be lacking to each of us, in which case there reverts to me $\frac{1}{2}a$, whereas our chances will be become equal. But I have one chance against one to win the first point or to lose it; I have therefore equal chances to obtain $\frac{7}{8}a$ or $\frac{1}{2}a$, that which is worth to me $\frac{11}{16}a$ according to the first Proposition. Eleven parts of the stake are due me therefore against five to my adversary.

PROPOSITION VII

Suppose that two points are lacking to me again and four to him.

Having won the first point, I will have therefore again 1 point to win against 4; having lost, again 2 against 3. I have therefore equal chances to have $\frac{15}{16}a$ or $\frac{11}{16}a$, that which is worth to me $\frac{13}{16}a$ according to the first Proposition. There results from it that it is more advantageous to have two points to win against four, than one against two. Because in this last case: namely the case of one point against two, my share is $\frac{3}{4}a$ according to the 4th Proposition, which is less than $\frac{13}{16}a$.

PROPOSITION VIII

Suppose now that three persons play together and that one point is lacking to the first, as also the second, but that two of them are lacking to the third.

In order to calculate the share of the first player, it is necessary next to take account of that which would revert to him if he himself or if one of the two others won the first point. If he won it, he would obtain the stake that I call a. If the second won this point, the first would have nothing, because the second would have terminated the game. If the third won, one point would be lacking still to each of the three: consequently the first, as also each of the two others, will have right to $\frac{1}{3}a$. The first has therefore 1 chance to obtain a, 1 chance to obtain 0, and 1 chance to obtain $\frac{1}{3}a$ (because each of the three has the same chance to win the first point), this which is worth to him $\frac{4}{9}a$ according to the second Proposition. To the second player reverts therefore the same division, that is, $\frac{4}{9}a$, so that there remains $\frac{1}{9}a$ for the third. One will be able to find directly the share of this last and to calculate starting from there the share of the others.[14]

[14]See Appendix I.

PROPOSITION IX[15]

In order to calculate the share for each of a given number of players, of whom some points in given number are lacking for each of them separately, it is necessary first of all to render account of that which would revert to the one of whom one wishes to know the division in the case either to him and to those or each of the others in his turn would have won the first point following. By adding all these divisions and by dividing the sum by the number of players one finds the sought share of the player considered.

Suppose that 3 persons A, B, and C play together and that there 1 point is lacking to A, 2 to B and equally 2 to C. One wishes to know what share of the stake that I call q reverts to player B.

We must examine first of all of what shares B would have right, if himself or A or C would have won the first point following. If A won it, it would have terminated the game. B would receive consequently 0. If B was himself winning it, 1 point to him would be lacking yet, as also A, while 2 of them will be lacking to C. B would have therefore right to $\frac{4}{9}q$ according to the eighth Proposition.

Finally, if C won the first point following, there 1 point would be lacking to A and to C, whereas two of them would be lacking to B; B consequently, would have right to $\frac{1}{9}q$ according to the same Proposition VIII.

It is necessary now to add the divisions which would revert to B under these three hypotheses, namely 0, $\frac{4}{9}q$ and $\frac{1}{9}q$. It comes to $\frac{5}{9}q$. By dividing this number by 3, the number of players, one finds $\frac{5}{27}q$. This is the just share of player B.

One demonstrates it by the second Proposition. Indeed, as B has equal chances to obtain 0, $\frac{4}{9}q$ or $\frac{1}{9}q$, he possesses, according to this Proposition, so to speak $\frac{0+\frac{4}{9}q+\frac{1}{9}q}{3} = \frac{5}{27}q$. And it is certain that the divisor 3 is equal to the number of players.

But in order to know that which will revert in each case to each player when himself or any one of the others will have won the first point following, it is necessary first of all to make the calculation for the most simple cases and next, by departing thence, for the cases following. Because likewise the last case we have considered would not be able to be amenable to calculation without first the eighth Proposition had been resolved, in which the points lacking were in the number of 1, of 1 and of 2, similarly also the share of each player in the case where the numbers of points missing are 1, 2, 3 are not able to be calculated unless one seeks first that division in the case of the numbers 1, 2, 2, as we just did it, as also in the one of the numbers 1, 1, 3, that which is able to be done according to the eighth Proposition. It is in following this method that one finds the corresponding divisions in the numbers of the following table, as also in an infinity of others.

Table for three players.

Points which are lacking to them.	1.1.2	1.2.2	1.1.3	1.2.3
Their divisions.	4.4.1	17.5.5	13.13.1	19.6.2
	9	27	27	27

[15]This proposition is wanting in the manuscript sent to van Schooten 20 April 1656. It must have been added either in 1656 after 6 May, date of the Piece No. 289, or else in 1657.

