De usu algorithmi infinitesimalis in arte coniectandi specimen*

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§ 1. When recently I reflected concerning an argument, the examination of which I shall delay to the next opportunity, I have fallen into a question which concerns the art of conjecture. Because indeed this question of conjecture is able to appear utterly unrelated from the other argument, I have supposed not incongruous to send it ahead separately, and in such a way as if to pave a road to the next investigations. And indeed so much more willingly, I have resolved that I do this, because our soon to be explicated method itself appears to be able to apply to some extent toward new principles to be formed and established in the art of conjecture, with respect to which I do not yet know, by applications and by so much more with the worthy attention of the geometers. As often as certainly it happens, that the condition of things may be changed completely by a successive lot and by a variable one, just as when tickets, distinguished with different inscribed numbers, are successively extracted from an urn, and also laws are sought for the different things brought into existence from that place, one after the other, through determined changes, infinitesimal calculations are able to be used profitably toward accomplishing that work, only if any variation whatsoever is able to be considered as infinitely small, with respect to which it can happen, as long as the remaining number of tickets in the urn is very large. For then oneness is able to be had as if infinite for the small element; the same is supported by the arithmetical hypothesis of infinities, which mathematicians have used before having had discovered the differential and integral calculus. Truly I think such argument in the most abstract of propositions to have the need for explication; therefore to this thing itself I hasten with illuminating examples; but first I will make use of common analysis, and then I will descend to the use of the infinitesimal algorithm.

§ 2. *Problem*. Let tickets in equal number be deposited in an urn and two at a time be inscribed by a number so that the one is paired with the other and the two constitute one pair; but different pairs are assumed inscribed with different numbers, so that single pairs by themselves in turn are able to be distinguished; next tickets are extracted by lot from the urn one after the other; by which fact, it is demanded, with a given number of tickets remaining in the urn, how many complete pairs should be in it probably and

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likewise how many tickets should be with its companion or should remain bereft from its partner?

Solution. Let the number of all tickets be $= 2n$
and so of all pairs $= n.$
but after the extraction having been made and repeated the number of tickets in the urn remaining $= r;$
next the number of pairs remaining $= x$
and so the number of joined tickets deprived of its partner $= r - 2x.$
Thus with this established, I ordain one further ticket to be extracted, so that the number of tickets in the urn becomes $= r - 1;$
next that extracted ticket to be produced either will be of solitary class, or of joined class;
but the number of generating cases for the former event is $= r - 2x,$
and for the latter event $= 2x;$
if the former will have happened, the number of remaining pairs even now will be $= x;$
if the latter, that number will be diminished by one unit; and so the previously mentioned values x and $x-1$ are multiplied, according to the fundamental rule of the art of conjecture, by the corresponding number of cases, and the sum of the products divided by the sum of the cases, by which fact we will have

$$\frac{(r-2x)x + 2x(x-1)}{r}$$

or simply

$$x - \frac{2x}{r};$$

this last formula shows us, with whatever new one of the tickets being extracted, I have reduced the remaining pairs number in the urn by the quantity $2x/r$.

Now it is easy, on the pattern of the previous lemma, to define the successive values of x from the beginning until the end, by setting successively for r the numbers

$$2n; \quad 2n-1; \quad 2n-2; \quad 2n-3; \quad 2n-4; \quad 2n-5 \quad \text{etc.}$$

namely because immediately from the beginning there is

$$x = n,$$

there will be after the first extraction of a ticket

$$x = n - 1;$$

after the second extraction

$$x = n - 1 - \frac{2(n-1)}{2n-1}$$

or

$$= \frac{(2n-3) \times (2n-2)}{4n-2};$$

after the third extraction

$$x = \frac{(2n-4)\times(2n-3)}{4n-2}$$

after the fourth

$$x = \frac{(2n-5)\times(2n-4)}{4n-2} \text{ etc.}$$

From the succession of these values it is understood after so many extractions $(2n-r)$, that is, repeated as many times as units are contained in $2n-r$, to be

$$x = \frac{(r-1)r}{4n-2}.$$

Q.E.I.

