General Researches on the Mortality and the Multiplication of the Human Race

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1. The registers of the births and of the deaths at each age, which are published in several places every year, provide so many of the different questions on the mortality and the multiplication of the human race, that it would be too lengthy to report all of them. Now, one depends for the most part in a way on the others, that having developed one or two of them, all the others themselves are equally determined. Since the solutions must be drawn from the mentioned registers, it is remarked that these registers differ greatly, according to the diversity of the towns, villages and provinces where they were prepared; and for the same reason, the solutions of all these questions are very different according to the registers on which they are found. This is why I propose me to treat herein in general the greater part of these questions, without limiting myself to the results that the registers of a certain place provide; and afterwards, it will be easy to make application to each part as one wishes.

2. Now, I observe first that all these questions taken in general depend on two hypotheses; which being well fixed, it is easy to draw from them the solution of all. I will name the first *the hypothesis of mortality*, by which one determines how many, of a certain number of men who are born at the same time, will be still alive after each number of elapsed years. Here, the consideration of multiplication does not enter at all in the computation, and it is therefore necessary to constitute the second hypothesis, that I will name the one of *multiplication*, and by which I indicate by how much the number of all men is increased or diminished during the course of one year. This hypothesis depends therefore on the quantity of marriages and on fertility, while the first is found on the vitality or the power of living, which is inherent in man.

I. HYPOTHESIS OF MORTALITY

3. For the first hypothesis, we conceive some number N of infants who were born at the same time; and I denote the number of them who will still be alive at the end of one year by $(1)N$, of those who will remain at the end of two years by $(2)N$, of three years by $(3)N$, of four years by $(4)N$ and thus in sequence. These are the general signs that I employ in order to denote how the number of men born at the same time decreases

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successively; which will have for each climate and each kind of life particular values. Nevertheless, one is able to remark that the numbers indicated by

$$(1), \quad (2), \quad (3), \quad (4), \quad (5) \quad \text{etc.}$$

constitutes a decreasing progression of fractions, of which the greatest, (1), is smaller than unity; and when one continues the terms to beyond 100, they decrease so strongly that they vanish near entirely. Because, if not one of 100 millions of men attains the age of 125 years, it is necessary that the term (125) be less than $\frac{1}{100\,000\,000}$.

4. Having established for a certain place by a sufficiently great number of observations the values of the fractions (1), (2), (3), (4) etc., one is able to solve a number of the questions that one proposes ordinarily on the probability of human life. First of all it is evident, if the number of infants born at the same time $= N$, then, according to probability, there die of them in all the years as many as this table denotes of them:

from	0	years to	1	there die of them	$N-(1)N$
"	1	"	2	"	$(1)N-(2)N$
"	2	"	3	"	$(2)N-(3)N$
"	3	"	4	"	$(3)N-(4)N$
"	4	"	5	"	$(4)N-(5)N$

And as of this number N there are again probably alive $(n)N$ at the end of n years, it is necessary that the number of deaths after this term of n years be $= N-(n)N$. After this remark, I will give the solution of the following questions.

1. QUESTION

5. *A certain number of men, of whom all are of the same age, being given, to find how many of them are probably yet alive after a certain number of years.*

We suppose that there are M men who have the same age of m years and that one demands, how many of them probably live yet after n years. Let one put

$$M = (m)N,$$

as to have $N = \frac{M}{(m)}$, where N denotes the number of all the infants born at the same time, of which there remain still alive M after m years. Now, of these same number there will be probably still alive $(m+n)N$ after $m+n$ years since their birth and therefore after n years since the time proposed. Therefore, the number sought in the question is

$$= \frac{(m+n)}{(m)}M;$$

or after n years, there will probably be still as many alive of M men who are all now m years old.

Therefore, it is probable that the number of men, M, all m years of age, there die of them

$$\left(1 - \frac{(m+n)}{(m)}\right)M,$$

before there elapse n years.