Points which are lacking to them.	1.1.4	1.1.5	1.2.4	1.2.5
Their divisions.	40.40.1	121.121.1	178.58.7	542.179.8
	81	243	243	729

Points which are lacking to them.	1.3.3	1.3.4	1.3.5
Their divisions.	65.8.8	616.82.31	629.87.13
	81	729	729

Points which are lacking to them.	2.2.3	2.2.4	2.2.5	2.3.3	2.3.4	2.3.5
Their divisions.	34.34.13	338.338.53	353.353.23	133.55.55	451.195.83	1433.635.119
	81	729	729	243	729	2187

In regard to dice, one is able to pose these questions: namely in how many times one is able to accept to throw with one die a 6 or else one of the other numbers; similarly how many times 2 six with 2 dice or 3 six with 3 dice. And also other questions still. In order to resolve them, it is necessary to remark that which follows.

First one is able to make with one die six different equally probable coups[16] (throws). For I suppose that the die has the form of a perfect Cube.

Next one is able to make 36 different throws with 2 dice, which are also equally probable. Indeed, with each throw of the first die each of the 6 throws of the second die is able to be combined. And 6 times 6 make 36.

Likewise there are 216 throws of 3 dice. Because with each of the 36 throws of the 2 dice is able to be combined any one of the 6 throws of the third. And 6 times 36 are 216.

One finds in the same fashion that there are 6×216 or 1296 throws of four dice; and that one is able to continue thus to calculate the number of throws for any number of dice: one multiplies by 6 the preceding number of throws, each time one adds a new die.

Next, it is necessary to know that with two dice one is only able to make one throw of 2 or of 12 points, and 2 throws of 3 or 11 points. Indeed, if we name the dice A and B respectively, it is evident that in order to throw 3 points A is able to give one ace and B a 2, or else B an ace and A a 2. Similarly in order to obtain 11 points, A is able to give 5 and B 6, or else A 6 and B 5. The throw of 4 points is triple, namely A 1, B 3, or A 3, B 1, or A 2, B 2. The throw of 10 points is equally triple. The one of 5 or of 9, quadruple. The one of 6 or of 8 points, quintuple. The one of 7 points, sextuple.

$$\text{With 3 dice one finds for} \left\{ \begin{array}{c} 3 \text{ or } 18 \\ 4 \text{ or } 17 \\ 5 \text{ or } 16 \\ 6 \text{ or } 15 \\ 7 \text{ or } 14 \\ 8 \text{ or } 13 \\ 9 \text{ or } 12 \\ 10 \text{ or } 11 \end{array} \right\} \text{points} \left\{ \begin{array}{c} 1 \\ 3 \\ 6 \\ 10 \\ 15 \\ 21 \\ 25 \\ 27 \end{array} \right\} \text{different throws.}$$

PROPOSITION X

To find how many times one is able to accept to throw a six with one die.

[16]Translator's note: We render the word "coups" by throw in this case, since it consistently refers to the desired outcome of the cast of one or more dice.

It is certain that the player who accepts to throw a 6 in a single throw has 1 chance to win the stake and 5 to lose. Because there are 5 throws against him and not more than one for him. We call the stake a. There is therefore 1 chance to obtain a and 5 chances to obtain nothing, that which according to the second Proposition is worth $\frac{1}{6}a$ to him. There remains $\frac{5}{6}a$ for the one who engage him to throw the die. The one who plays one game of a single throw, is therefore able to put only 1 against 5.

The share of the one who wagers to throw a 6 in two throws, is calculated in the following fashion. If he throws a 6 the first time, he wins a. If he misses his throw, he has yet one of them, that which is worth to him $\frac{1}{6}a$ according to the preceding calculation. But he has only one chance to throw a 6 on the first throw and he has 5 to miss this throw. There is therefore at the beginning 1 chance to obtain a and 5 chances to obtain $\frac{1}{6}a$, this which is worth to him $\frac{11}{36}a$ according to the second Proposition. There remains $\frac{25}{36}a$ for the one who wagers against him. The one who plays with 2 throws is able therefore to put 11 against 25, that which is less than 1 against 2.

Starting from this result, one calculates in the same manner that the share of the one who wagers to throw a 6 in three throws is $\frac{91}{216}a$. He is able therefore to set 91 against 125, that is nearly 3 against 4.

The share of the one who plays to 4 throws is $\frac{671}{1296}a$. He is able therefore to set 671 against 625, that is to say more than one against 1.

The share of the one who plays to 5 throws is $\frac{4651}{7776}a$; he is able to wager 4651 against 3125; that is to say nearly 3 against 2.

The division of the one who plays with 6 throws is $\frac{31031}{46656}a$, and he is able to put 31031 against 15625, that is to say nearly 2 against 1.

