Solution to the fourth problem exchanged between Huygens and Hudde

The players, A and B, cast a fair coin in a game of Heads or Tails with A being first to cast. Thus the sequence of plays proceed as ABABAB... At the beginning of the game, the pot is empty. If a Tail is cast, the player must put a ducat into the pot. If a Head is cast, the player takes a ducat from the pot if it is not empty. Eventually a ducat must be placed into the pot. The game then continues until the pot is empty.

The game must always end in an even number of trials beginning from the cast of the first Tail. For example, there is only one game of length 2, `TH`; one game of length 4, `TTHH`; and two games of length 6, `TTTHHH` and `TTHTHH`.

If Player A should cast the first Tail, the sequence `TTTHHH` is worth -1 ducat to Player A since he would deposit 1 ducat with each Tail and receive 1 ducat with the Head. On the other hand, the sequence `TTHTHH` is worth 1 ducat to Player A since he would deposit 1 ducat with the Tail and receive 2 ducats for the Heads. Since sequences of the same length are equiprobable, the outcomes of length 6 are worth on average 0 to Player A. This symmetry must prevail for all sequences. *Therefore, the expected value of game to Player A, given he has cast the first Tail, is 0.*

Consequently, the expected value of the game to Player A is determined solely by the time at which the first Tail is cast.

If A casts the first Tails then it occurs with one of the following sequence of casts: `T, HHT, HHHHT, HHHHHHT, ....` Now with these A pays 1 ducat with probability $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots$ respectively. Therefore the expected *loss* of A in this case is

$$\sum_{i=0}^{\infty} \frac{1}{2^{(2\,i+1)}} = \frac{2}{3}.$$

If B casts the first Tails, then it must occur with one of the following sequences of casts: `HT, HHHT, HHHHHT, ....` With these B loses 1 ducat with probability $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$ respectively. Therefore the expected *loss* of B is

$$\sum_{i=1}^{\infty} \frac{1}{4^i} = \frac{1}{3}.$$

Let E_A and E_B denote the respected expected values of Players A and B before any money has been deposited. Clearly $E_A + E_B = 0$. The position of Player A, if Player A has just cast the first Tail, equals the position of Player B, if Player B has just cast the first Tail. Therefore

$$E_A + \frac{2}{3} = E_B + \frac{1}{3}.$$

Combining these equalities yields $E_A = -1/6$.