2. QUESTION

6. *To find the probability that a man of a certain age be still alive after a certain number of years.*

Let the man in question be m years old, and let one seek the probability that the man be alive at the end of n years. We conceive of M men of this same age, and since, after n years, there will be probably still alive $\frac{(m+n)}{(m)}M$, the probability that the proposed man finds himself in this number will be

$$= \frac{(m+n)}{(m)}.$$

Therefore, the probability that the man should come to die before the end of these n years is

$$1 - \frac{(m+n)}{(m)}.$$

And therefore, the aspiration that this man is able to have of not dying in the interval of the following $n+m$ years, is to the dread of dying in this same interval as $(m+n)$ to $(m)-(m+n)$. Thus, the aspiration will surpass the dread if $(m+n) > \frac{1}{2}(m)$, and the dread will be more founded if $(m+n) < \frac{1}{2}(m)$. Now, the dread will equal the aspiration if $(m+n) = \frac{1}{2}(m)$.

3. QUESTION

7. *One demands the probability that a man of a certain age will die in the course of a given year.*

Let the man in question be the age of m years, and let one require the probability that he attain the age of n years, but that he dies before he reaches to the age of $n+1$ years. In order to find this probability, we conceive a great number M of men of the same age, and having $M = (m)N$ and $N = \frac{M}{(m)}$, there will be $\frac{(n)}{(m)}M$ men who attain the age of n years and $\frac{(n+1)}{(m)}M$ who attain the age of $n+1$ years; there will die therefore probably in the course of this year

$$\frac{(n)-(n+1)}{(m)}M;$$

and therefore, the probability that the proposed man finds himself in this number will be

$$= \frac{(n)-(n+1)}{(m)}.$$

Hence it is evident, in order that this same man die between the year n and the year $n+v$ of his age, the probability will be

$$\frac{(n)-(n+v)}{(m)}.$$

Now, in order that this man die one marked day of the proposed year, the probability will be

$$= \frac{(n)-(n+1)}{365(m)}.$$

If the question is of an infant newly born, one only has to write 1 instead of the fraction (m).

4. QUESTION

8. *To find the term in which a man of a given age is able to hope to survive, of the kind that it is equally probable that he die before this term as after.*

Let the age of the man in question be m years and the one he is able to aspire to attain z years, which it is the question to find. Now, the probability that he arrives to this age $= \frac{(z)}{(m)}$, the probability that he dies before the term will be $= 1 - \frac{(z)}{(m)}$. Therefore, since the one and the other probability ought to be the same, we will have this equation

$$\frac{(z)}{(m)} = 1 - \frac{(z)}{(m)},$$

and therefore $(z) = \frac{1}{2}(m)$, from which is easy to find the number z, as soon as one has determined by observations the values of all the fractions

$$(1), \quad (2), \quad (3), \quad (4), \quad (5), \quad (6) \quad \text{etc.};$$

because one will see first of all which, (z), will be the half of the proposed (m).

Having found this number z, one names the interval $z - m$ *the power of the life of a man* of m years.

5. QUESTION

9. *To determine the life annuity that it is just to pay to a man of any age all the years, until his death, for a sum which will have been advanced first.*

We conceive M men who have all the same age of m years, and let each pay first the sum a; that which will provide a fund $= Ma$. Let the sum x be that one ought to pay to each all these years, so long as he is alive; and after one year the fund ought to pay

$$\frac{(m+1)}{(m)}Mx,$$

after two years

$$\frac{(m+2)}{(m)}Mx,$$

after three years

$$\frac{(m+3)}{(m)}Mx,$$

and thus in succession.

Now, calculating that the fund is placed at 5%, a sum S payable after n years is worth at present only $\left(\frac{20}{21}\right)^n S$; but, in order to render our determination more general, we suppose that a sum S increases by interest in a year to λS; and $\frac{1}{\lambda}$ will be that which we have marked by $\frac{20}{21}$, and a sum S payable at the end of n years is worth at present only $S : \lambda^n$. Hence, one will prepare the following calculations:

	one ought to pay	that which is worth at present
after 1 year	$\frac{(m+1)}{(m)}Mx,$	$\frac{(m+1)}{(m)} \cdot \frac{Mx}{\lambda}$
after 2 years	$\frac{(m+2)}{(m)}Mx,$	$\frac{(m+2)}{(m)} \cdot \frac{Mx}{\lambda^2}$
after 3 years	$\frac{(m+3)}{(m)}Mx,$	$\frac{(m+3)}{(m)} \cdot \frac{Mx}{\lambda^3}$
	etc.;	etc.

