Notes to accompany
De duratione media matrimoniorum

Richard J. Pulskamp
Department of Mathematics and Computer Science
Xavier University, Cincinnati OH.

October 30, 2009

§ 1. Deparcieux presents the data of Wargentin in Table IV of the 1760 "Addition a l'essai sur les probabilités de la durée de la vie humaine," a supplement to his work, "Essay sur les Probabilités de la durée de la vie humaine...," Paris, 1746. This table, which appears to be equivalent to the one cited by Bernoulli, begins with 10,000 births of which 4,933 are males and 5,067 are females. During the first year of life, Wargentin reports that 1,623 males and 1,438 females die.

§ 6. Halley's table gives the number living up to age 84. Since the number living at age 20 in this table is 598 and since Bernoulli wishes to begin with 500 marriages at this age, he must rescale the number living at each age. He does so by multiplying the number living at each age by $1000/598$.

§ 7. Let l_x denote the number living at each age in Bernoulli's table. Inspection of the last few values of this table indicate clearly how Halley's was extended.

x	84	85	86	87	88	89
l_x	33	27	22	17	13	9

Note that the successive differences are 6, 5, 5, 4, and 4.

The number of marriages surviving at each age is given by the formula

$$\frac{l_x(l_x - 1)}{4n - 2}$$

where $n = 500$, n being the number of marriages at age 20.

The number of living widows or widowers is given by the formula

$$\frac{(2n - l_x)l_x}{2n - 1}.$$

Bernoulli's values for both the number of marriages and the number of living widows do not quite agree with these computed values. But recall that he has smoothed the values slightly besides rounding. The decline in intact marriages and the distribution of widows by age are shown in Figure 1 below.

§ 9. Half of the marriages will be broken when the number of remaining marriages is 250 which is seen to occur between the entry for 42 years old and 43 years old. By linear interpolation, the number of months must be $\frac{254-250}{11} \times 12 = 4.4$ which Bernoulli takes to be $4\frac{1}{2}$ months. Since all marriages begin with spouses age 20, half will be destroyed in 22 years $4\frac{1}{2}$ months.

§ 11. Figure 2 displays the mean duration of marriages begun at each age starting from age 20. It clearly shows the linear relationship observed by Bernoulli. The odd behavior for ages in the early eighties is due to the extension of Halley's table by Bernoulli.

§ 14. The sum of all widows is 10,330 up to and including age 89, not 10,326 as Bernoulli states. Regardless, Bernoulli takes l0,350 as the complete count. The total of all married persons is 23,834 by his previous computation. Consequently these possess the ratio of 2.35 to 1.

§ 17. The entries to the table are obtained in a very easy manner. We begin from Halley's Table with the number of men and women living at the ages of 40 and beyond and 20 and beyond respectively. To obtain Bernoulli's values, each of Halley's is multiplied by the appropriate proportion so the number at each beginning age is 500.

		Halley		Bernoulli	
Age		Surviving		Surviving	
Men	Women	Men	Women	Men (s)	Women (t)
40	20	445	598	$445 \times \frac{500}{445}$	$598 \times \frac{500}{598}$
41	21	436	592	$436 \times \frac{500}{445}$	$592 \times \frac{500}{598}$
42	22	427	586	$427 \times \frac{500}{445}$	$586 \times \frac{500}{598}$
43	23	417	579	$417 \times \frac{500}{445}$	$579 \times \frac{500}{598}$
44	24	407	573	$407 \times \frac{500}{445}$	$573 \times \frac{500}{598}$

The number of remaining marriages (x), Column III, is just the product of the number of surviving men (s) and women (t) divided by 500. Figure 3 shows the distribution of these variables over time.

§ 19. The expected duration of marriage again shows strong linearity. For which see Figure 4. However, Bernoulli's table does not correspond exactly to the computed values.

