Continuatio argumenti de mensura sortis ad fortuitam successionem rerum naturaliter contingentium applicata*

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§ 1. In our previous studies concerning that argument we have examined the hypothesis thus far having the appearance of truth in the first place to consider, that the falsehood of it will have commenced to come in the midst of suspicion finally after countless experiments; I understand the equal proclivity of nature to forming each sex. Now certainly all experts admit with one voice, nature to favor the male sex more than the other or at least to this point to have favored more always. Certainly can it be that it has happened by blind chance or by the command of a natural law? For my part the prior is possible, the other certainly by far most having the appearance of truth and most probable; let us disregard the words and let us ponder the thing itself. Thus the value of the work will be as of the individual events which are able to happen, let us examine the probability of the cases for this other hypothesis, with respect to which nature is more fecund in forming a male descendent, than in the other and it in any given ratio but the same uniformly which I shall name a to b. I have encountered a new question, more amplified than the first infinities, besides the expectation concisely expressed with an elegant and sufficiently simple formula, which now I shall explain.

§ 2. Again let there be, just as in the second paragraph of the preceding dissertation, the number of annual births $= 2N$ and, in order that we may indicate the thing in a mathematical term, let us put for any birth the sex to be determined in this manner, that into an urn tickets have been restored some black for the male sex others white for the next sex to be specified; the number of black tickets will be $= a$, of white tickets $= b$: moreover the extracted ticket will announce the sex of any birth next restoring the ticket into the urn; because if by this manner the sex for $2N$ births is determined, it is sought how much be the probability that the number of boys become $= m$ precisely and the number of girls $= 2N - m$ exactly. Now the theory of combinations will give, if only all of the disposed will have been arranged, the following more general formula,

*Continuation of the argument concerning the measurement of chance applied to the random succession of natural events.

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which certainly sets up the value of the sought probability:

$$\frac{2N.(2N-1).(2N-2).(2n-3)\dots(2N-m+1)}{1.2.3.4\dots m.2^{2N}}\times\left(\frac{a}{b}\right)^m\times\left(\frac{2b}{a+b}\right)^{2N}$$

§ 3. I have marveled at the simplicity of the manner, by which this more general theory embraces the other sent ahead by us on behalf of each sex having equal power: for with $a=b$ put immediately it is observed to happen

$$\left(\frac{a}{b}\right)^m\times\left(\frac{2b}{a+b}\right)^{2N}=1$$

and exactly to produce the same formula plainly, as we have set forth in the second paragraph of the first dissertation. But if a very small inequality intervenes between a and b, immediately thenceforth a conspicuous difference will arise between the probabilities computed under each hypothesis, as often as the numbers for N are assumed larger; namely both probabilities for the same numbers m and N are as unity to the number

$$\left(\frac{a}{b}\right)^m\times\left(\frac{2b}{a+b}\right)^{2N},$$

which ratio generally recedes to a ratio of equality completely as the magnitude of the number m and this quality makes available a criterion to us not at all to be despised in distinguishing a law of nature, if natality tables, for many years, are at hand. Look at an example of this thing.

The number of all births will be= 20000 or $N=10000$ and let the discussion be of the special case, by which that sum of births from each sex is divided into two parts perfectly equal to each other: we will have $m=10000$; let be put $a/b=1.055$, which value corresponds to observations not badly: thus it becomes

$$\left(\frac{a}{b}\right)^m\times\left(\frac{2b}{a+b}\right)^{2N}=\left(\frac{1055}{1000}\right)^{10000}\times\left(\frac{2000}{2055}\right)^{20000}=\frac{1}{1296}.$$

Therefore the probability, which for this case is however a very small thousandth, will be one thousand two hundred ninety six greater if only there will have been $a/b=1$ than if there is $a/b=1.055$; but we have seen in our preceding dissertation in the seventh paragraph the probability for the first position to be $\frac{1}{177}$; therefore the probability for the second position will be $=\frac{1}{177}\times\frac{1}{1296}$ or $\frac{1}{229392}$. If we add all to this very small probability, in which the number m is suppressed below the number N, however much the prior is able to become by the methods to ten thousand as the number m vanishes completely, yet the sum of all these probabilities for ten thousand cases is not stronger than the probability is for a single case, where it is assumed $m=N$; whence it is deduced, if the question will have been how much the probability be that at London more girls may be born than little boys within a year or in equal number at least, this probability to be less than $\frac{1}{11469}$; but it is certainly similar, that it may happen just about once in any course of 12000 years. A table exists here and there, where annually from the year 1664 to the year 1758 the number is indicated so many of little girls as of baptized little boys at London in the Episcopal Church, where it is to see never within 95 years to have happened that the number of girls was equal, still less greater, than the number of boys, although the number of all annual baptisms notably was less than 20000 and truly it should have been able to happen much more easily; in the year 1703 the girls have approached most to equality with the boys; certainly 7683 girls have been baptized and 7765 little boys; but of course this small difference has been most difficult to overcome by far.