One is able to pursue these calculations successively for each number of throws. But one is able thus to move forward by a greater leap, as we will indicate in the following Proposition; without which the Calculation would become too long.

PROPOSITION XI.

To find how many times one is able to accept to throw 2 sixes with 2 dice.

The one who plays with a single throw, has 1 chance to win, that is to say to have a, against 35 chances to lose, that is to say to have 0, expecting that there are 36 throws. So that he has $\frac{1}{36}a$ according to the second Proposition.

As for the one who plays to two throws, he wins a if he throws 2 sixes the first time. If he misses his throw the first time, there remains to him yet one of them, that which is worth to him $\frac{1}{36}a$ according to that which we just said. But he has only one chance to throw 2 sixes the first time against 35 chances to miss his throw. There is therefore at the beginning 1 chance to obtain a and 35 chances to obtain $\frac{1}{36}a$, this which is worth to him $\frac{71}{1296}a$ according to the second Proposition. There remains $\frac{1225}{1296}a$ for the one who engages him to throw. We are able to find starting there the chance or the share of the one who plays to four throws; one is able to pass over the case of the game in 3 throws.

Indeed, the one who plays to 4 throws, obtains a, if he throws 2 sixes one of the first two times; otherwise, there remains to him still 2 throws, that which is worth to him $\frac{71}{1296}a$ according to the preceding calculation. But according to the same calculation he has 71 chances to throw 2 sixes one of the first two times, against 1225 chances to miss them. There is therefore in starting 71 chances to obtain a and 1225 chances to obtain $\frac{71}{1296}a$; that which, according to the second Proposition, is worth to him $\frac{178991}{1679616}a$. There remains

$\frac{1500625}{1679616}a$ for the one who plays against him. Their chances are therefore one to the other as 178991 is to 1500625.

Departing from there, one finds in the same manner the chance of the one who wagers to throw one time 2 sixes in 8 throws. Next, from it hence further, the chance of the one who plays with 16 throws. And hence from the chance of this last, joined to that of the one who plays with 8 throws, one finds the chance of the one who plays to 24 throws. In this calculation, as the concern is mainly to seek for what number of throws the chances of the two players begin to become equal, one is able to skip one part of the last Digits of the numbers which without this would become very great. I find that the one who plays to 24 throws has still a slight disadvantage, and that one is not able to accept the game with advantage only in playing to 25 throws at least.

PROPOSITION XII.

To find the number of dice with which one is able to accept to throw 2 sixes on the first throw.

This is equivalent to wishing to know how many throws of one die alone one is able to count to throw two times a 6. According to that which we have demonstrated above, the one who would accept to throw 2 sixes in 2 throws, would have right to $\frac{1}{36}a$.

As for the one who would play to 3 throws, if his first throw was not a 6, there would remain to him still 2 throws which both must be 6; this as we have said to be worth $\frac{1}{36}a$. But if his first throw is a 6, it is no longer necessary for him but to throw a single six in the two following throws, that which according to the tenth Proposition is worth to him $\frac{11}{36}a$. Now, it is certain that there is 1 chance to throw a 6 on the first throw against 5 chances to miss it. There is therefore at the beginning 1 chance to obtain $\frac{11}{36}a$ and 5 chances to obtain $\frac{1}{36}a$, that which according to the second Proposition is worth $\frac{16}{216}a$ or $\frac{2}{27}a$ to him. Taking thus each time one throw more, one finds that one is able to accept with advantage to throw 2 sixes with a die in 10 throws or with 10 dice in one throw.

PROPOSITION XIII.

Under the hypothesis that I play a throw with two dice against another person with the condition that if there comes 7 points, I will have won, but that she will have won if there come 10 from it, and that we will divide the stake into equal parts if there comes another thing, to find the share which reverts to each of us.

As among the 36 throws that one is able to make with 2 dice, there are 6 with 7 points and 3 with 10 points, there remain 27 of them in which make win neither the one nor the other win. In this last case, we have each right to $\frac{1}{2}a$. But otherwise, I have 6 chances to win, that is to say to have a, and 3 chances to lose, that is to say to have 0; that which according to the second Proposition is worth $\frac{2}{3}a$ to me for this case. I have therefore at the beginning 27 chances to have $\frac{1}{2}a$ and 9 chances to have $\frac{2}{3}a$; that which according to the second Proposition is worth $\frac{13}{24}a$ to me. And there remains $\frac{11}{24}a$ for the other player.

PROPOSITION XIV.

If another player and I throw turn by turn 2 dice with the condition that I will have won as soon as I will have thrown 7 points and he as soon as he will have thrown 6, while I permit him the first throw, to find the ratio of my chance to his.

