§II. ON THE CALCULUS OF PROBABILITIES

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1. I ask pardon with the geometers to return again to this subject. But I swear that the more I think, the more I am confirmed in my doubts on the principles of the ordinary theory; I desire that one clear up these doubts, & that this theory, either that one change some principles, or that one conserve it such as it is, or at least exposed distort in a manner to no longer leave any cloud.

2. I suppose that there are n different ways to bring about *heads*, & n different ways to bring about *tails*; I bring about *heads* on the first toss; is it probable that the impulse which will give me again *heads* on the second toss, will be *precisely* the same, as the one which has given it to me on the first toss? It seems to me as no. Now in this case, there will be no more than $n-1$ ways to bring about *heads* on the second toss, while there are again n to bring about *tails* on this second toss. There is therefore already a little more probability for *tails* on the second toss, as for *heads*.

3. This reasoning becomes again stronger if one has brought about *heads* many times in sequence. One will say perhaps that the number n is infinite, as much for *heads* as for *tails*, & that thus $n-m$ (m being finite) is counted always$= n$. It will be no less true, it seems to me, that the more the number m of tosses will be great, the more it will be probable that the toss which must follow, will be found in the *sequence* which has not yet been begun.

4. One objects that, if it is very little probable that *heads*, for example, does not happen 20 times in sequence, it is that there are $2^{20}-1$ combinations where *heads* will not happen in this way; & that for the same reason, if it is little probable that the same event not happen 20 times in sequence, it is that there are $2^{20}-1$ combinations for which it does not happen. But could not this little probability also be from another reason, from this that there are in nature some continuously active causes, which tend to change the state of it in each instant, & which do not permit that the same event happen a great number of times in sequence, & even a sufficiently small number of times? This reasoning has been developed by M. Beguelin in the Mém. de Berlin of 1767.

5. One says: *tails & heads in particular* are equally possible. Therefore taken *successively*, they are also equally possible. Is the consequence just? It is very certain that, *mathematically* speaking, any result does not depend on those which have preceded, & have no influence on those which follow, & that for this reason one must suppose, in the *mathematical* analysis, all the effects equally possible; but *physically* speaking, is this true, & does experience not prove the contrary? It is likewise this that one supposes in certain calculations of probabilities where the experience is sufficiently clear to us. It is possible, for example, mathematically & even to the rigor physically speaking, that 100 persons born together, & likewise well constituted, all arrive to old age, since each in particular can

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happen, & likewise aspire to it. However as experience has apprised us to the contrary, one bases on this experience the calculation of probabilities of the duration of life, & to the one of tontines & of life-annuities. Now experience teaches us the same, it seems to me, that a same event never happens a great number of times in sequence. Why therefore is there no regard in the calculus of probabilities?

6. We suppose that 2^{100} players cast a piece into the air 100 times in sequence; it is necessary, either that in two of these sequences of 100 casts, *heads & tails* are found each without mixing, & have consequently happened 100 times in sequence, or that there are at least two of the other sequences of casts (where *heads & tails* are found together) which are repeated. Now I believe, as I have already said in Book IV of these *Opusc*. pag. 299, that one can wager without fear that the two sequences where *heads & tails* could be found without mixing, will not take place & that thus there will be one or two, at least, of the other sequences, which will be found repeated two or more times.

7. It seems that in the problem of Petersburg, & in most of the others, there is some kind of contradiction to add altogether the partial expectations. In fact, if one must win, for example, only on the second toss, it is clear that one will not have won on the first. One can not therefore have at the same time the expectation to win on the first, & the expectation to win on the second. In order to have the total expectation, is it necessary to add the partial expectations which seem to exclude the one from the others? I do not wish to conclude from here that the result of this addition is not correct; see Book IV *Opusc*. page 300, art. 18. I say only that on this point the theory expresses, at least, in an obscure & little satisfactory manner.

8. I suppose that one plays at *heads & tails* in two tosses, & that one must play two tosses whatever happens. The probability that *heads* will happen on the first toss, is $\frac{1}{2}$; the probability that *heads* will happen on the second, by supposing, as one does here, that one plays this second toss in all the cases, is again $\frac{1}{2}$, at least according to the common theory. Therefore according to this same theory, the probability that *heads* will happen, at least one time, in two tosses, is $\frac{1}{2} + \frac{1}{2} = 1$, that is to say is equal to the certitude that *heads* will happen on the first toss. Now I ask if this is true, or at least if a similar result, founded on similar principles, is very proper to satisfy the intellect. This question is so much more natural, that in the problem of Petersburg, the sum of the probabilities $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. to infinity, is in fact equal to the certitude 1, that is to say, to the certitude 1 for limit, as in fact the one must be, since the more one will play the tosses, the more it is probable & approaching to certitude, that heads will happen. But by this same reason the sum of the probabilities should not be $\frac{1}{2} + \frac{1}{2}$ in the case preceding, since this sum must not be $= 1$.

9. In this same problem of Petersburg, suppose that the écus promised by one of the players to the other, instead of being increasing according to the progression $1,\ 2,\ 4,$ &c. is decreasing according to the progression $1,\ \frac{1}{2},\ \frac{1}{4},$ &c. the expectation, according to the common theory, will be $\frac{1}{2} + \frac{1}{2\cdot 4}+$&c.$> \frac{1}{2}$. Now is this estimation very just? Because if the game is by a single toss, I must give only $\frac{1}{2}$ écu, because I can win only one écu; in the second case, where one supposes that one plays by many tosses, I must & can thus win only one écu; why therefore this difference of lot?