§ 3. The same value is repeated, if it proceeds in retrograde order, so that now the preceding term is determined from any term coming after; if of course the indication of the letters x and r remains the same, what the value of the term in the preceding paragraph will be, of which the index is $r+1$, or of the antecedent term

$$= \frac{r+1}{r-1}x;$$

let now successive natural numbers be established for the letter r, by beginning with two, or from the case $r=2$, and let for this case $x=a$; thus there will be successively

$$x=a; \quad x=\frac{3.2}{2}a; \quad x=\frac{4.3}{2}a; \quad x=\frac{5.4}{2}a; \quad \text{etc.}$$

whence now it is clear, for whatever number r, to be

$$x = \frac{r\times(r-1)}{2}a;$$

therefore only this remains to be done, that the number a be determined, when only two tickets are remaining; but then there is one case, where two tickets are equal to themselves and so they compose one pair, while there are $2n-2$ cases, in which two remaining tickets are divided; whence it makes

$$a = \frac{1}{2n-1}$$

the same value is discovered out of it, because to set $r=2n$ there is $x=n$; indeed with these values substituted into the equation

$$x = \frac{r\times(r-1)}{2}a$$

it produces

$$n = \frac{2n\times(2n-1)}{2}a,$$

whence in return

$$a = \frac{1}{2n-1}:$$

because if then for a only the said value is substituted, it is had generally, as before,

$$x = \frac{r \times (r-1)}{4n-2}.$$

§ 4. In order that we illustrate our presently named rule by some example, we put two decks of playing cards (deux jeux de cartes) and both composed of 52 cards or leaves, and the two thoroughly shuffled together: therefore we will have 52 pairs of cards and also $n = 52$, but the total number of cards will be 104; at random 13 cards are removed; 91 cards will remain in the bundle and so there will be $r = 91$, whence

$$x = \frac{r \times (r-1)}{4n-2} = 39\frac{156}{206};$$

therefore from the beginning as many pairs in general are destroyed as cards are extracted. If indeed 52 cards are extracted, it will happen $x = 12\frac{90}{103}$; therefore some will contend with hope of gain among the remaining 52 cards there to be at very least 12 pairs, however it would be a slightly detrimental contest on behalf of 13 pairs for the players. Indeed I will not delay with this inquiry for a long time; nor have I wished to omit the direct solution to the problem, however easy; certainly during these times the most respected study of the art of conjecture is being neglected and despised too much, because to separate most questions whether belonging to morals or politics, teaches to bring about the maximum advantage to human actions and to guide them with prudence. Now I proceed to other observations more accommodating to my present purpose.

§ 5. When the tickets are extracted in uniform succession from the urn, the number r diminishes gradually as that of some flowing quantity, yet this idea of flowing is not able to be asserted to fit perfectly, because the decrementing of the remaining tickets in the urn proceed by means of steps through integer numbers until a number smaller than unity; and if the relation between the abscissa r and the application x is constructed exactly, the scale of the relations will be polygonal, not curved; but it is observed, the polygon is taken to approach so much more to a curve where a smaller line is taken for unity, or insofar as it reverts to the same place, where the greater number of the units is r, and finally indeed to establish the idea of a curve, if the number of units are considered as if boundless.

Now then, according to the pattern of these precepts, we shall have the numbers n and r as infinite or even greatly in magnitude, and so we shall be able without error, at least perceptibly, to change the equation in the paragraph of the third or second part

$$x = \frac{r \cdot (r-1)}{4n-2}$$

into this other

$$x = \frac{rr}{4n};$$

while these are advanced, I shall reveal the method, by which route the latter equation

$$x = \frac{rr}{4n}$$

is able to be discovered immediately and easily by the infinitesimal calculus.

§ 6. When the number r decreases by the element dr, that decrement falls according to the number of tickets either unjoined, of which the number is $r - 2x$, or joined, of which the number is $2x$: in the first case the number of non-pairs takes the decrement and makes $dx = 0$; in the other case the total decrement dr drops according to the diminution of the number x and makes also $dx = dr$; therefore we have $r - 2x$ cases which makes the element $dx = 0$ and $2x$ cases, which likewise makes the element $dx = dr$, whence, by the fundamental rule of the art of conjecture, makes true the value of the element

$$dx = \frac{2xdr}{r};$$

hence

$$\frac{dx}{x} = \frac{2dr}{r};$$

but this last equation is going to be integrated, with the added constant, so that having established $x = n$ makes $r = 2n$; whence such complete equation arises

$$\log x - \log n = 2\log r - 2\log 2n$$

or

$$\log\frac{x}{n} = 2\log\frac{r}{2n}$$

or, by the assumed numbers,

$$\frac{x}{n} = \frac{rr}{4nn},$$

or finally

$$x = \frac{rr}{4n};$$

which I have demonstrated.

§ 7. While indeed the use of this method may reveal more, we shall add another condition to our problem and of such kind indeed, as again the argument with the most recent change to be deduced has supplied to me.