Let x be the value of my chance, and a the stake. The chance of the other player has therefore the value $a - x$. It is evident also that each time that it is his turn to throw, my chance will be anew the value x. But each time that it is my turn to throw, my chance must have a superior value, put y. Now, seeing that among the 36 throws that one is able to make with 2 dice, there are 5 which are able to give 6 points to my adversary and to make him win the game, and 31 throws to his disadvantage, that is to say that would bring forth my turn to throw, I have 5 chances to have 0 when he throws the first time, and 31 chances to have y; that which according to the third Proposition,[17] is worth to me $\frac{31y}{36}$. But we have put that my chance was worth x at the beginning the game. So that $\frac{31y}{36} = x$, hence $y = \frac{36x}{31}$. We have put besides that my chance is worth y, when it is my turn to throw. But when I throw, I have 6 chances to have a, seeing that there are 6 throws of 7 points which make me win; and I have 30 chances to make the turn of my adversary return, that is to say to have for my share x. The value y is therefore equivalent to 6 chances to have a and 30 chances to have x; this which, according to the third Proposition, is worth $\frac{6a+30x}{36}$ to me. This expression being equal therefore to y, and y according to that which precedes to $\frac{36x}{31}$, it is necessary that $\frac{30x+6a}{36}$ be equal to $\frac{36x}{31}$, whence one draws $x = \frac{30a}{61}$, the value of my chance. Consequently, the chance of my adversary would be $\frac{30a}{61}$. The ratio of our chances is therefore 31 to 30.

I end by making some Propositions yet follow.

I. A and B play together with 2 dice with the following condition: A will have won if he throws 6 points, B if he throws 7 of them. A will make first a single throw; next B 2 successive throws; then anew A 2 throws, and thus in sequence, until the one or the other will have won. One demands the ratio of the chance of A to that of B? Answer: as 10355 is to 12276.[18]

II. Three players A, B and C take 12 tokens of which 4 are white and 8 black; they play with this condition that the one will win who will have first, in choosing blindly, drawn a white token, and that A will choose first, B next, then C, then anew A and, thus the rest, in rotation. One demands the ratio of their chances?[19]

III. A wagers against B, that of 40 cards, of which 10 of each color, he will draw 4 of them in a way to have one of each color. One finds in this case that the chance of A is to that of B as 1000 is to 8139.[20]

[17]This is the only time that Proposition III is cited in the present treatise.

[18]One owes this problem to Fermat (Letter No. 301). After having solved Prop. XIV of Huygens, transmitted to him through the intermediary Carcavy (Letter No. 297), Fermat will invent this more complicated problem, which he made to arrive to Huygens in the same way (see the letter of 22 June 1656 of Carcavy, Letter No. 302 and the Appendix T. XIV). Huygens will send his solution to Carcavi by his letter of 6 July 1656, Letter No. 308. There are solutions by Montmort, Bernoulli, Struyck and Spinoza.

[19]In the "Ars Conjectandi", Jacques Bernoulli distinguishes three different interpretations that one is able to state on this problem. In first place one is able to suppose that each black token which has been drawn is put back among the other tokens, such that the players are always to choose among 4 white and 8 black tokens. In second place, when the tokens are not replaced, one is able to suppose that the players take all together 12 tokens, as well as each player take 12 tokens in order to draw one until it is his turn to play.

Huygens has in mind the first of these three interpretations, while Hudde adopts the second in his solution which he sent to Huygens in 1665. Bernoulli solves the problem for each of the three interpretations. Monmort, deMoivre and Struyck occupy themselves all with the first and second interpretations, the third is indeed a little forced, and giving the solutions to which they correspond.

[20]This problem is due to Fermat; see Letter No. 301. Huygens in his letter to Carcavi gives the numerical solution without adding his analysis (Letter No. 308). There are solutions by Monmort, Bernoulli and Struyck.

IV. One takes as above 12 tokens of which 4 are white and 8 black. A wagers against B that among 7 tokens that he will draw from them blindly, there will be found 3 white. One demands the ratio of the chance of A to that of B.[21]

V. Having taken each 12 tokens, A and B play with 3 dice on this condition that to each throw of 11 points, A must give a token to B, but that B must give 1 to A on each throw of 14 points, and that the one there will win who will be the first in possession of all the tokens. One finds in this case that the chance of A is to that of B as 244140625 is to 282429536481.[22]

[21] The problem admits two different interpretations. Solved by Monmort, de Moivre, Bernoulli and Struyck.

[22] This problem was proposed by Pascal to Fermat. Huygens received communication in a letter of Carcavi, dated 28 September 1656, (Letter No. 336.) He sent the solution to Carcavi on 12 October 1656, (Letter No. 342) Later, probably in 1676, Huygens will occupy himself anew with the problem and will elaborate a solution (Appendix VI). Bernoulli, Monmort, de Moivre and Struyck each solved the problem.