§ 20. a) If both spouses are of age 55, the mean duration of marriage is 9 years, 9 months whereas if the husband is 75, but the wife 55, the mean duration is 4 years 10 months. The first is essentially the double of the second.

b) From § 11, we have the mean duration of marriages where both spouses are of age 20 equal to $23\frac{5}{6}$ years. From § 19, when the husband is 40 and the wife 20, the mean duration is $17\frac{11}{12}$ years.

$$23\frac{5}{6} \times 500 = 11917$$
$$17\frac{11}{12} \times 500 = 8958$$

c) From Halley's mortality table, we know that if we had 500 young men of age 20, by age 40 only 372 would survive, or very nearly 75%. If there were parity in the number men and women of each age, but men of age 40 married women of age 20, only 75% of the women could have husbands. The remaining women would either remain single all their lives or marry one of the widowers.

d) Bernoulli is reasoning as follows:

$$\text{Total lives} = \text{Widows} + 2 \times \text{Marriages}.$$

In the stationary population where both spouses wed at age 20, he has computed the number of intact first marriages as $500 \times 23\frac{5}{6}$ years or $11,917$. He further estimates the number of widows as 10,350 of which number half, or 5175, are men and the other half women. Consequently, the total population is

$$10,350 + 2 \times 11917 = 34,184.$$

We might observe that the sum of Column I, the number living, is 34,666 or 482 lives greater. Assuming that deaths occur mid-year, on average, this sum should be correct by subtracting 500 to be 34,166.

If, on the other hand, men of age 40 marry women of age 20, the sum of the widowers is 2154 (true value 2143) and the sum of the widowers is 7949 (true value 7945) which he increases by 197 to 8146. Here, rather than multiply the mean duration of marriages of $17\frac{11}{12}$ years by 500 so that we have 8,958 marriages, he instead sums Column III to obtain 1907. Now, in this case

$$\text{Widows} + \text{Widowers} + 2 \times \text{Marriages} = \text{Total Population}$$
$$5175 + 8146 + 2 \times 1907 = 28,714$$

The total population is 28,507; here no adjustment for deaths is indicated.

e) The derivation of the number of widowers and widows is straightforward. Let s_x and t_x indicate the number of men and women living at time x respectively. We let n denote the initial number of marriages. The integrals can be discretized as

$$-\int \frac{tds}{n} = -\sum_x \frac{t_x(s_{x+1} - s_x)}{n}$$
$$-\int \frac{sdt}{n} = -\sum_x \frac{s_x(t_{x+1} - t_x)}{n}$$

The first few values are displayed in the following table:

	Surviving		Decrease of the	
Time	Men	Women	Husbands	Wives
1	500	500	$\frac{-500(490-500)}{500}$	$\frac{-500(495-500)}{500}$
2	490	495	$\frac{-495(479-490)}{500}$	$\frac{-490(490-495)}{500}$
3	479	490	$\frac{-490(468-479)}{500}$	$\frac{-479(485-490)}{500}$
4	468	485	$\frac{-585(457-468)}{500}$	$\frac{-468(480-485)}{500}$
5	457	480		

Continuing the table in this manner and summing the two rightmost columns, we obtain the total number of deceased husbands to be 348 and of wives to be 153. This total is obtained with each decrease rounded to the nearest integer.