§ 4. The formula exposed at the end of the second paragraph exhibits the nature of our argument excellently. Under the first hypothesis, where is put $a = b$, the probability decreases from the middle toward one of the two extremities; under the second hypothesis, where is put $a > b$, first it increases to a certain term beyond which it decreases; near the middle, where $m = N$, the probability under the first hypothesis exceeds greatly the probability under the other hypothesis; because certainly in the former it decreases in the latter it increases, the place will be where the same probability is under each hypothesis, the other place where the probability, under the latter hypothesis, may be double, triple, quadruple &c. now these places or values I shall define m; but it is required, that the factor

$$\left(\frac{a}{b}\right)^m \times \left(\frac{2b}{a+b}\right)^{2N}$$

be put successively equal to 1, 2, 3, 4 &c. and thence the number m is determined. Let us begin with the first equation and we will find

$$m = \frac{2N(\log(a+b) - \log(2b))}{\log a - \log b}.$$

Let us call this first value A and thus we will have successively

$$\begin{aligned} m &= A \\ m &= A + \frac{\log 2}{\log a - \log b} \\ m &= A + \frac{\log 3}{\log a - \log b} \\ m &= A + \frac{\log 4}{\log a - \log b} \end{aligned}$$

Thus generally there will be

$$m = A + \frac{\log \phi}{\log a - \log b}$$

if it should be desired that the two probabilities must themselves be as $1 : \phi$. Let us descend to a numerical example.

§ 5. Let there be further $N = 10000$ and $\frac{a}{b} = 1055/1000$; there will be $A = 10134$ and generally

$$m = 10134 + \frac{\log \phi}{\log 1.055}$$

or, with the common logarithm employed,

$$m = 10134 + 43 \log \phi$$

whence if there is put successively

$\phi = 1$	there will be had	$m = 10134$
$\phi = 2$		$m = 10147$
$\phi = 3$		$m = 10154$
$\phi = 4$		$m = 10160$
$\phi = 5$		$m = 10164$
$\phi = 10$		$m = 10177.$

It is evident from this here how the ratio which intercedes between the probabilities for the two positions $\frac{a}{b} = 1$ and $\frac{a}{b} = 1.055$ increases immensely. It is understood likewise with respect to which how often the number of boys born increased by forty-three is located so often is the ratio between the two proportional probabilities multiplied tenfold.

§ 6. The relation between ϕ and m relates to the logarithm so that by the most simple operation the number m is able to be indicated, for which the ratio ϕ may obtain the given value. Let, for the sake of an example, for the numbers assumed in the preceding paragraph, m indicating the number of boys, which happens ten times one thousand by a thousand thousand times more easily, specify $\frac{a}{b} = 1.055$ than specify $\frac{a}{b} = 1$. In this example let $\phi = 10,000,000,000$ and $\log \phi = 10$ therefore (§ 5) $m = 10134 + 430 = 10564$ by which one must not be amazed by the almost extraordinary distinction of the probabilities for the case, by which the number of boys crosses the center, by the small number 564 among 20000. But if therefore with the rarest case it will have happened that among 20000 births 10564 little boys had been counted and exactly 9436 girls, who, understanding of these things, will establish the nature according to each sex the proclivity to be formed absolutely equal? It is certain anyhow, a case of this kind to become more probable by $10,000,000,000$ times, if there will have been $\frac{a}{b} = 1.055$, ever so much also then certainly scarcely it merits to restore among the possibles, seeing that the probability of that case alone is only $= \frac{1}{1416000}$. If further therefore that least probability is increased, in order that it comprehends all cases, in which the number of little boys crosses the number 10564, scarcely thence it will become ten times greater, so much I am able to decide without the established calculus; thus each probability will become only $= \frac{1}{141600}$, by which neglected it will be permitted to affirm under no circumstances about to be that the number of boys born may ascend to 10564 out of 20000 births, even if to the male sex its prerogative may be assigned, as the value $\frac{a}{b} = 1.055$ indicates.

§ 7. We have seen presently, what very little appearance of the truth there is, that for 20000 births the number of boys ever may ascend to 10564 and of girls truly be kept down to 9436 and thus the difference between both sexes may spread to 1128, even if nature may favor the male sex over the other in ratio 1055 to 1000. Thus curious of this thing I have consulted the London table cited above, which the celebrated *Süssmilch* reckons in the second part of his distinguished work,[1] to which at the end many tables

[1] J. P. Süssmilch, "Die Göttliche Ordnung in den Veränderungen des menschlichen Geschlechts, aus der Geburt, Tod, und Fortpflanzung desselben erwiesen," Berlin 1741 (1st edition).

of this kind have been added: I have examined years: where the number of boys most surpassed the girls; the remarkable ones to me have been seen the year 1676, where 6552 little boys born or rather baptized and 5847 little girls are indicated, next the year 1698, where 8426 males and 7627 females; finally the year 1717, where 9630 males and 8845 of the other sex are indicated. But I have shown the differences between each sex to be changed in subduplicated ratio of the numbers $2N$ as the same probability is returned: therefore by this correction employed I have discovered none out of the three alleged years to be, which are worthy of greater attention, than if for 20000 births the excess of boys over girls will have been just about 900, which excess wants much yet from 1128.