Now, equity requires that all these sums reduced to the present time be equal to the entire fund Ma, whence one derives this equation

$$a = \frac{x}{(m)} \left(\frac{(m+1)}{\lambda} + \frac{(m+2)}{\lambda^2} + \frac{(m+3)}{\lambda^3} + \frac{(m+4)}{\lambda^4} + \text{etc.} \right),$$

and therefore, that which the fund ought to pay per year to each one of the interested parties is

$$x = \frac{(m)a}{\frac{(m+1)}{\lambda} + \frac{(m+2)}{\lambda^2} + \frac{(m+3)}{\lambda^3} + \frac{(m+4)}{\lambda^4} + \text{etc.}}.$$

Knowing therefore the values of all these fractions $(1), (2), (3)$ etc., it is easy to find the sum x that agrees to each age of m years, referring to the given interest.

6. QUESTION

10. *When the interested parties are some infants newly born and when the payment of the life annuities must begin only when they will have attained a certain age, to determine the amount of these life annuities.*

We suppose that one pays the sum a for each infant newly born and that he must receive the annuity only when he will have attained the age of n years; that after this time, one pays for all years the sum x which it is necessary to determine. Computing therefore the interest as before, one will arrive at this equation

$$a = x \left(\frac{(n)}{\lambda^n} + \frac{(n+1)}{\lambda^{n+1}} + \frac{(n+2)}{\lambda^{n+2}} + \frac{(n+3)}{\lambda^{n+3}} + \text{etc.} \right),$$

which gives

$$x = \frac{a}{\frac{(n)}{\lambda^n} + \frac{(n+1)}{\lambda^{n+1}} + \frac{(n+2)}{\lambda^{n+2}} + \frac{(n+3)}{\lambda^{n+3}} + \text{etc.}}.$$

Whence it is evident that one such annuity is able to become a powerful benefit and that a man, when he will have attained a certain age, is able to enjoy the considerable income at little cost, during all his life.

11. All these questions will resolve themselves therefore easily, as soon as one will know the values of the fractions (1), (2), (3), (4) etc., which depend as much on the climate as on the way of life; also one has noticed that these values are different for the two sexes,[1] in a manner that one would know nothing to determine in general. Now,

[1] Euler indirectly refers to the work of Nicholaas Struyck.

in order to conclude these observations, one comprehends easily that it is necessary to use a large number of them, which spreads evenly over all sorts of persons; and in this respect, one would not know how to be served with the registers of the pensions life annuity, which begin with some children below one year. For first, one could not regard these children as newly born, and the majority have probably escaped already the dangers of the first month; and next, one will enlist scarcely any children of a fragile constitution, in a manner that one must regard as chosen the children for whom one takes life annuities. So, the values of our fractions that one will conclude from the registers of the life annuities will be infallibly too great, especially in consideration of the first years. However, since it is necessary to regulate the annuities on such registers rather than on the real mortality, I will append the values of our fractions such as one extracts them from some observations of M. Kersseboom.[2]