We put of course, the tickets restored into the urn to be divided in half into two classes, just as into black and white, so that before extraction any black whatsoever has its white partner, or else designated with similar sign by a number. Thus again the integer number of pairs will be $= n$ and all tickets will be $= 2n$ in black and white equally divided into two parts; however we imagine in addition, one or the other ticket of a class to have more inclination to exit according to a given law of whatever sort either constant or variable, so that blacks and whites are extracted in unequal numbers from the urn. We establish exactly the number of remaining black tickets in the urn $= s$ and of white $= t$; now is demanded a second time, how many pairs x among all the tickets, whites and blacks, are probably to be remaining in the urn?

For this question to be solved it is proper to examine each class of tickets separately and to subdivide them into two parts, of which one of the two contains tickets still joined, the other broken: we investigate concerning the blacks, of which the number $= s$; that number may be divided into two parts x and $s - x$; so any black ticket whatever of the part x will have its white partner, while any ticket whatever of the other one of the parts $s-x$ will be bereft its partner; but since the black tickets, each and every, are set to the same fate exposed equally, the decrement will be dx, insofar as it arises from the decrement ds,

$$= \frac{xds}{s},$$

because certainly there are x cases, to which the value of that decrement is $= ds$, and $s-x$ cases, to which it is $= 0$; similarly the decrement to the number x, which arises by diminution to the number t, is

$$= \frac{xdt}{t},$$

so the whole decrement dx, which arises from both classes by diminution, makes

$$= \frac{xds}{s} + \frac{xdt}{t};$$

but if

$$dx = \frac{xds}{s} + \frac{xdt}{t}$$

there will be

$$\frac{dx}{x} = \frac{ds}{s} + \frac{dt}{t};$$

because indeed from the beginning the single numbers, s and t are equals to the number n, the complete equation will be

$$\log\frac{x}{n} = \log\frac{s}{n} + \log\frac{t}{n};$$

whence, from the assumed number, there arises

$$\frac{x}{n} = \frac{st}{nn}$$

or

$$x = \frac{st}{n}.$$

§ 8. If it is imagined, the black tickets to be extracted with infinite ease, there will be $t = n$ constantly and $x = s$ will happen exactly, the truth of which corollary is clear by itself; for no white ticket will be extracted and any remaining black whatsoever in the urn of necessity will have its partner: the same prevails, if of the white it is understood, what we have supposed only of the black. But certainly, if both classes of tickets are being extracted with equal facility, then we shall find in regard to the question in the sixth paragraph; for there will be $s = t = \frac{1}{2}r$ and hence

$$x = \frac{rr}{4n},$$

clearly as we have discovered in the alleged paragraph six.

§ 9. By no means with difficulty again it is understood by reason of our method, but if, in the place of two linked tickets, three separate tickets or four separate ones joined or inscribed by number are put in the same place and from the gathered tickets three or four distinguished classes are formed with distinct colors, and finally it is established again, before the extraction had taken place, howsoever many classes to contain n tickets; then if it is supposed after extraction to be remaining in the urn s tickets in the first class, t tickets in the second, u tickets in the third, and finally z tickets in the fourth if there are four classes, there to be made of three numbers, which entire will have remained in the urn,

$$= \frac{stu}{nn}$$

or of four parts

$$= \frac{stuz}{n^3}$$

and so again with higher combinations: but we suppose n, s, t, u, z very great numbers and as if infinite all the time.

§ 10. One case remains, because again it pertains to our principle in some way. Namely when in paragraph seven we have imagined separate facility in the black and white tickets, which are extracted from the urn and what must bring about that the tickets belonging in one class or the other remain in unequal number in the urn, it will not be by the thing to add a few words, which pertains more closely to this hypothesis.

We shall observe therefore, either always a relation to be known between s and t, that is, between the black and white tickets remaining in the urn and then the relation to be able to be desired between the aforementioned facilities, whether that relation will be the same constant or with whatever manner of variability, or this latter relation to be known, from which the former relation is deduced.

§ 11. Let there be given therefore, for any extraction whatsoever, a proclivity for going out on behalf of the black tickets to a similar proclivity on behalf of the white tickets, as ϕ to 1, ϕ must be understood as any number whatsoever either constant or a given law of variability, and the denominations employed previously may be retained; thus it is well-known to be

$$ds : dt = \phi s : t,$$

because certainly decrements ds and dt for any imminent extraction whatsoever follow the ratio composed from the ratio of s to t and from the ratio of the facilities, to either class of the corresponding tickets, which next it is evident from our definition of the different facilities themselves; hence we deduce

$$\phi = \frac{tds}{sdt}$$

and this equation will serve the interests of either of the two relations determined from other givens.