Nevertheless I shall not leave unsaid the especially rare year 1749 in view of all the others, where namely baptized 7288 little boys and so many 6172 little girls are indicated; we have here the excess of boys $= 1116$, while the sum of all births was at least $= 13460$; therefore the aforesaid excess 1116 must be increased in subduplicate ratio of the numbers 13460 to 20000, by that reduction accomplished the aforesaid excess is changed to 1361; now certainly that excess notably surpasses the excess 1128, which not without reason we have supposed in the very long series of more than thousands of years is going to happen once with difficulty; therefore I persuade myself an error to have crept into one or the other of the numbers 6172 and 7288; without doubt much more easily it is, that in tables stuffed with so many numbers and most often copied an error may creep in sometime or other than that an extraordinary inequality may find a place; on the other hand I suppose in the place of 6172 little daughters to have been written 6972; certainly with this change accomplished the relation of one number between either sex becomes especially probable, which had been not so much impossible. I have examined the error suspected of the numbers, which indicates the sum of the baptized young daughters within the ten years from the year 1741 to the end of the year 1750; in the table for the sum 70322 is put, which used with my correction agrees perfectly to the thing itself; therefore the error has been committed either by the writer or by the typesetter, nor do I doubt in fact the annals of London are going to confirm my conjecture.

It is permitted to add a word concerning the baptism table, which the same author produces page 13 for the capital city of Austria; a single inspection of it for me has provoked anger; it contains nothing, not even I fear to say, except pure fictions, written down with wandering pen, how deceitful is this table, without our calculations it is scarcely understood nor am I amazed, because the celebrated *Süssmilch* must have considered it worthy to insert into his extraordinary work, producing the reported has restored unrestrained confidence to each one.

§ 8. It is understood from the previous, that with the assumed number

$$m > \frac{2N(\log(a+b) - \log(2b))}{\log a - \log b}$$

the probability, under the hypothesis $a > b$, may surpass excessively the probability under the hypothesis $a = b$; the contrary is obtained when it is assumed

$$m < \frac{2N(\log(a+b) - \log(2b))}{\log a - \log b}.$$

Of course, with the significance of the letter ϕ retained, now the ratio between each probability will be expressed by $\frac{1}{\phi}$ and since there is

$$\log\frac{1}{\phi} = -\log\phi,$$

there will be had (§ 4)

$$m = A - \frac{\log\phi}{\log a - \log b};$$

and, for example $m = 10000$, there becomes (§ 5)

$$m = 10134 - 43\log\phi,$$

where now ϕ designates, how much the probability, under the hypothesis $a > b$, must be surpassed by the probability under the hypothesis $a = b$. Let be put again $\phi = 10,000,000,000$ and there becomes $m = 10134 - 430$ or $m = 9704$.

Thus understand this thing. There is sought, under the hypothesis $a = b$, the probability that the number of boys is $= 9704$
and that probability will be discovered just about $= \frac{1}{177\times 8000}$
or $= \frac{1}{141600}$;
if it is assumed of this least small fraction $\frac{1}{10,000,000,000}$,
it will be held for the hypothesis $a = 1.055b$ the probability that the number of boys is $= 9704$. Smallness of this kind is hardly understood by the mind. Finally if all and individual probabilities are united into a sum that the number of boys is suppressed below 9704 and there is provided by this summation the probability to become ten times greater,
the combined probability becomes $= \frac{1}{1,416,000,000,000,000}$,
by which disregarded it is permitted to affirm, it not be able to happen that the number of boys cross the limits 10564 and 9704, if it will have been $a = 1.055b$ nor the number of girls the limits 9436 and 10296.

§ 9. Now I am undertaking another question, by what then is the number m the annual male offspring given with greatest probability in view of all the rest? Truly, under the hypothesis $a = b$, in the first paper I have assumed without demonstration, because at that time the thing is clear by itself, to be made $m = N$. But when an inequality between a and b is supposed, the present question assumes another form. We will recur to the formula exposed at the end of the second paragraph, which for any number m expresses its probability, certainly

$$\frac{2N.(2N-1).(2N-2).(2n-3)\dots(2N-m+1)}{1.2.3.4\dots m.2^{2N}} \times \left(\frac{a}{b}\right)^m \times \left(\frac{2b}{a+b}\right)^{2N}$$

In that formula the value certainly of the indefinite factor begins to decrease immediately and moreover the number m is assumed $> N$; but in fact certainly the other variable factor $\left(\frac{a}{b}\right)^m$ seeing that it proceeds to grow continuously, it is evident to be able to have a position, where the product is a maximum out of the two factors; hence in some way or other as if eccentricity. It will be permitted to define that place of maximum probability from it certainly, because the probability must be the same for the two indices nearest to m and $m+1$. Moreover with the assumed

$$\frac{2N.(2N-1).(2N-2).(2n-3)\dots(2N-m+1)}{1.2.3.4\dots m.2^{2N}} = S$$

the probability for the index m is

$$= S \times \left(\frac{a}{b}\right)^m \times \left(\frac{2b}{a+b}\right)^{2N}$$

and equally for the index $m+1$ the probability emerges

$$= \frac{2N-m}{m+1} \times S \times \left(\frac{a}{b}\right)^{m+1} \times \left(\frac{2b}{a+b}\right)^{2N}.$$

Therefore with an equation made between the two probabilities, it is discovered

$$\frac{2N-m}{m+1} \times \frac{a}{b} = 1$$

or

$$m = \frac{2Na-b}{a+b},$$

because certainly the term $2Na$ is as if incomparably greater than b, it will be able simply to put

$$m = \frac{2Na}{a+b}$$

and the number of girls or

$$2N - m = \frac{2Nb}{a+b}$$

so that the two numbers themselves are in the ratio a to b, that which confirms our formulas exceptionally.

