Solution to the first problem exchanged between Huygens and Hudde

The players, A and B, cast a fair coin in a game of Heads or Tails with A being first to cast. Thus the sequence of plays proceed as `ABABAB...`. If a Tails is cast, the player must put a ducat into the pot. If a Heads is cast, the player takes the entire pot. At the beginning of the game, the pot is empty.

Therefore, if, say, the sequence of plays `TTTTH` were observed, Heads was cast by A, B had put 2 ducats into the pot and so A gains 2 ducats. If, say, the sequence `TTTH` were observed, Heads was cast by B, A had put 2 ducats into the pot and so A loses 2 ducats.

Hudde's solution:

Hudde assumed that if the game began with the cast of a Heads, the game was over with no gain to A. Thus all games must necessarily proceed as a sequence of Tails followed by a single Heads.[1]

The sequences `TH, TTTH, TTTTTH,...` result in losses to A of 1, 2, 3,... ducats respectively and occur with probabilities

$$\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \ldots$$

and therefore, the sequences of such type result in an expected gain to A of

$$\sum_{k=1}^{\infty} \frac{k}{4^k} = -\frac{4}{9}.$$

The sequences `TTH, TTTTH, TTTTTTH,...` result in gains to A of 1, 2, 3,... ducats respectively and occur with probabilities

$$\frac{1}{2^3}, \frac{1}{2^5}, \frac{1}{2^7}, \ldots$$

and therefore, the sequences of such type result in an expected gain to A of

$$\sum_{k=1}^{\infty} \frac{k}{2^{2k+1}} = \frac{2}{9}.$$

Consequently, Player A has an expected gain of $-2/9$. That is, Player A should expect to lose this quantity.

Huygens' solution:

Huygens assumed that the game continued until at least one ducat had been put into the pot. Thus, in addition to the sequences discussed above we must also include the sequences of the form `H...HT...TH`. That is, sequences which consist of an initial sequence of one or more Heads, followed by a sequence of one or more Tails and then terminating with a Head.

These can be divided into two types: (1) those where an even number of Heads are observed before the first Tails and (2) those where an odd number are observed.

Type 1. Here the first Tail is cast by A. In this case, the expected gain to A, when this Tail is cast, is $-\frac{2}{9}$ by the previous. Therefore, the expected gain to A for

[1]Initially, Hudde also assumed that each player would stake at first opportunity but 1 ducat. Under this hypothesis he correctly derived the expectation of player A to be $-\frac{1}{6}$.

this type is the sum of $-\frac{2}{9}$ times the probability of each initial sequence of Heads. In other words, $-\frac{2}{9}$ times the sum

$$\sum_{k=0}^{\infty} \frac{1}{4^k}.$$

This latter sum is $\frac{4}{3}$ and so the expected gain is $-\frac{8}{27}$.

Type 2. In this case the first Tail is cast by B. If we ignore the initial sequence of Heads but for the last one, the sequence of casts must be of the form `HT...TH`. With the sequences `HTH, HTTTH,...` Player A receives 1, 2,... ducats with probability

$$\frac{1}{2^3}, \frac{1}{2^5}, \ldots$$

respectively. With the sequences `HTTH, HTTTH,...` Player A must pay 1, 2,... with probability

$$\frac{1}{2^4}, \frac{1}{2^5}, \ldots$$

respectively. It therefore follows that the expected gain of Player A, given that Player B casts the first Tails immediately after his cast of Heads is

$$\sum_{k=1}^{\infty} \frac{k}{2^{2k+1}} - \sum_{k=1}^{\infty} \frac{k}{4^{k+1}} = \frac{1}{9}.$$

We must again multiply by $\frac{4}{3}$ to account for the intial sequence of Heads. Thus the expected gain is $\frac{4}{27}$.

If now, the expected gains of the two types are combined, we obtain $-\frac{4}{27}$.