The Solution of a curious Question in the Science of Combinations

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I. The question, which I undertake to explain here, is revealed thus: *Given a sequence of however many letters a, b, c, d, e etc., let the number of which be n, to find in how many ways the order of them can be changed, so that none of them are found in the same position, which it occupied in the beginning.*

It is manifest immediately, that if the last condition is omitted and the number of all permutations completely is sought exactly, it will be the product of all numbers from unity up to n. Moreover, it is proper all those orders to be excluded, where any letter was occupying its initial position, whence the number of permutations, which I seek, is less than $1 \cdot 2 \cdot 3 \cdots n$.

II. In order that we may investigate into the solution of this question, we consider first the simplest case, from which method next we deduce the solution for a large number of letters as many as you wish to be drawn. And indeed first, if only the letter a be put forward, it is evident to have no variant position. To put forward two letters ab it has one variation in position, namely ba. Moreover for three letters abc only two variations are able to be given, which are

$$bca, \quad cab.$$

While if four letters $abcd$ be given, three cases occur here, in which either b or c or d obtain the first position; of the cases therefore, in which b is located in the first position, they admit three variations of the remaining three, which are adc, dac, cda; therefore just as many variations likewise will be had, if the first position is assigned to the letter c, than d is assigned to the first position, and thus they are able to have in all nine variations of position, which are:

$$\begin{array}{c|c|c} badc & cadb & dabc \\ dbac & cdab & dcab \\ bcda & cdba & dcba \end{array}$$

III. We unfold in the same manner the case of five letters $abcde$, where the first position is able to hold either b or c or d or e. Therefore b would occupy the first position,

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH but we give the second position to the letter a and to the remaining three, c, d and e, they admit two variations, which are $badec$, $baecd$. But when the third seat is assigned to this a, they admit three variations of the remaining three, which therefore we represent $bcaed$, $bdaec$, $beacd$. In like manner if the fourth seatis assigned to the letter a, now too they have three variations of positions, which are $bedac$, $bcead$, $bdeac$. Finally if the fifth position is assigned to the letter a, the three variations will be following: $bcdea$, $bdeca$, $bedca$. Therefore provided that the first position is given to the letter b, eleven variations will be given in all; just as many indeed besides occur, if either c or d or e are placed in the first position. Whence we conclude to have in all four times eleven or 44 variations of positions for the five letters $abcde$.

IV. But if on the other hand we wish to advance to more letters in similar manner, enumeration of all cases would become exceedingly difficult and laborious, moreover hazardous; whence it will be investigated by us in a sure method, of which by aid the number of variations would always be able to be assigned accurately, the number of letters may be as great as is wished.

To this end it will help greatly to name a suitable character for assistance, by which number the multitude of all variations for whatever number of letters proposed is indicated. Therefore let this character

$$\mathit{\Pi} : n$$

denote the number of all variations, which n letters admit, and since the cases, in which n is either 1, or 2, or 3, or 4, or 5, we now have obtained, since now we have known to be

$$\mathit{\Pi} : 1 = 0, \quad \mathit{\Pi} : 2 = 1, \quad \mathit{\Pi} : 3 = 2, \quad \mathit{\Pi} : 4 = 9 \quad \text{and} \quad \mathit{\Pi} : 5 = 44;$$

whence it it is clear by progressing further, the number of variations soon to extend to an immense degree.

V. Now let us seek with this character established immediately in general the number of all variations for the number of letters $= n$, which therefore will be $\mathit{\Pi} : n$; where the entire trouble with it reverts back, in order that it may be investigated, in what way that required number is constructed from the preceding, which are

$$\mathit{\Pi} : (n-1), \quad \mathit{\Pi} : (n-2), \quad \mathit{\Pi} : (n-3) \quad \text{etc.}$$

Indeed let us establish the reasoning in a similar way, for which purpose we previously made use. First namely we will consider the cases, where the letter b is located in the first position: for it is easily understood, how many variations they will have produced for this case, likewise just as many are going to be produced, if any other letter is located in the first position; whence it is understood, whatever number of variations had been found, as long as the letter b obtains the first position, it multiplied by $n-1$ is going to provide the number of all possible variations, and therefore the value of the character $\mathit{\Pi} : n$.

VI. However here meet two cases to be unfolded, according as the letter a either holds the second position or some other. Therefore let us locate a in the second position and it will be investigated, how many variations of the remaining letters c, d, e, f etc. are going to be admitted; of which seeing that the number is $n-2$, the number of variations by hypothesis will be $\Pi : (n-2)$. Let us further position a in the third position or any other position, and now the the question arises, how many variations the letters b, c, d, e, f etc. admit; where it is noted in the variations of them the letter b is not able to occur further, because now it occupies the first position, but with the position of it with regard to the variations to enter the letter a; and thus it will in the same way, and if with the first position rejected, the variations of the letters a, c, d, e, f etc. may be sought; since the number of which is $n-1$, the multitude of all variations by hypothesis will be $\Pi : (n-1)$. Consequently, as long as the letter b is located in the first position, the number of all variations will be

$$\Pi : (n-2) + \Pi : (n-1).$$

VII. Now it is manifest by itself each just as many variations are about to be produced, if any whatsoever of the remaining letters is written in the first position; whereby seeing that of all these letters, the first a excluded, the number is $n-1$, the number of all variations will be completely

$$(n-1)\Pi : (n-2) + (n-1)\Pi : (n-1),$$

which therefore is the value of the sought formula $\Pi : n$, thus it is

$$\Pi : n = (n-1)\Pi : (n-1) + (n-1)\Pi : (n-2)$$

or

$$\Pi : n = (n-1)(\Pi : (n-1) + \Pi : (n-2)).$$

And thus of the two characters immediately preceding the highest, namely

$$\Pi : (n-1) + \Pi : (n-2),$$

multiplied by $n-1$, always will give the following character $\Pi : n$, of which with the help of the rule the progression, how the numbers of variations are established individually for the numbers of letters, until it was pleasing, is able to be continued easily.