§ 10. Thus therefore in our example, where we have put $2N = 20000$, the maximum probability falls at the number $m = 10268$; beyond and before this location the probability decreases, from the beginning at least most slowly, then more quickly, next enormously. We shall note here in passing what that point, concerning which §§ 4 and 5 we have asserted, where the probability is the same under each hypothesis, it is in the position at the center between the two points of maximum probability under each hypothesis; it is namely for this point $m = 10134$, which number is the center between 10000 and 10268 and this property generally takes place, if the difference between a and b is small.

§ 11. It serves chiefly to consider the aforementioned position, which makes

$$m = \frac{2Na}{a+b}$$

and for which the maximum probability arises, just as if a fixed point and thus all calculations to assume that the distance from this point is considered just as if from the center of strength: which in fact have been said they not yet support the computation enough: therefore the work is given that for whatever given index m the probability may be determined from some definite formula, in all events as near as possible since it cannot happen in all rigor. I have undertaken this path in the first paper certainly not without success; where concerning the thing in the first place the fifth paragraph appears; next the eighteenth.

§ 12. Let be assumed, for the sake of brevity, $\frac{2Na}{a+b} = M$,
thus so that M denotes the number of boys given with maximum probability
and let the index be assumed $m = M + \mu$
where μ will record the excess of the index above the number corresponding to the maximum probability:
let the probability be assumed in addition, for the index $M + \mu$ $= \pi$,
which true value now we have indicated above in the second and ninth paragraph but expressed by an indefinite formula, incomputable for large numbers. The mind considers to examine again, can it be that to this indefinite formula another nearest the actual and definite is able to be substituted; we will consider the quantities μ and π just as if coordinate variables. But it is clear from the indefinite formula itself the probability to be, for the index nearest $M + \mu + 1$,

$$= \frac{2N - m}{m + 1} \times \frac{a}{b} \times \pi$$

or

$$\frac{2N - m - \mu}{M + \mu + 1} \times \frac{a}{b} \times \pi,$$

which if it is subtracted from the preceding probability π the difference of one or the other probability will be

$$= \pi - \frac{2N - M - \mu}{M + \mu + 1} \times \frac{a}{b} \times \pi.$$

For certainly a second time I will suppose this difference to be compared with the difference of the two nearest indices, that is, compared with unity, just as $-d\pi$ to $d\mu$, which certainly without any sensible error is able to be supposed on account of the proximate relationship of the two indices; moreover this supposition furnishes the following equation

$$-\frac{d\pi}{\pi} = \left(1 - \frac{2N - M - \mu}{M + \mu + 1} \times \frac{a}{b}\right) d\mu.$$

In that equation for the quantity M I shall restore the value of it

$$\frac{2Na}{a + b},$$

that by so much more the quantities, which in the end of the calculation are able to be neglected, are able to be distinguished from one another; that fact becomes by restatement

$$-\frac{d\pi}{\pi} = \left(1 - \frac{2Na : (a + b) - \mu a : b)}{2na : (a + b) + \mu + 1}\right) d\mu$$

or

$$-\frac{d\pi}{\pi} = \frac{\mu + 1 + \mu a : b}{2Na : (a+b) + \mu + 1} d\mu$$

or finally

$$-\frac{d\pi}{\pi} = \frac{\mu + 1 + \mu a : b}{M + \mu + 1} d\mu.$$

§ 13. Thus the aforementioned equation is integrated so that with $\mu = 0$ assumed it happens $\pi = Q$, where by Q I understand the maximum probability, which has location as index M or $\frac{2Na}{a+b}$. Thus it produces

$$\log\frac{Q}{\pi} = \frac{a+b}{b}\mu - \frac{a+b}{b}(M+1)\log\frac{M+\mu+1}{M+1} + \log\frac{M+\mu+1}{M+1}.$$

Because certainly this equation must be serviceable only to the examples to be computed, in which the number μ is very much less than the number M, seeing that in the others the probability nearly vanishes and is not deserved that a ratio is considered of it, out of the thing there will be in the next to last term the quantity

$$\log\frac{M+\mu+1}{M+1}$$

to convert into series; in this series it will suffice to have considered the first three terms and thus to assume

$$\log\frac{M+\mu+1}{M+1} = \frac{\mu}{M+1} - \frac{1}{2}\left(\frac{\mu}{M+1}\right)^2 + \frac{1}{3}\left(\frac{\mu}{M+1}\right)^3;$$

in this way thus the equation will be able to be set

$$\log\frac{Q}{\pi} = \frac{a+b}{a}\left(\frac{\mu\mu}{2(M+1)} - \frac{\mu^3}{3(M+1)^2} + \log\frac{M+\mu+1}{M+1}\right)$$

or with c assumed for the number of which the hyperbolic logarithm is unity

$$\frac{Q}{\pi} = \frac{M+\mu+1}{M+1} c^{\frac{a+b}{b}\left[\frac{\mu\mu}{2(M+1)} - \frac{\mu^3}{3(M+1)^2}\right]}$$

or

$$\frac{\pi}{Q} = \frac{M+1}{M+\mu+1} c^{-\frac{a+b}{b}\left[\frac{\mu\mu}{2(M+1)} - \frac{\mu^3}{3(M+1)^2}\right]}$$