If $\phi = 1$, that is, if the facility from each part of the two is similar, it makes

$$sdt = tds.$$

or

$$\frac{dt}{t} = \frac{ds}{s};$$

or

$$\log\frac{t}{n} = \log\frac{s}{n};$$

whence

$$t = s.$$

If

$$\phi = \frac{g}{h} = \text{a constant number of whatever kind,}$$

it makes

$$\frac{g}{h} = \frac{tds}{sdt},$$

whence

$$\frac{gdt}{t} = \frac{hds}{s}$$

or

$$g\log\frac{t}{n} = h\log\frac{s}{n}$$

or

$$t = n\left(\frac{s}{n}\right)^{h:g.}$$

If $\phi = \frac{n}{n+s}$,
it has

$$\frac{n}{n+s} = \frac{tds}{sdt}$$

or

$$\frac{ndt}{t} = \frac{n+s}{s}ds,$$

whence

$$n\log\frac{t}{n} = s - n + n\log\frac{s}{n}$$

and

$$n\log\frac{t}{s} = s - n$$

or

$$t = c^{(s-n):n}s,$$

where c indicates the number of which the hyperbolic logarithm is unity.

If again there is put $t = \frac{ss}{n}$
there will be

$$\phi = \frac{1}{2}$$

or if $t = s^f : n^{f-1}$
it makes

$$\phi = \frac{1}{f}.$$

If finally $t = \frac{2ss-ns}{n}$
it makes

$$\phi = \frac{2s-n}{4s-n}.$$

§ 12. I do not deny, our next problem, it is proposed just as in the seventh paragraph, to admit an easy solution from other principles; indeed such facility arises from that place, because each of two numbers s and t I have assumed as knowns, when nevertheless each of the two is unknown, since simply the number of tickets remaining in the urn is given first of the black next of the white; thus certainly the sum of the numbers s and t is given; but neither is recognized by itself and in the end they will be elicited out of the given function ϕ. Thus the problem assumes another appearance by far, and I see not yet, in what way it is able to admit a separate solution from our. But our solution itself will be established in this fashion.

We see in the eleventh paragraph, to be

$$\phi = \frac{tds}{sdt};$$

we have therefore

$$\frac{ds}{s} = \frac{\phi dt}{t},$$

hence

$$\log \frac{s}{n} = \int \frac{\phi dt}{t};$$

let c be put for the number, of which the hyperbolic logarithm is unity, and so there will be

$$\frac{s}{n} = c^{\int \phi dt:t}.$$

Because thus the value of the number s is being expressed by a function of the number t, the product st will be simply expressed by a function of the number t; now let again the given sum of all the remaining tickets in the urn $= r$ and there will be $s + t = r$ and because it gives s through t, the equation is had between t and r of which with help the number t will be changed into a function of the number r, so that finally the quantity st/n or the sought quantity x is considered as a pure function expressed of the number r. This general solution is laborious and intricate and it is realized out of the very innate character of the formulas, a solution to the problem by common analysis is not able to be deduced; truly most calculation in the cases is eased exceptionally, as now we shall see.

§ 13. Let us set uniquely to belong to the proclivity of a black ticket by itself, that it is extracted, doubly much, as is in regard to any white ticket whatsoever; so there will be

$$\phi = 2 = \frac{tds}{sdt};$$

whence

$$\frac{2dt}{t} = \frac{ds}{s}$$

and

$$s = \frac{tt}{n};$$

because indeed

$$s + t = r,$$

now there will be

$$\frac{tt}{n} + t = r,$$

likewise

$$x = \frac{st}{n} \qquad \S\ 7$$

$$= \frac{t^3}{nn}$$

or

$$t = \sqrt[3]{nnx};$$

hence the equation

$$\frac{tt}{n} + t = r$$

is changed into this

$$\sqrt[3]{nxx} + \sqrt[3]{nnx} = r,$$

which reduction again gives the sought equation:

$$x = \frac{\left(-\frac{1}{2}n + \sqrt{nr + \frac{1}{4}nn}\right)^3}{nn}.$$

There will be, for example, $r = \frac{5}{16}n$, it will happen $x = \frac{1}{16}n$, but under the hypothesis of equality of the proclivity for both classes of tickets to exit, there will be in the same place for example $x = \frac{25}{1024}n$, a ratio each of two of the values is as 16 to 25. There will be in addition $s = \frac{1}{16}n$ and $t = \frac{1}{4}n$ while under the other hypothesis it was $s = t = \frac{5}{32}n$. Thus it is clear abundantly, that if out of the sole origin of inequalities between the values of s and t our problem had been solved, it had been able to prevail scarcely otherwise than we have created it.