Figure 1: Marriage undertaken when both spouses of age 20

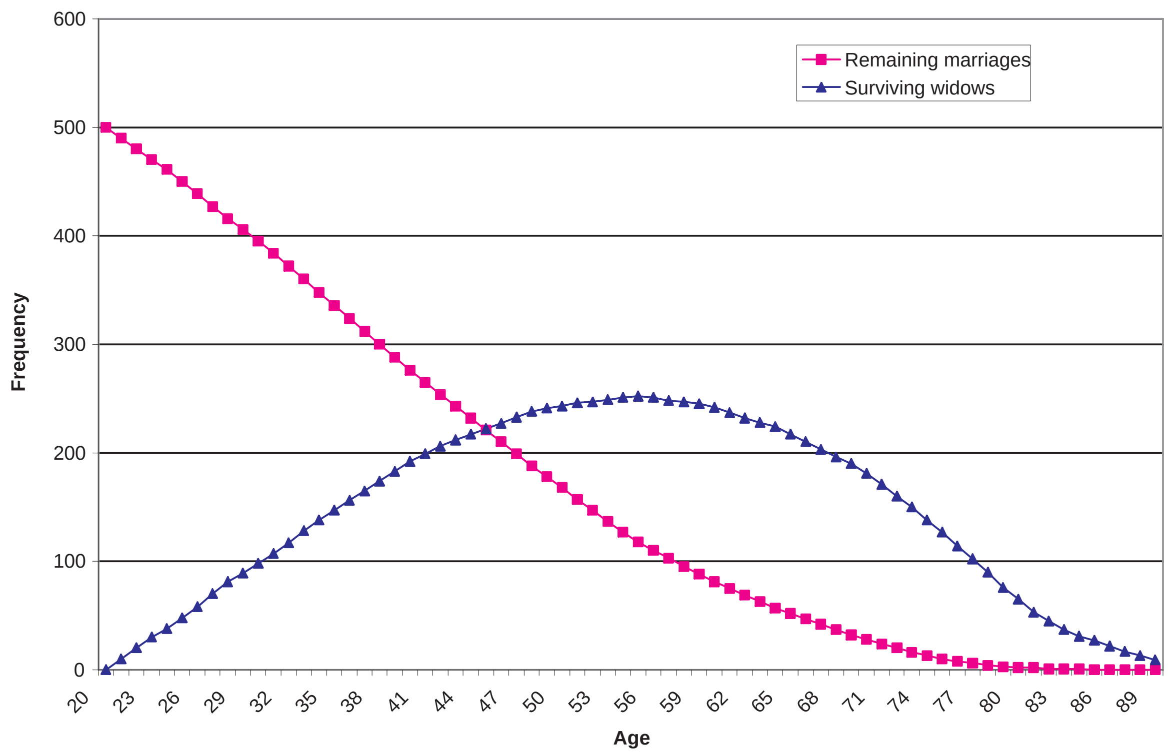

Figure 2: Mean duration of marriages

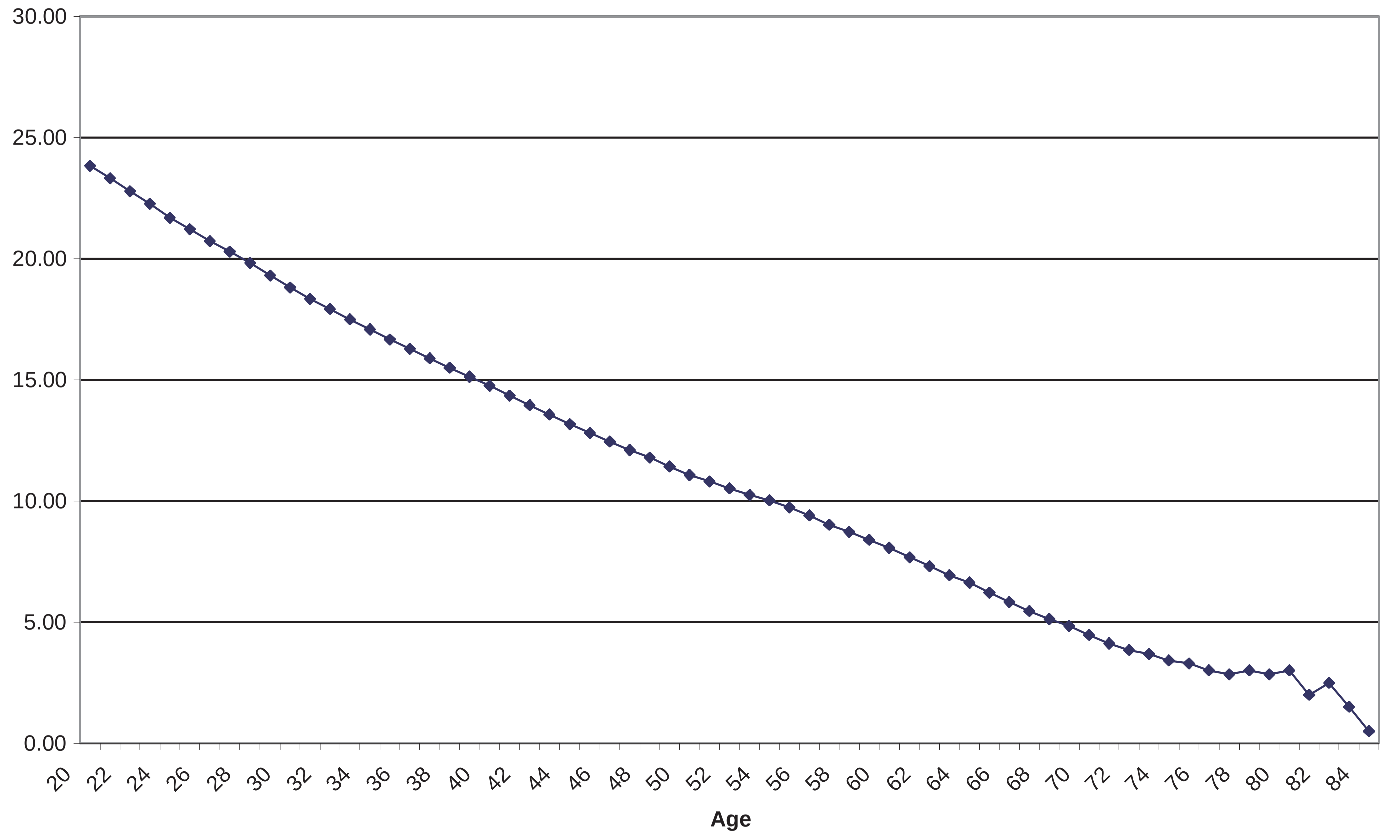