Because if we wish to relax more from scrupulosity, it will be permitted with that simple formula,

$$\frac{\pi}{Q} = c^{-\frac{a+b}{2b} \times \frac{\mu\mu}{M}}$$

Certainly this last formula is perfectly the same with that, which I have exhibited in the first paper § 18 and which I have confirmed in the following very small table; with on the other hand $a = b$ assumed, it happens likewise $M = N$ and thus there appears simply

$$\frac{\pi}{Q} = \frac{1}{c^{\frac{\mu\mu}{N}}}$$

§ 14. The aforementioned equation will be so much more accurate, as the difference between a and b is supposed less and as the number μ is likewise less, because each is appropriate enough to our principle; therefore it will be worthwhile to expose this equation to last examination.

Let the value indicated for the letter M of the one in the twelfth paragraph be restored, of course $\frac{2Na}{a+b}$;
thus the exponent will be

$$\frac{a+b}{2b} \times \frac{\mu\mu}{M} = \frac{(a+b)^2}{4ab} \times \frac{\mu\mu}{N};$$

let $b = 1$ and $a = 1 + \alpha$ be assumed, where α is put much smaller than unity; there will be had

$$\frac{(a+b)^2}{4ab} = \frac{4 + 4\alpha + \alpha\alpha}{4 + 4\alpha} = 1 + \frac{\alpha\alpha}{4 + 4\alpha};$$

here clearly the term added to unity is able to be neglected and indeed to be assumed $\frac{(a+b)^2}{4ab} = 1$,
by which fact the exponent becomes

$$\frac{a+b}{2b} \times \frac{\mu\mu}{M} = \frac{\mu\mu}{N}$$

and thus there is able simply to be put

$$\frac{\pi}{Q} = c^{-\frac{\mu\mu}{N}} \quad \text{or} \quad = \frac{1}{c^{\frac{\mu\mu}{N}}}.$$

Thus therefore, for each value $\frac{a}{b}$, the probability is expressed without change in the same manner, but, from the location of the distance of the term from the middle, the distance from the given term with the maximum probability is understood through μ, which special character certainly merits each attention.

§ 15. And also the probability itself of the term, which is a maximum, with the varying ratio $\frac{a}{b}$, remains very nearly the same for the same number μ and the same number N, and this other property is no less worthy to observe as well as it illustrates our entire argument excellently. Certainly I demonstrate it thus.

Let by hypothesis $a = b$,
the maximum probability $= q$,
what the condition is when it is assumed $m = N$;
next, for the same number N, let there be assumed a very small inequality between a and b and under that other hypothesis let the maximum probability be designated $= Q$;
but this § 9 happens at the index $\frac{2Na}{a+b}$:
let the difference be supposed between the two indices N and $\frac{2Na}{a+b}$,
which will be $= \frac{a-b}{a+b}N$.
Thus the probability of the term will be (under the hypothesis $a = b$), of which the index is indicated by $\frac{2Na}{a+b}$

$$= q : c^{\left(\frac{a-b}{a+b}\right)^2 N},$$

because namely there should be assumed for μ $\frac{a-b}{a+b}N$;

certainly this last probability, if it is multiplied by $\left(\frac{a}{b}\right)^m \times \left(\frac{2b}{a+b}\right)^{2N}$

it will give, by strength of the second paragraph, the probability Q for the same index $\frac{2Na}{a+b}$,

which will have been given by the maximum probability under the alternate hypothesis. Thus therefore there will be had

$$Q = \frac{q}{c^{\left(\frac{a-b}{a+b}\right)^2 N}} \times \left(\frac{a}{b}\right)^m \times \left(\frac{2b}{a+b}\right)^{2N}$$

where it is supposed $m = N + \mu$

or for this work $m = \frac{2Na}{a+b}$,

and by this fact by substitution there happens

$$Q = \frac{q}{c^{\left(\frac{a-b}{a+b}\right)^2 N}} \times \left(\frac{a}{b}\right)^{2Na:(a+b)} \times \left(\frac{2b}{a+b}\right)^{2N}.$$

Now I shall demonstrate, but if a and b differ slightly between themselves, the factor is able to be reckoned

$$\left(\frac{a}{b}\right)^{2Na:(a+b)} \times \left(\frac{2b}{a+b}\right)^{2N} = c^{\left(\frac{a-b}{a+b}\right)^2 N}.$$

thus as $Q = q$ is able to be assumed; to this end there should be put $b = 1$ and $a = 1+\alpha$ again by understanding by α a very small fraction; thus there happens

$$\frac{2Na}{a+b} = \frac{2+2\alpha}{2+\alpha}N = \left(1 + \frac{1}{2}\alpha - \frac{1}{4}\alpha\alpha + \frac{1}{8}\alpha^3\right) N;$$

therefore, with the hyperbolic logarithms assumed, the aforementioned equality to be demonstrated vanishes in this other

$$\left(1 + \frac{1}{2}\alpha - \frac{1}{4}\alpha\alpha + \frac{1}{8}\alpha^3\right) N \log\frac{a}{b} + 2N\log\frac{2b}{a+b} = \left(\frac{a-b}{a+b}\right)^2 N.$$

