Solution to the third problem exchanged between Huygens and Hudde

The players, A and B, cast a fair coin in a game of Heads or Tails with A being first to cast. Thus the sequence of plays proceed as `ABABAB...`. If a Tails is cast, the player must put a ducat into the pot. If a Heads is cast, the player takes the entire pot. At the beginning of the game, the pot contains an unknown quantity K, each player contributing half. We seek K so as to make the game fair.

Therefore, if, say, the sequence of plays `TTTTH` were observed, Heads was cast by A, B had put 2 ducats into the pot and so A gains 2 ducats. If, say, the sequence `TTTH` were observed, Heads was cast by B, A had put 2 ducats into the pot and so A loses 2 ducats.

Solution: Suppose A casts the first Heads. It must occur with one of the following sequence of casts: `H, TTH, TTTTH, ...` Now with these A gains $\frac{K}{2}, \frac{K}{2}+1, \frac{K}{2}+2\ldots$ with probability $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots$ respectively. Therefore the expected gain of A in this case is

$$\sum_{i=0}^{\infty} \frac{\frac{K}{2}+i}{2^{(2\,i+1)}} = \frac{K}{3} + \frac{2}{9}.$$

If B casts the first Heads, then it must occur with one of the following sequences of casts: `TH, TTTH, TTTTTH, ...` With these A loses $\frac{K}{2}+1, \frac{K}{2}+2, \frac{K}{2}+3, \ldots$ with probability $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$ respectively. Therefore the expected loss of A is

$$\sum_{i=1}^{\infty} \frac{\frac{K}{2}+i}{4^{i}} = \frac{K}{6} + \frac{4}{9}.$$

In order that the game be fair, it is necessary that these two quantities be equal. Thus $K = \frac{4}{3}$. Each player must set $\frac{2}{3}$ into the game.