Solution to Huygen's Second Exercise[1]

There are 12 counters of which 8 are Black and 4 White. The first player to draw a White wins. Here the turns proceed as `ABCABC....` There are at least two interpretations of the problem.

1. Huygens assumed drawing *with replacement*. On each draw, White comes forth with probability $p = 1/3$ and Black with probability $q = 2/3$. Player A wins if the sequence of draws proceed as `W, BBBW, BBBBBBW, ...` Their respective probabilities are $p, q^3p, q^6p, \ldots$ Thus the probability that Player A wins is given by

$$\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{3k} = \frac{9}{19}.$$

In like manner, Player B wins if the sequence of draws proceed as `BW, BBBBW, ...` It is easy to see that the probability that B wins is given by

$$\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{3k+1} = \frac{6}{19}.$$

Finally, Player C wins if the sequence of draws proceed as `BBW, BBBBBW, ...` so that the probability that C wins is given by

$$\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{3k+2} = \frac{4}{19}.$$

Of course, these three probabilities must sum to 1. Thus the ratio of chances is as $9 : 6 : 4$.

2. Hudde assumed drawing *without replacement*. Player A wins if one of the three sequences `W, BBBW` or `BBBBBBW` occur. The probability of this event is

$$\frac{4}{12} + \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{4}{9} + \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8}\cdot\frac{3}{7}\cdot\frac{4}{6} = \frac{231}{495}.$$

Player B wins if one of the three sequences `BW, BBBBW, BBBBBBBW` occur. The probability of this event is

$$\frac{8}{12}\cdot\frac{4}{11} + \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8} + \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8}\cdot\frac{3}{7}\cdot\frac{2}{6}\cdot\frac{4}{5} = \frac{159}{495}.$$

Finally, Player C wins if one of the three sequences `BBW, BBBBBW` or `BBBBBBBBW` occur. The probability of this event is

$$\frac{8}{12}\cdot\frac{7}{11}\cdot\frac{4}{10} + \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8}\cdot\frac{4}{7} + \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8}\cdot\frac{3}{7}\cdot\frac{2}{6}\cdot\frac{1}{5}\cdot\frac{4}{4} = \frac{105}{495}.$$

Thus the ratio of chances is as 231 : 159 : 105 or 77 : 53 : 35.

3. Another variation is to assume each player has his own set of tokens from which he draws without replacement.

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