Because if now further the values for a and b are substituted $1+\alpha$ and 1 and the quantities $\log\frac{a}{b}$; $\log\frac{2b}{a+b}$ and $\left(\frac{a-b}{a+b}\right)^2$ are converted into series, with the terms neglected in which α transcends the third dimension, it will produce

$$\log\frac{a}{b}, \quad \text{or} \quad \log(1+\alpha) = \alpha - \frac{1}{2}\alpha\alpha + \frac{1}{3}\alpha^3;$$

next

$$\log\frac{2b}{a+b} = -\frac{1}{2}\alpha + \frac{1}{8}\alpha\alpha - \frac{1}{24}\alpha^3;$$

finally

$$\left(\frac{a-b}{a+b}\right)^2 = \frac{1}{4}\alpha\alpha - \frac{1}{4}\alpha^3;$$

but with these terms substituted, if the multiplications are actually performed with the terms neglected again, in which α transcends the third dimension, the identical equation is obtained perfectly. Therefore without any hesitation $Q = q$ is able to be reckoned. Thus our argument, which at first glance appeared greatly dark, shines forth with unexpected light; for the entire work had been put into it, that for whatever value $\frac{a}{b}$ the index μ is counted from the term given by the maximum probability, or that for μ the excess of boys is taken above the number $\frac{2Na}{a+b}$: by this fact the variations of the probabilities in each example will be most nearly the same for the same numbers μ assumed either affirmatively or negatively. But also probabilities themselves, where they are maximums, in each example, without any scruple, the same are able to be reckoned, if only the difference between a and b is not taken exceedingly large. Thus likewise the ratio of that quantity is apparent, of which we have made mention § 10 at the end.

§ 16. From the previous it is well-known, to the extent that the probabilities are able to be determined as nearly as possible for whatever number of births, for whatever number of boys and for whatever ratio between a and b. Behold the entire process! The maximum probability may be sought first for the hypothesis $a = b$, which by the power of the seventh paragraph of the earlier paper

$$= \frac{1.12826}{\sqrt{4N+1}}$$

and this probability will remain nearly the same, when it is the maximum, for each other ratio between a and b; indeed the maximum probability will be able to be put more simply

$$= \frac{0.56413}{\sqrt{N}}.$$

Next the number is assumed $\frac{2Na}{a+b}$, which indicates the number of boys delighting with maximum probability, by which fact whatsoever other number of boys expressed by the formula is given $\frac{2Na}{a+b} + \mu$; I say the probability to be nearly

$$= \frac{0.56413}{\sqrt{N}} \times \frac{1}{c^{\mu\mu:N}}.$$

Nor do I think this thing to be able to be made more suitable and likewise more accurate, when the number N is great. If that number will have been very small, the entire work with total rigor will be completed by the power of the formula expressed at the end of second paragraph. If finally the middle ones, one has seen by analysis, one wishes to prefer either formula to the other.

§ 17. I shall illustrate the entire process by a single example selected from the tables of Süssmilch, although somewhat less suitable on account of the enormity of our number μ: of course the celebrated Süssmilch in the second part of his work, to which at the end many tables have been added, page 13 table IV reports the example for the capital city of Austria in the year 1728, with respect to which 3102 little girls and 2020 boys had been born; therefore here

$$N = \frac{3102 + 2020}{2} = 2561$$

and put again $\frac{a}{b} = 1.055$
there happens $\frac{2Na}{a+b} = 2651;$
whence $\mu = 2020 - 2651 = -631;$
therefore for the most special case itself the probability is

$$= \frac{0.56413}{\sqrt{2561}} \times \frac{1}{c^{155}};$$

certainly that small fraction is less, than unity applied to unity prefixed to sixty-nine *zeros*, of which sort the smallness eludes every conception indeed if we add all probabilities, in which the number of boys is assumed to descend below 2021, hardly thence the aforementioned probability will be tripled; therefore if the question will be how much is the probability that among 5122 births the number of boys is reduced below 2021, I say the number is greater than three prefixed with sixty-eight *zeros*, to be able to be contested against one not to be that this happens; can it be that it will have happened in Vienna and can it be that with so many events repeated another similar monstrosity will have happened, just as the cited table reports, others may judge. Nor is it concealed to be able to happen, that in Vienna little daughters and also little boys are created more easily and more frequently; for in the same table it is reported for the year 1724, the baptized girls to have been only 1422 but 3005 little boys, any such enormous inequality contrary to the other, if again it be subjected to calculation, it will be held as morally impossible by everyone. Certainly the irregularities of such size are able to be ascribed neither to a natural law nor to chance in any way.

§ 18. But if the entire number of births is very small, then the cases which appear most extraordinary, are much less improbables than it is able to be presumed; I shall set forth an example, which is certain to me. Certainly in the year 1763 in a small town of the Basil Region, of which the name is Liechstahl,[2] 20 little sons and 37 girls had been born. The distribution of births of this kind, where the number of girls is nearly the double of the little boys, truly it is not able to not be greatly rare, but on account of the paucity of births it holds nothing, because with the aforesaid examples it is able to be compared in any manner. Behold the numerical calculation.

