Notes to accompany "De Usu algorithmi infinitesimalis in arte coniectandi specimen"

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An urn contains n white-black pairs of tickets. These pairs are distinguishable by being numbered consecutively. Hence the urn contains $2n$ tickets altogether.

Let some tickets be drawn at random so that r remain in the urn at the end of this process. Suppose further that the urn now contains x pairs.

If now a new ticket is extracted, it may break a pair with probability $\frac{2x}{r}$ and not do so with probability $1 - \frac{2x}{r}$.

Clearly, if X_r is the random variable representing the number of pairs in the urn when there are r tickets in the urn,

$$\begin{aligned}\mathrm{E}(X_{r-1}|X_r = x) &= \frac{2x}{r}(x-1) + \left(1 - \frac{2x}{r}\right)x \\ &= x - \frac{2x}{r} \\ &= x\left(1 - \frac{2}{r}\right)\end{aligned}$$

Now $x_{2n} = n$ by construction. Therefore

$$\mathrm{E}(X_{2n-1}) = n - 1 = \frac{(2n-2)(2n-1)}{4n-2} = \frac{(r-1)r}{4n-2}$$

and

$$\mathrm{E}(X_{2n-2}) = (n-1)\left(1 - \frac{2}{2n-1}\right) = \frac{(2n-3)(2n-2)}{4n-2} = \frac{(r-1)r}{4n-2}$$

since $X_{2n-1} = n - 1$ with probability 1. With the extraction of the second ticket, X_r

may vary. However, we note

$$\begin{aligned}
\mathrm{E}(X_{r-1}) &= \sum_x \mathrm{E}(X_{r-1}|X_r = x)\Pr(X_r = x) \\
&= x\left(1 - \frac{2}{r}\right)\Pr(X_r = x) \\
&= \left(1 - \frac{2}{r}\right)\mathrm{E}(X_r)
\end{aligned}$$

and so, we may conclude, by induction on r,

$$\mathrm{E}(X_r) = \frac{r(r-1)}{4n-2}$$

In addition,

$$\mathrm{E}(X_{r+1}) = \frac{r+1}{r-1}\mathrm{E}(X_r)$$

and

$$\mathrm{E}(X_2) = \frac{1}{2n-1}.$$

Example. We may shuffle together 2 decks of 52 playing cards with distinguishable backs. Here $n = 52$, $2n = 104$. If 13 cards are removed, $r = 91$ and

$$\mathrm{E}(X_{90}) = \frac{91(90)}{4 \cdot 52 - 2} = 39\frac{156}{206}$$

If 52 are removed,

$$\mathrm{E}(X_{51}) = \frac{52(51)}{4 \cdot 52 - 2} = 12\frac{90}{103}$$

Bernoulli notes in this first case that removing 13 cards removes nearly 13 pairs, on average. In the second, removing half the cards leaves about 13 pairs.

If we suppose n and r are very great, then $r - 1 \approx r$, $4n - 2 \approx 4n$. Therefore

$$\mathrm{E}(X_{r-1}) = \frac{r^2}{4n} \text{ very nearly.}$$

Bernoulli derives this by solving a differential equation. If $\Delta r = 1$, then $\Delta x = 0$ with probability $\frac{r-2x}{r}$ and $\Delta x = \Delta r$ with probability $\frac{2x}{r}$. Therefore

$$\Delta x = \frac{r-2x}{r} \cdot 0 + \frac{2x}{r} \cdot \Delta r = \frac{2x}{r}\Delta r$$

Passing from differences to differentials,

$$dx = \frac{2x}{r}dr.$$

Hence

$$\frac{dx}{x} = \frac{2dr}{r}$$

subject to the initial condition that $x = n$ when $r = 2n$. The solution to this equation is $x = \frac{r^2}{4n}$.

Consider now that there are s black tickets and t white tickets in the urn. We must have $s + t = r$. Let x be the number of pairs in the urn. If $\Delta x = 1$, then

$$\begin{aligned} \Delta x &= \Delta s \text{ with probability } \frac{x}{s} \\ &= t \text{ with probability } \frac{x}{t} \end{aligned}$$

so, proceeding as before,

$$dx = \frac{x}{s} ds + \frac{x}{t} dt$$

subject to the initial condition that $x_{2n} = s = t$. This gives $x = \frac{st}{n}$. We note, of course, that if $s = t = \frac{1}{2}r$, then $x = \frac{r^2}{n}$.

In the same manner, if there are 3 or 4 special classes, with s, t, u, z members of these special classes in the urn when there are r tickets in the urn, then

$$\mathrm{E}(X) = \frac{stu}{n^2} \quad \text{and} \quad \mathrm{E}(X) = \frac{stuz}{n^3}.$$

In every case, we should look at this expectation as, for example, the last

$$\mathrm{E}(X) = \left(\frac{s}{n}\right) \left(\frac{t}{n}\right) \left(\frac{u}{n}\right) \left(\frac{z}{n}\right) n$$

Bernoulli now supposes

$$\frac{ds}{dt} = \phi \frac{s}{t}$$

That is, $\frac{ds/s}{dt/t} = \phi$ or $\frac{ds}{s} = \phi \frac{dt}{t}$ where ϕ is a function of s and t.

If we take ϕ a constant, then

$$t = n \left(\frac{s}{n}\right)^{1/\phi}$$

since the initial condition is $s = t = n$.

Bernoulli takes $\phi = \frac{n}{n+s}$. This gives

$$t = s e^{\frac{s-n}{n}}.$$

and mentions some special cases:

If $t = s^2/n$, then $\phi = \frac{1}{2}$.

if $t = s^f/n^{f-1}$, then $\phi = \frac{1}{f}$, or, using the first result,

if $\phi = 1/f$, then $t = n\left(\frac{s}{n}\right)^f$.

Examples. If $\phi = 1$, $s = t = r/2$ and $x = \frac{st}{n}$ gives $x = \frac{r^2}{4n}$ or, if $r = kn$,

$$x = \frac{k^2}{4} n.$$

If $\phi = 2$, $s = t^2/n$, $s + t = r$, and $x = st/n$, gives $x = t^3/n^2$. This produces

$$x = \frac{\left(-\frac{1}{2}n + \frac{1}{2}\sqrt{n^2 - 4nr}\right)^3}{n}$$

If we put $r = kn$,

$$x = \frac{n}{8}\left(-1 + \sqrt{4k + 1}\right)^3$$

Taking $k = \frac{5}{16}$,

$$\begin{array}{ll} \phi = 1 & \phi = 2 \\ x = \dfrac{25}{1024}n & x = \dfrac{1}{64}n \\ t = s = \dfrac{5}{32}n & t = \dfrac{1}{4}n \\ & s = \dfrac{1}{16}n \end{array}$$