$(1)= 0.804$	$(25)= 0.552$	$(49)= 0.370$	$(73)= 0.145$
$(2)= 0.768$	$(26)= 0.544$	$(50)= 0.362$	$(74)= 0.135$
$(3)= 0.736$	$(27)= 0.535$	$(51)= 0.354$	$(75)= 0.125$
$(4)= 0.709$	$(28)= 0.525$	$(52)= 0.345$	$(76)= 0.114$
$(5)= 0.688$	$(29)= 0.516$	$(53)= 0.336$	$(77)= 0.104$
$(6)= 0.676$	$(30)= 0.507$	$(54)= 0.327$	$(78)= 0.093$
$(7)= 0.664$	$(31)= 0.499$	$(55)= 0.319$	$(79)= 0.082$
$(8)= 0.653$	$(32)= 0.490$	$(56)= 0.310$	$(80)= 0.072$
$(9)= 0.646$	$(33)= 0.482$	$(57)= 0.301$	$(81)= 0.063$
$(10)= 0.639$	$(34)= 0.475$	$(58)= 0.291$	$(82)= 0.054$
$(11)= 0.633$	$(35)= 0.468$	$(59)= 0.282$	$(83)= 0.046$
$(12)= 0.627$	$(36)= 0.461$	$(60)= 0.273$	$(84)= 0.039$
$(13)= 0.621$	$(37)= 0.454$	$(61)= 0.264$	$(85)= 0.032$
$(14)= 0.616$	$(38)= 0.446$	$(62)= 0.254$	$(86)= 0.026$
$(15)= 0.611$	$(39)= 0.439$	$(63)= 0.245$	$(87)= 0.020$
$(16)= 0.606$	$(40)= 0.432$	$(64)= 0.235$	$(88)= 0.015$
$(17)= 0.601$	$(41)= 0.426$	$(65)= 0.225$	$(89)= 0.011$
$(18)= 0.596$	$(42)= 0.420$	$(66)= 0.215$	$(90)= 0.008$
$(19)= 0.590$	$(43)= 0.413$	$(67)= 0.205$	$(91)= 0.006$
$(20)= 0.584$	$(44)= 0.406$	$(68)= 0.195$	$(92)= 0.004$
$(21)= 0.577$	$(45)= 0.400$	$(69)= 0.185$	$(93)= 0.003$
$(22)= 0.571$	$(46)= 0.393$	$(70)= 0.175$	$(94)= 0.002$
$(23)= 0.565$	$(47)= 0.386$	$(71)= 0.165$	$(95)= 0.001$
$(24)= 0.559$	$(48)= 0.378$	$(72)= 0.155$	

Now, since this table is prepared on some selected children who have even survived already some months since their birth, if one wishes to apply it to all the children newly born in a town or province, it is necessary to decrease all these numbers a certain part, in order to take into account the high mortality to which the infants are subject immediately after their birth. But we will take this correction more surely of the observations which contain the multiplication already, that I am going to consider myself.

[2]Of the table laid out by Kersseboom, one deduces some rates which in four places differ slightly from those used by Euler; these are: $(5) = 0.68857$, which one ought to abbreviate to 0.689; next $(8) = 0.652$; $(30) = 0.508$; $(90) = 0.007$; $(91) = 0.005$.

II. HYPOTHESIS OF THE MULTIPLICATION

12. It is the principle of the propagation on which this hypothesis is found; whence it is first apparent that, if there are born every year as many children as there die of men, the number of all the men will always stay the same and that there won't be then multiplication. But, if the number of the children who are born every year surpasses the number of deaths, every year will produce an increase in the number of the living, that will be equal to the excess of the newborns over the deaths. Now, this increase will change to decrease, when the number of the deaths surpasses the one of newborns. Thence, we will have three cases to consider: the first, where the number of the men remains constantly the same; the second, where it increases every year; and the third, where it decreases every year. Therefore, if M indicates the number of all the men who live to the present and mM the number of those who live the following year, the first case will take place if $m = 1$, the second if $m > 1$, and the third if $m < 1$; in a way that all the cases could be included in the general coefficient m.

13. Now, having fixed the principle of the propagation which depends on the marriages and fecundity, it is apparent that the number of the children who are born during the course of one year, must hold a certain relationship to the number of all the living men. Whence it follows that, if the number of the living always remains the same, there will be born every year the same number of children; and if the number of the living increases or decreases, the number of births must increase or decrease by the same reason. Therefore, by comparing together the number of all the newborns during several successive years, according to whether this number stays the same, or that it increases or decreases, one will conclude from it if the number of all the men stays the same, or if it is increasing or decreasing. There joining the principle of mortality, it is also clear that the number of deaths during one year must hold a certain relationship as much to the one of all the living as to the one of the newborns.