We have namely

$$2N = 57$$

and exactly (further with $\frac{a}{b} = \frac{1055}{1000}$ remaining) $\frac{2Na}{a+b} = 29.26;$
therefore $\mu = 20 - 29.26 = -9.26$
and $\frac{\mu\mu}{N} = 3.01;$
thus $c^{\mu\mu:N} = 20.08$
therefore the sought probability

$$= \frac{0.56413}{20.08\sqrt{28.5}} = \frac{1}{190},$$

[2]*Translator's note*: This would be the present day Liestal which lies about 20 km southeast of Basel.

which at least it is worth for the case itself, has been such. Therefore for 57 births it is probable that within 190 years that case most itself, such as has happened, may happen once. Because certainly it has been remarkable from the unique paucity of boys, rightly the probabilities of single cases should be added, where the number of boys is suppressed as yet more and then the sum of all these probabilities ascends just about to $\frac{1}{70}$; and indeed if the number of all annual observations will have been 70, it has been probable that it itself may happen once because it has happened, certainly that from 57 births the male offspring may be depressed below 21; I have made the other reduction of the probability $\frac{1}{190}$ to $\frac{1}{70}$ in passing; furthermore I confess the numerical calculations, on account of the smallness of the number N and the relative magnitude of the number μ, indeed not exactly to delight with accuracy; nevertheless I contend the errors to be very slight, which are able to be neglected easily.

§ 19. Now we understand further, because by computing the number μ from number, at which the maximum probability falls, the probability is the same for the same number μ, whatever ratio may intercede between a and b, if only this ratio recede very little from equality, we understand, it is said, that also the sum of the probabilities for the same number μ of the terms must be the same moreover in the first schedule, I have given a table by aid of this I have determined the limits between which, that the number of boys may subsist, the contest is equal; now I say the same limits to be able to be assumed for whatever slightly unequal ratio between a and b if only it happens that the maximum probability may fall in the middle of these limits. But we have seen in the last paragraph of the twelfth paragraph under the hypothesis $a = b$ and also $2N = 20000$, the fact that these limits are $9952\frac{3}{4}$ and $10047\frac{1}{4}$, which are equidistant from the number 10000, each evidently by the number $47\frac{1}{4}$; therefore now I say the fact that with the ratio a to b changed and with it put $= \frac{1055}{1000}$, with the same number of births remaining, a similar condition will happen within the limits

$$9952\frac{3}{4} + 268 \quad \text{and} \quad 10047\frac{1}{4} + 268$$

or within the limits

$$10220\frac{3}{4} \quad \text{and} \quad 10315\frac{1}{4}$$

equidistant from the term

$$10268$$

given by the maximum probability (§ 10) where the common distance again is $47\frac{1}{4}$. Therefore in return the probability is equal that the number of boys may cross or not cross these limits. I have been amazed at the very narrow width of these terms.

But also further in the preceding paper I have taught in the thirteenth paragraph, the distances of the limits to be diminished most nearly in subduplicate ratio of the numbers, which indicate the sum of all births. Thus for 5000 births an equal contest will be most nearly, to be that the number of boys is no larger than

$$2500 + 67 + 24 \quad \text{or} \quad 2591$$

nor less than

$$2500 + 67 - 24 \quad \text{or} \quad 2543$$

which limits are indicated simply

$$2567 \pm 24.$$

If in France the total annual offspring is assumed 600000 the mean male offspring will be $= 308040$ and it will be just about an equal wager, the excess or defect of the enumerated masculine offspring will not be greater than 260, if, it is compared with the middle position, with $\frac{a}{b} = 1.055$ put evidently.

§ 20. I hoped these inquiries concerning the equally probable limits will be useful, that a more prudent and more accurate judgment may be able to be produced concerning the true natural law or concerning the true proportion between the numbers a and b: can it be that of lands everywhere? can it be that it is constant for each time itself? can it be that all variations, which are remaining, are to be ascribed to chance? can it be that natural law itself permits some variation?

Even yet concerning this I hesitate: an exceedingly small difference between a and b is seen and an involvement of the influence of chance too much than that by the maximum number of observations is able to be determined accurately: finally the numbers themselves, which should have been able to be observed exactly, are not more reliable from every suspicion. The ratio $a : b$, which I ascribe to natural law, at any rate is not able to be deduced otherwise than out of a great number of observations; but many deductions of this sort are able to be deduced; the most simple method, by which it is assumed a to b to be, as the sum of the boys born to the sum of the young daughters born, to me until now seems to be preferred to all the rest. Nevertheless I think the criteria not to be despised, which have been placed within limits enriched with equal probability; here concerning the thing I shall say somewhat more eloquently.