Figure 3: Number of marriages and widows.
Women of age 20, Men of age 40

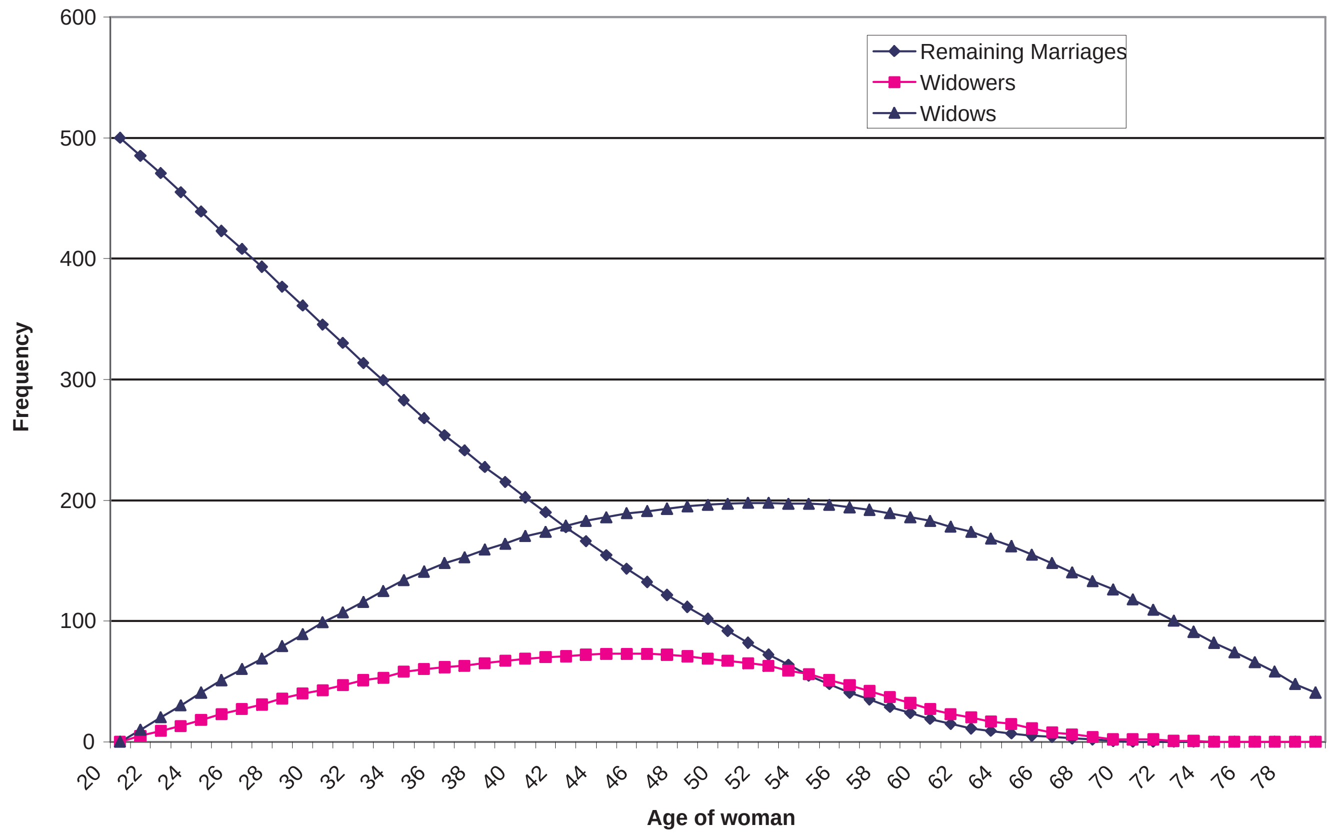

Figure 4: Expected duration of marriage
Men of age 40, Women of age 20