14. Since these two principles of mortality and of propagation are independent of one another and since I have considered the first independently of the other, one is able to represent this also, without the first mixed in it. Because, supposing the number of all the living at the same time $= M$, the number of infants who are produced in the space of a year will be put $= \alpha M$, in a way that α is the measure of the propagation or fecundity. But it is difficult to draw from this place the consequences which concern the multiplication and the other phenomena which depend upon it. The reasoning will become more clear, if we introduce first in the calculation the number of infants who are born every year, to which if we join the hypothesis of mortality, we will be able to conclude the value of α. Therefore reciprocally, the number of the births depends at the same time on the two hypotheses of mortality and fecundity; and thence, one will draw next without difficulty the solution to all the other questions which one proposes ordinarily in treating this material.

15. Since I suppose that the rule of mortality always remains the same, I will suppose a similar constancy in fecundity; of a sort that the number of the infants who are born every year is always proportional to the number of all the living. Therefore, if the number of all the living remains the same, one will have also every year the same number of births; and if the number of all the living is increasing or decreasing, the number of annual births will increase or decrease by the same reason.

Therefore let N be the number of infants born during the course of a year and nN that of the infants born the following year; and since the ratio which has changed the number N to nN subsists still, it is necessary that from any year to the following the number of births increases in the ratio of 1 to n. Consequently, the third year there will be born n^2N, the fourth n^3N, the fifth n^5N and thus in sequence; or else, the number of annual births will constitute a geometric progression, either increasing or decreasing or of equality, according as $n > 1$ or $n < 1$ or $n = 1$.

16. Therefore we put that, in a village or province, the number of infants born in this year be $= N$, and of those who will be born the next year $= nN$, and also the following according to this progression:

	the number of births
at present	N
after one year	nN
after two years	n^2N
after 3 years	n^3N
after 4 years	n^4N
etc.	

and if we suppose that after 100 years each of the men who exist at present are no longer alive, there will be none, after 100 years, of the other living but those who remain yet living of these births. Therefore, joining the hypothesis of mortality, one will be able to determine the number of all the men who will be living after 100 years. Now, since there will be born this year $n^{100}N$, one will have the ratio of the births to the number of all the living.

17. In order to render this more clear, observe how many men will be still living, after one hundred years, from the births of all the years preceding.

	Number of births	After 100 years there are living still
at present	N	$(100)N$
after 1 year	nN	$(99)nN$
after 2 years	n^2N	$(98)n^2N$
after 3 years	n^3N	$(97)n^3N$
$\vdots$	$\vdots$	$\vdots$
after 98 years	$n^{98}N$	$(2)n^{98}N$
after 99 years	$n^{99}N$	$(1)n^{99}N$
after 100 years	$n^{100}N$	$n^{100}N$

Therefore, the number of all the living after 100 years will be

$$= n^{100}N\left(1 + \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \frac{(5)}{n^5} + \text{etc.}\right).$$

18. The terms of this series will vanish in the end, by virtue of the hypothesis of mortality; and since the number of all the living has a certain ratio to the number of births during the course of a year, the multiplication from one year to another, which

comes to be supposed as 1 to n, we discover this ratio. Because, if the number of all the living is $= M$ and the number of infants who are procreated by them during the course of one year is put $= N$, we will have

$$\frac{M}{N} = 1 + \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \frac{(5)}{n^5} + \text{etc.}$$

Therefore, if we know the ratio $\frac{M}{N}$ and if we join it to the hypothesis of mortality or the values of the fractions (1), (2), (3), (4) etc., this equation determines reciprocally the ratio of the multiplication, $1 : n$, from one year to another. However, one sees well that this determination would not known to be developed in general; but, for each hypothesis of mortality, if one calculates the ratio $\frac{M}{N}$ for several values of n and if one lays out a table of them, it will be easy to assign reciprocally for each given ratio $M : N$, which expresses the fecundity, the annual increase of all the living, which is the same as that of the births.

19. We suppose therefore that the hypothesis of mortality or the fractions

$$(1), \quad (2), \quad (3), \quad (4), \quad (5) \quad \text{etc.}$$

are known, similarly that the hypothesis of fecundity or the ratio of all the living, M, to the number of infants, N, who are procreated by them during a year; one will know if the number of men remains invariable, or if there is an increase or a decrease. Because, if we put the number of all the living the next year $= nM$, those of the living at present being $= M$, it is necessary to extract the value of n from the equation found

$$\frac{M}{N} = 1 + \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \frac{(5)}{n^5} + \text{etc.},$$

and supposing known the solution of this equation, it is indifferent if one knows the fecundity $\frac{M}{N}$ or the multiplication $1 : n$, the one being determined by the other, with the help of the hypothesis of mortality.

