"A Problem concerning Chance in Play"

Leonhard Euler*

Adversaria mathematica† H4, 1740–1750?, pp. 186–187

A has a coins, but B b coins. It is being thrown with dice, and as often as the throw occurs, of which the probability of appearing is $\frac{1}{m}$, then B surrenders a coin to A; but as often as the different throw happens, of which the probability of appearing is $\frac{1}{n}$, then A surrenders a coin to B. With this condition they contend so long, until they give up all coins to one or the other, who then gains 1 deposit. It is demanded before the contest begins the expectation of both of them.

Answer

The expectation of A will be

$$= \frac{n^b(m^a - n^a)}{m^{a+b} - n^{a+b}}$$

The expectation of B will be

$$= \frac{m^a(m^b - n^b)}{m^{a+b} - n^{a+b}}$$

Therefore the chance of A : the chance of B will be as $n^b(m^a - n^a) : m^a(m^b - n^b)$.

Solution

With regard to the position of A the expectation of the received coin to the expectation of the lost coin is as n to m. This contest is continuing,

as long as A has coins	his expectation
0	0
1	$\alpha = \frac{n\beta + m0}{m+n}$
2	$\beta = \frac{n\gamma + m\alpha}{m+n}$
3	$\gamma = \frac{n\delta + m\beta}{m+n}$
4	$\delta = \frac{n\epsilon + m\beta}{m+n}$
5	ϵ
$\vdots$	$\vdots$
$a + b$	1

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†Mathematical notebook

In this sequence three repeating terms are X, Y, Z. There will be

$$nZ = (m+n)Y - mX$$

and there will be exactly the recurrent sequence; to which let be set

$$nZZ = (m+n) - m,$$

to be found. Which roots are

$$Z = 1 \text{ and } z = \frac{m}{n};$$

therefore the term will be the answer to general index x

$$= A + B\left(\frac{m}{n}\right)^x;$$

Let $x = 0$ be put; there will be

$$A + B = 0$$

Let $x = a+b$ be put; there will be

$$A + B\left(\frac{m}{n}\right)^{a+b} = 1$$

and hence

$$A = \frac{1}{1 - \left(\frac{m}{n}\right)^{a+b}} = \frac{-n^{a+b}}{m^{a+b} - n^{a+b}}$$

and

$$B = \frac{n^{a+b}}{m^{a+b} - n^{a+b}}$$

Wherefore as long as A has x coins, his lot will be

$$= \frac{n^{a+b}}{m^{a+b} - n^{a+b}}\left(\frac{m^x - n^x}{n^x}\right),$$

or the lot of A will be

$$= \frac{n^{a+b-x}(m^x - n^x)}{m^{a+b} - n^{a+b}}.$$

Wherefore from the beginning, provided that A has a coins, his expectation will be

$$= \frac{n^b(m^a - n^a)}{m^{a+b} - n^{a+b}}$$

Certainly the expectation of B will be

$$= 1 - \frac{n^b(m^a - n^a)}{m^{a+b} - n^{a+b}} = \frac{m^{a+b} - m^a n^b}{m^{a+b} - n^{a+b}}$$

Q.E.I.

Example

A cast with two dice favorable to A IX, 4 ways,
A cast with two dice favorable to B VII, 6 ways,
therefore $m : n = 3 : 2$. Let $b = 2$, there will be

$$\text{lot of A : lot of B} = 2^2(3^a - 2^a) : 3^a(3^2 - 2^2) = (4 \cdot 3^a - 4 \cdot 2^a) : 5 \cdot 3^a$$

If $b = 2$ and $a = \infty$, there will be

$$\text{chance of A : chance of B} = 4 : 5$$

Therefore, even if A assumes countless coins, nevertheless the lot of B will be better.

Scholion

In order that the expectations of the two are equal, there must be

$$m^a n^b - n^{a+b} = m^{a+b} - m^a n^b$$

or

$$2m^a n^b = m^{a+b} + n^{a+b};$$

and if the ratio $m : n$ is given, there will be

$$a = \frac{b \ln n - \ln(2n^b - m^b)}{lm - ln}$$

Let $m = 9, n = 5, b = 12$; there will be

$$a = \frac{12 \ln 5 - \ln(2 \cdot 5^{12} - 9^{12})}{\ln 9 - \ln 5}$$

Therefore it is understood, in order that the equality of the lots is able to appear, it is necessary, that there be

$$2n^b > m^b \text{ or } 2\left(\frac{m}{n}\right)^b > 1$$

that is

$$l2 - b \ln \frac{m}{n} > 0$$

and therefore

$$b < \frac{\ln 2}{\ln m - \ln n}$$