§ 21. Where the sum of the births is greater, by it it is safer for determining the ratio $a : b$. Through the entire 95 years at London 737629 little sons and 698958 girls have been born; whence it is established best

$$\frac{a}{b} = \frac{737629}{698958} = 1.055$$

(in the writings of Süssmilch with a small error 1.054 is put see page 21.) From this mean value the observations recede at any time notably, although an entire ten year period is received: the ten year period 1721...1730 produces 89217 girls and 92813 young sons and thus $\frac{a}{b} = 1.040$, which value is least among all periods of ten years; the maximum is for the seven year period 1664...1670, in which 37283 girls and 40306 males have been born; whence $\frac{a}{b} = 1.081$; out of the summation of the aforesaid period of ten years and of seven years emerges $\frac{a}{b} = 1.054$, which value agrees enough with the ordinary hypothesis. The ten year period 1681...1690 indicates greater aberration; for $\frac{a}{b} = 1.097$ is put; but an error has been committed and 1.056 should be put in the place of 1.091. of this method the errors by themselves reveal themselves; but likewise of the other errors incurred, which in no way are able to be divined, they create dread. Nevertheless let us put certainly the value $\frac{a}{b} = 1.055$ and there should have been for the period of ten years 1721...1730, $2N = 182031$ (the number 92814 should have been put in place of the number 92813 indicating the sum of male offspring); thus it produces the number

$$\frac{2Na}{a+b} = 93451,$$

which rejoices with maximum probability for the number to be indicated of male offspring; but it has happened by lot that that number was only 92814; therefore the aberration due to chance here has been $= 637$ for the entire generation 182031; now let us put the greater generation yet by far, just as 4,000,000; but we have seen everywhere the aberrations for the same degree of probability to be in subduplicate ratio of the generations; therefore the aberration now will be equally probable

$$= 637 \times \sqrt{\frac{4000000}{182031}} = 2986:$$

but now there is further

$$\frac{2Na}{a+b} = 2053528,$$

which number indicates the most probable value of male offspring, with $\frac{a}{b} = 1.055$ put and then the number 1946472 of all little daughters arises; let us subtract the error 2986 due to chance from the number of boys and let us add the same to the number of young daughters, we will have the numbers 2050542 and 1949458, for each sex, which are able to happen with the same facility for the generation 4,000,000 as well as the numbers 92814 and 89217 have happened for the generation 182031; but there is $1949458 : 2050542 = 1000 : 1052$. Therefore, if even it is certain with $\frac{a}{b} = 1.055$ put, it will be nevertheless able to happen in order that even in the generation of 4,000,000 infants $\frac{a}{b} = 1.052$ may be revealed; further the error arising by chance is able to happen in excess with the same facility and then the ratio $\frac{a}{b} = 1.058$ may happen. Therefore the observations of two hundred years established in London, although they may have been most accurate, will not yet be sufficient for removing 0.006 hesitation in regard to establishing the natural law or ratio $\frac{a}{b}$. Wanderings to and fro are able to be much greater in by far fewer generations, as the tables confirm.

§ 22. I shall add one thing concerning the middle or equally probable limits of aberrations due to chance; namely again let the number be

$$\frac{2Na}{a+b} \pm \mu,$$

where

$$\frac{2Na}{a+b}$$

expresses the number of boys according to the law of nature and μ the aberration; we have seen for $2N = 20000$ there to be

$$\mu = 47\tfrac{1}{4}$$

and for whatever other value

$$\mu = 47\tfrac{1}{4}\sqrt{\tfrac{2N}{20000}} = 0.4725\sqrt{N};$$

therefore it will be equally probable that μ is greater or lesser than $0.4725\sqrt{N}$ and for many years the fortuitous events must answer to this law not badly, thus as one may happen nearly as often as another, not even the very small inequalities may destroy this quality of the value $\frac{a}{b}$. I have consulted therefore in the table of London the period of ten years 1721…1730 and with the calculation put I have had success almost beyond expectation. Behold the London table defended with my calculation, where the fifth column supposes

$$\frac{a}{b} = 1.055$$

or

$$\frac{2Na}{a+b} = \frac{1055}{2055} \times 2N;$$

but the seventh column[3] supposes

$$\frac{a}{b} = 1.040$$

and precisely

$$\frac{2Na}{a+b} = \frac{1040}{2040} \times 2N:$$

Year	Girls	Boys	Sum $2N$	$\frac{1055}{2055} \times 2N$	Aberration μ		$\frac{1040}{2040} \times 2N$	Aberration μ	
1721	8940	9430	18370	9431	$+\ 1$	NB	9365	$-\ 65$	
1722	9014	9325	18339	9414	$+\ 89$		9349	$+\ 24$	NB
1723	9392	9811	19203	9858	$+\ 47$	NB	9790	$-\ 21$	NB
1724	9468	9902	19370	9944	$+\ 42$	NB	9875	$-\ 27$	NB
1725	9198	9661	18759	9630	$-\ 31$	NB	9563	$-\ 98$	
1726	9203	9605	18808	9655	$+\ 50$		9588	$-\ 17$	NB
1727	9011	9241	18252	9370	$+129$		9305	$+\ 64$	
1728	8155	8497	16652	8548	$+\ 51$		8489	$-\ 8$	NB
1729	8324	8736	17060	8758	$+\ 22$	NB	8697	$-\ 39$	NB
1730	8512	8606	17118	8788	$+182$		8727	$+121$	

This very small table confirms excellently our entire theory, so clear as approached near. In the sixth column the aberration 47 has been noted by the sign NB, although such small quantity crosses the defined limits; in the sixth column the positive signs and in the eighth column the negative signs prevail, while yet the aberration to each part is able to rise with equal facility; it itself of the extraordinary, which has happened by chance, I attribute to the efficiency of chance. Certainly I shall not delay a long time with these content with the method exposed, by which likewise several other related arguments will be able to be treated with success.

[3]*Translator's note*: Over the years displayed in the table, the total number of boys is 92814 and the total number of girls is 89217. Hence the ratio is 1.040.