Solution to the second problem exchanged between Huygens and Hudde[1]

The players, A and B, draw repeatedly with replacement from individual sets of White and Black tokens. A is the first to draw. Thus the sequence of plays proceed as `ABABAB...`. A has a set of 2 White and 1 Black. B has c White tokens and d Black. If a Black is drawn, the player must put a ducat into the pot. If a White is drawn, the player takes the entire pot. At the beginning of the game, the pot is empty.

Therefore, if, say, the sequence of plays `BBBBW` were observed, White was drawn by A, B had put 2 ducats into the pot and so A gains 2 ducats. If, say, the sequence `BBBW` were observed, White was drawn by B, A had put 2 ducats into the pot and so A loses 2 ducats.

Let $p = \frac{c}{c+d}$ be the proportion of White tokens for Player B and let $q = 1 - p$.

Hudde's solution:

Hudde assumed that if the game began with the drawing of White, the game was over with no gain to A. Thus all games must necessarily proceed as a sequence of Blacks followed by a single White.

The sequences `BW, BBBW, BBBBBW,...` result in losses to A of 1, 2, 3, ... ducats respectively and occur with probabilities $\frac{p}{3}, \frac{pq}{9}, \frac{pq^2}{27}, \ldots$ and therefore the sequences of such type result in an expected loss to A of

$$\sum_{k=1}^{\infty} \frac{kpq^{k-1}}{3^k} = \frac{3p}{(p+2)^2}.$$

The sequences `BBW, BBBBW, BBBBBBW,...` result in gains to A of 1, 2, 3,... ducats respectively and occur with probabilities

$$\frac{2q}{9}, \frac{2q^2}{27}, \frac{2q^3}{81} \ldots$$

and therefore, the sequences of such type result in an expected gain to A of

$$\sum_{k=1}^{\infty} \frac{2kq^k}{3^{k+1}} = \frac{2q}{(q-3)^2} = \frac{2(1-p)}{(p+2)^2}.$$

Consequently, Player A has an expected gain of $\frac{2-5p}{(2+p)^2}$. That is, Player A should expect to lose this quantity.

In order that this be a fair game, we must have this expected gain equal to 0. So $p = \frac{2}{5}$.

[1]Prepared by Richard Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. February 7, 2009

Generalization of Hudde's version of the problem and its solution:

Let Player A have a White tokens and b Black tokens. Put $P = \frac{a}{a+b}$ and $Q = 1 - P$. By the above, it is easy to see that the expected loss of A for sequences of the form BW, BBBW, BBBBBW, ..., where B draws the White, is

$$\sum_{k=1}^{\infty} kQp(Qq)^{k-1} = \frac{pQ}{(Qq-1)^2}$$

and that the expected gain of A for sequences of the form BBW, BBBBW, BBBBBBW, ..., where A draws the White, is

$$\sum_{k=1}^{\infty} kP(Qq)^{k} = \frac{PQq}{(Qq-1)^2}.$$

Thus the game is fair if $PQq = pQ$.

This is the solution of Hudde: $P = \frac{p}{q}$. We conclude $p = \frac{P}{P+1}$.

Huygens' solution:

Huygens assumed that the game continued until at least one ducat had been put into the pot. Thus, in addition to the sequences discussed above we must also include the sequences of the form W...WB...BW. That is, sequences which consist of an initial sequence of one or more Whites, followed by a sequence of one or more Blacks and then terminating with a White.

These sequences can be divided into two types: (1) those where an even number of Whites are observed before the first Black and (2) those where an odd number are observed. We proceed directly to the general case.

Type 1. Here the first Black is drawn by A. In the case, the expected gain to A, when this Black is cast, is

$$\frac{PQq - pQ}{(Qq-1)^2}$$

by the previous. Therefore, the expected gain to A for this type is the sum of $\frac{PQq-pQ}{(Qq-1)^2}$ times the probability of each initial sequence of Heads. In other words, $\frac{PQq-pQ}{(Qq-1)^2}$ times the sum $\sum_{k=0}^{\infty}(Pp)^k$. This latter sum is $\frac{1}{1-Pp}$ and so the expected gain is

$$\frac{Q(Pq-p)}{(Qq-1)^2(1-Pp)}.$$

Type 2. In this case the first Black is drawn by B. If we ignore the initial sequence of Whites but for the last one, the sequence of draws must be of the form WB...BW. With the sequences WBW, WBBBW, WBBBBBW, ..., Player A draws the White and receives 1, 2, 3,... ducats with probability $P^2q, P^2Qq^2, P^2Q^2q^3, \dots$ respectively. With the sequences WBBW, WBBBBW, ... Player B draws the White and Player A must pay 1, 2, ... with probability $PQqp, PQ^2q^2p, \dots$ respectively. It therefore follows that the expected gain of Player A, given that Player B draws the first Black immediately after A's draw of White is

$$\sum_{k=1}^{\infty} P^2kQ^{(k-1)}q^k - \sum_{k=1}^{\infty} Ppk(Qq)^k = \frac{Pq(P-pQ)}{(Qq-1)^2}.$$

We must again multiply by $\frac{1}{1-Pp}$ to account for the initial sequence of Whites. Thus the expected gain is

$$\frac{Pq(P-pQ)}{(Qq-1)^2(1-Pp)}.$$

If now, the expected gains of the two types are combined, we obtain

$$\frac{Pq(P-pQ)+Q(Pq-p)}{(Qq-1)^2(1-Pp)}.$$

Thus, the game is fair if this quantity vanishes.

Replacing Q and q in the numerator by the corresponding expressions in P and p, we obtain a quadratic in p, namely, $(-P^2+P)p^2+(-P+P^2-1)p+P$. Solving for p we obtain the general solution

$$\frac{(1+P-P^2)+(-5P^2+2P^3+2P+P^4+1)^{\frac{1}{2}}}{2(P^2-P)}.$$

Substituting 2/3 for P yields $p=\frac{11}{4}-\frac{\sqrt{73}}{4}$ or approximately 0.614. Consequently the ratio of p to q is $\frac{1+\sqrt{73}}{6}$. The 8th convergent of the continued fraction expansion of this ratio is $\frac{1193}{750}$.

Under the solution of Hudde, the expectation of A is $\frac{PQq-pQ}{(Qq-1)^2}$ and under the solution of Huygens it is $\frac{Pq(P-pQ)+Q(Pq-p)}{(Qq-1)^2(1-Pp)}$. We can examine the expectation of A for different choices of P and p.

For example, if A has two White and 1 Black tokens and B has 1 White and 2 Black tokens, then we have $P=\frac{2}{3}$ and $p=\frac{1}{3}$ (and, of course, $Q=\frac{1}{3}$ and $q=\frac{2}{3}$). Hudde's expression[2] gives $\frac{3}{49}$ and Huygens' gives $\frac{207}{343}$. If A has 10 White and 1 Black and B has 10 White and 11 Black, then Hudde's expression gives 0 and Huygens' $\frac{105}{131}$.

[2]Hudde gives 9/245 in his correspondence, but this is clearly erroneous.