VIII. Because for which reason it may be evident more easily, let us begin with the simplest cases and let us exhibit the values of the character $\Pi : n$ in the following table:

$$\begin{aligned}
\Pi : 3 &= 2(\Pi : 2 + \Pi : 1) = 2 \cdot (1+0) = 2,\\
\Pi : 4 &= 3(\Pi : 3 + \Pi : 2) = 3 \cdot (2+1) = 9,\\
\Pi : 5 &= 4(\Pi : 4 + \Pi : 3) = 4 \cdot (9+2) = 44,\\
\Pi : 6 &= 5(\Pi : 5 + \Pi : 4) = 5 \cdot (44+9) = 265,\\
\Pi : 7 &= 6(\Pi : 6 + \Pi : 5) = 6 \cdot (265+44) = 1854,\\
\Pi : 8 &= 7(\Pi : 7 + \Pi : 6) = 7 \cdot (1854+265) = 14833,\\
\Pi : 9 &= 8(\Pi : 8 + \Pi : 7) = 8 \cdot (14833+1854) = 133496,\\
\Pi : 10 &= 9(\Pi : 9 + \Pi : 8) = 9 \cdot (133496+14833) = 1334961.
\end{aligned}$$

IX. Let us arrange these numbers $\Pi : n$, according to their indices n, in the following sequence:

n	1,	2,	3,	4,	5,	6,	7,	8,	9,
$\Pi : n$	0,	1,	2,	9,	44,	265,	1854,	14833,	133496.

And if now we should consider this sequence more carefully, we will discover an exceptional relation, by which whatever number is referred back to the preceding, as the following table reveals:

$$\begin{array}{rl} 2 = 3 \cdot 1 & -1, \\ 9 = 4 \cdot 2 & +1, \\ 44 = 5 \cdot 9 & -1, \\ 265 = 6 \cdot 44 & +1, \\ 1854 = 7 \cdot 265 & -1, \\ 14833 = 8 \cdot 1854 & +1, \\ 133496 = 9 \cdot 14833 & -1, \\ \text{etc.} & \end{array}$$

Therefore by the great benefit of such observation it is permitted to continue our progression more easily, provided that whatever limit is always a certain multiple of the preceding, either increased or decreased by unity; and so in general there will be

$$\Pi : n + n\Pi : (n-1) \pm 1.$$

Where it is observed the $+$ sign to prevail, if n was an even number, indeed the $-$ sign, when n was an odd number.

X. The remarkable will be seen, in what way of these two laws of progression hold between themselves; moreover from this latter law the prior is derived easily. For with put

$$\Pi : n = n\Pi : (n-1) \pm 1$$

still there will be in a similar way

$$\Pi : (n-1) = (n-1)\Pi : (n-2) \mp 1.$$

Let these two formulas be added, in order that the ambiguous signs $+$ or $-$ by much destroy themselves, and the sum will be

$$\Pi : n + \Pi : (n-1) = n\Pi : (n-1) + (n-1)\Pi : (n-2),$$

whence it follows to become

$$\Pi : n = (n-1)\Pi : (n-1) + (n-1)\Pi : (n-2),$$

which is the prior law of progression itself.

Certainly nonetheless it is not easy to derive the latter law from the former; yet nevertheless the thing will succeed, if certainly we begin with the simplest cases, by observing, with respect to which $\Pi : 1 = 0$ and $\Pi : 2 = 1$. For hence there will be

$$\Pi : 3 = 2\Pi : 2 = 3\Pi : 2 - 1,$$

whence there is

$$3\Pi : 2 = \Pi : 3 + 1.$$

Now since there is from the prior law: $\Pi : 4 = 3\Pi : 3 + 3\Pi : 2$, if here in the position $3\Pi : 2$ only the value discovered is substituted, there will appear

$$\Pi : 4 = 4\Pi : 3 + 1,$$

whence there is

$$4\Pi : 3 = \Pi : 4 - 1.$$

Now the next relation was $\Pi : 5 = 4\Pi : 4 + 4\Pi : 3$; where if in the position only the discovered value $4\Pi :\ 3$ is written, it will produce

$$\Pi : 5 = 5\Pi : 4 - 1$$

and therefore

$$5\Pi : 4 = \Pi : 5 + 1.$$

But the next relation is: $\Pi : 6 = 5\Pi : 5 + 5\Pi : 4$; in which if in the position of the latest part the value previously discovered is substituted, it will be

$$\Pi : 6 = 6\Pi : 5 + 1$$

and therefore

$$6\Pi : 5 = \Pi : 6 - 1,$$

which value substituted into the next relation $\Pi : 7 = 6\Pi : 6 + 6\Pi : 5$ it presents

$$\Pi : 7 = 7\Pi : 6 - 1$$

and therefore further; whence it is clear enough, in what way the latter law is derived from the prior.