1. QUESTION

20. *The hypotheses of mortality and fecundity being given, if one knows the number of all the living, to find how many there will be at each age.*

Let M be the number of all the living and N the number of infants who are procreated during a year; and by the hypothesis of mortality, one will know the ratio of the annual multiplication $1 : n$. Now, knowing the value of n, it is easy to conclude from § 17 that there will be, among the number M,

N	infants newly born,
$\frac{(1)}{n}N$	aged one year,
$\frac{(2)}{n^2}N$	aged two years,
$\frac{(3)}{n^3}N$	aged 3 years,
$\frac{(4)}{n^4}N$	aged 4 years

and in general

$$\frac{(a)}{n^a}N \quad \text{aged } a \text{ years.}$$

Now, the sum of all the numbers taken together is $= M$.

2. QUESTION

21. *The same things being given, to find the number of men who will die in a year.*

Let M be the number of men who are living at present, it contains the infants who are born this year, of whom the number is $= N$; and the quotient $\frac{M}{N}$ will determine the annual increase, which is $1 : n$. Therefore, the next year, the number of living will be $= nM$, among which find the number of newly born $= nN$; the others, of which the number is $nM - nN$, are those who still living in the preceding year, of which the number was $= M$; whence it follows that there died of them

$$(1-n)M + nN.$$

Therefore, if the number of living is $= M$, there die during the course of one year $(1-n)M + nN$, while in this same time there are born N infants.

3. QUESTION

22. *Knowing how many the number of births and burials which happen during the course of one year, to find the number of all the living and their annual increase, for a given hypothesis of mortality.*

Let N be the number of births and O the number of burials which happen in a year; then, we put the number of all the living $= M$ and the annual increase $= 1 : n$, and the preceding solution provided us this equation

$$O = (1-n)M + nN.$$

Now, the hypothesis of mortality gives

$$\frac{M}{N} = 1 + \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \frac{(5)}{n^5} + \text{etc.}$$

Therefore, having by the first

$$M = \frac{O - nN}{1-n},$$

this value, being substituted into the other equation, gives

$$\frac{O-N}{N(1-n)} = \frac{N-O}{N(n-1)} = \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \text{etc.},$$

whence it is necessary to find the value of the number n.

23. If the number of burials, O, is equal to those of the births, N, of a kind that

$$N = (1-n)M + nN,$$

it is necessary absolutely that there be $n = 1$ or that the number of living remains always the same; and in this case, this number will be

$$M = N(1 + (1) + (2) + (3) + (4) + \text{etc.})$$

Now, if the number of births, N, surpasses those of the burials, O, in a way that $N - O$ is a positive number, the equation

$$\frac{N-O}{N(n-1)} = \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \text{etc.}$$

will give for n a value > 1, which indicates that the number of living is increasing. But, if the number of births, N, is smaller than those of the burials, O, our equation ought to be represented in this form

$$\frac{O-N}{N(n-1)} = \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \text{etc.},$$

whence one extracts for n a value smaller than 1, which indicates that the number of living is decreasing.

4. QUESTION

24. *The number of births and burials in a year being given, to find how many of each age there will be among the dead.*

Let N be the number of infants born during a year and O the number of deaths; and by the preceding question on will have the number of all the living, M, with the multiplication $1 : n$ from one year to the other. Thence, we consider how many men there will be living in each age, so many this year as the next year.

Number	this year	the following year
of newly born	N	nN
of age one year	$\frac{(1)}{n}N$	$(1)N$
of age two years	$\frac{(2)}{n^2}N$	$\frac{(2)}{n}N$
of age three years	$\frac{(3)}{n^3}N$	$\frac{(3)}{n^2}N$
etc.		etc.

Whence it is evident that there died of them during the course of this year:

	the number of deaths
to the end of one year	$(1-(1))N,$
from 1 year to two years	$((1)-(2))\frac{N}{n},$
from 2 years to 2 years	$((2)-(3))\frac{N}{n^2},$
from 3 years to 4 years	$((3)-(4))\frac{N}{n^3},$
from 4 years to 5 years	$((4)-(5))\frac{N}{n^4}$
etc.	

25. The number of all the deaths of this year being $= O$, one will have this equation

$$\frac{O}{N} = 1 - (1)\left(1 - \frac{1}{n}\right) - \frac{(2)}{n}\left(1 - \frac{1}{n}\right) - \frac{(3)}{n^2}\left(1 - \frac{1}{n}\right) - \text{etc.},$$

which agrees with this

$$O = (1-n)M + nN,$$

because

$$\frac{M}{N} = 1 + \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \frac{(5)}{n^5} + \text{etc.}$$

Therefore, knowing the hypothesis of mortality with the annual multiplication $1 : n$ and the number N of births of one year, one is able to determine how many men of each age will die probably during the course of one year.

5. QUESTION

26. *Knowing the number of all the living, similarly the number of births with the number of deaths at each age during the course of one year, to find the law of mortality.*

Let M be the number of all the living, N that of the births and O of the burials during the course of one year; and thence one will know first the annual multiplication

$$n = \frac{M-O}{M-N};$$

there is then for this year the number of deaths by the preceding question

at the end of one year	$\alpha = (1-(1))N,$
from 1 to 2 years	$\beta = ((1)-(2))\frac{N}{n},$
from 2 to 3 years	$\gamma = ((2)-(3))\frac{N}{n^2},$
from 3 to 4 years	$\delta = ((3)-(4))\frac{N}{n^3}$

etc.

27. Here is a way more certain than those of the life annuities in order to determine the law of mortality; and this determination will become the easiest, if one chooses a village or province where the number of burials equals those of the baptisms, in a way that $n = 1$; because then, it suffices to know the number of deaths at each age. But if is necessary quite to indicate that such a law of mortality ought to be extended only to the village or province from which one derives it. In another country there could take place a law entirely different; and one has observed, in particular, that in large towns the mortality is greater than in the smaller and in these greater than in the villages. If one gave the good trouble to establish the law of mortality and that of fertility for several places, one could extract from it a quantity of very important findings.

28. But it is necessary again to remark that, in this calculation that I come to develop, I have supposed that the number of all the living of a place remains the same, or that it increases or decreases uniformly, in a way that it is necessary to exclude so many extraordinary devastations, as the plague, war, famine, and the extraordinary increases, as of new colonies. It will be good also to choose such a place where all the newborns remain in the country and where strangers do not come to live and die there, that which would reverse the principles on which I have founded the preceding calculations. For places subjected to such irregularities, it would be necessary to extract from the registers exactly how many of all the living as deaths, and then, in following the principles that I have established, one would in the state apply the same calculations. Everything

always returns to these two principles, that of mortality and that of fecundity, which being one time well established for a certain place, it will not be difficult to resolve all the questions that one is able to propose on this matter, of which I have contented myself to report the principles.

29. I have treated also these questions in general, without the marking out to some particular place; now, in order to extract all the advantages, everything depends on a great number of observations made in several different places, as many of the number of all the living and of the newborns during one or several years, as of the number of deaths with their ages. As this is an article quite difficult to execute well, we must be very indebted to Mr. Sussmilch[3] , Counselor of the Superior Consistatory, which, after having surmounted the nearly invincible obstacles, have just to furnish us one such great number of such observations, which appear sufficient in order to decide most of the questions which are present in this research. And indeed, he extracted from it already himself so many important findings, that we would hope that he will carry by his attentiveness this science to the highest degree of perfection to which it is susceptible.

[3]Euler refers to the major work of JEAN-PIERRE SUSSMILCH (1707-1767). It bears the title *Die göttliche Ordnung in den Veränderungen des menschlichen Geschlechts, aus der Geburt, dem Tode und der Fortpflanzung desselben erwiesen*. (The divine order in the changes of the human generation, through the birth, the deaths and the procreation of the same established.) First edition, Berlin, 1741-1742, second edition 1761-1762. L. G. D.