It is true that in the second case I can yet wager something on the second toss, & that in the first case I can win nothing, since there is no second toss. But in the end, that which I could win in this second case on the second toss, is much less than one écu; I have never in the most favorable case, but an écu to expect; & as the quantity $\frac{1}{2\cdot 4}$ which one adds to the expectation $\frac{1}{2}$ of the first toss, supposes that this expectation will not be realized, must it be added?

10. Suppose that in the problem of Petersburg one plays to n tosses, & that all the rest remaining the same, the player must receive 2^{n-1} écus if *heads* happens only on the nth toss, & nothing if *heads* happens before; his expectation & his wager, consequently, will be $\frac{1}{2}$, as if he played only on one toss. Now is this just? & Is there a player who wished to give solely one half-écu, in order to receive 2^{50} écus if *heads* happened only at the end of 51 tosses?

11. In the VI Volume of the *Savans Etrangers*, M. de la Place showed easily that if the piece has greater tendency to fall on one side than the other, without one knowing which side (supposition very plausible), & if one plays to x tosses, for two écus on the first, for four on the second, for eight on the third, &c. the player must give to his antagonist less than x écus if $x < 5$; x écus if $x = 5$, & more than x écus if $x > 5$. Thus the supposition sufficiently probable, that the piece has greater tendency to fall on one side than the other, requires that the player give yet one greater sum, if $x > 5$, than in the ordinary solution of the problem of Petersburg. The difficulty is therefore again augmented by this supposition.

12. But by supposing the same, as one makes in all these games, that the piece has an equal tendency to fall on two sides, it is very certain that a person would not wish to give 20 écus, & even more, for a player in this game; the difficulty subsists therefore always without having yet, it seems to me, been resolved well, & it seems unable to be, as long as one will hold oneself uniquely to the received principles for the probability calculus.

13. One supposes in article 11 preceding, that the piece cast into the air has necessarily more propensity to fall on one side than the other; this supposition, although probable enough, is not however rigorous, & it is able to be made absolutely that the piece be constructed in a manner to fall indifferently on one or the other side; & in general, we suppose that it may have as much probability as one will wish, w, $1 - w$; w', $1 - w'$, &c. that heads will come, w', for example, being $\frac{1}{2}$, if *heads* has no more tendency to come on one side than the other; in this case the probability or wager would be, according to the ordinary principles, the sum of the quantities or series $2w[1+2(1-w)+4(1-w)^2\text{ \&c.}]+2(1-w)(1+2w+4w^2\text{\&c.})+2w'[1+2(1-w')\text{\&c.}]$, the whole divided by m that I suppose to be the number of probabilities w, $1-w$; w', $1-w'$, &c. in such a way that if one names Ω the function $2w[1+2(1-w)\text{ \&c.}]$, the total wager will be $\frac{\int \Omega dw}{m}$, from $w = 0$ until $w = 1$. But the difficulty would remain always the same, & the wager always infinite, as it is easy to prove it, in the case where the number of tosses would be indefinite; & the greater even after 5 tosses, than if one supposed w =simply $\frac{1}{2}$. In fact, since in taking any w from $\frac{1}{2}$ (exclusively) until 1, one has always the wager smaller than the number n of tosses if this number is < 5, equal if this number $= 5$, & greater if this number is greater than 5, it is clear that regarding the wager as the ordinate of a curve of which w is the abscissa, & the parameter n, this ordinate will be always $<$, or $=$, or $> n$, in the cases which one just said, & that, consequently, the total area of the curve divided by the total abscissa corresponding (that is to say, the true value of the wager) will give a quantity which will be $<$, or $=$, or $> n$.

14. In this same problem of Petersburg, the odds are 1 against 1 that I will win only one écu; because *heads* is able to come on the first cast, & in this case the game is ended. However, if one plays to 100 tosses, for example, it is necessary that I give 50 écus to the other player. Is it possible that there is equality in a parallel game, where it is necessary that I give 50 écus, where the odds are 1 against 1 that I will win only one écu, &, according to the ordinary theory, 63 against 1 that I will win only 32 écus, that is to say that I will not withdraw my stake? This would be so much worse if one would play to 1000, 10000, &c. tosses. It is very clear that the greater or lesser fortune of the player makes nothing

here, & that there is none who wished to play a similar game. One will say perhaps that this objection extends to the case where one would play only to two tosses, & where one should win 200 écus if *heads* happened only on the second toss. Because one would find the expectation $= \frac{1}{2} + 50$ écus; & I swear that in this case, & even in the preceding, one should give more than $\frac{1}{2}$ écu, although there are odds 1 against 1 that one will win only one écu; because one is able here to win 200 écus from the second toss, & in the other case 2 écus on the second, 4 on the third, &c. but it does not seem to me true that the ordinary theory has need here to be clarified or modified. Beyond the reasons brought above, we have again indicated besides one other reason plausible enough to defeat this theory; it is to regard in all the cases the *expectation* as the product of the expected sum by the probability that one will win this sum. It seems to us doubtful that this result is correct if the probability is very small, & that the probability $\frac{1}{10000}$ to win 5000 écus is the same as the probability $\frac{1}{2}$ to win one écu, as results from the ordinary principles. Under these principles, one names, it seems to me, inappropriately the *expectation* the product of the *expected sum* by the probability; it is the probability alone which forms the true *expectation*, & as the *expected sum*, as great as it be, does not augment this *probability*, it seems to me that one must not multiply this sum by the probability, in order to have that which one names the *expectation* of the player. In general, the more the probability to win is great, the more the player must give to his adversary, & the more the sum which he expects is great, the more also he must give to this same adversary; but that which he must give, must it be precisely in direct ratio of the expected sum, & of the probability? It is this which one supposes in the analysis of the games, & this which seems to me not incontestable.

15. Suppose that the characters cast on a floor give the word *Constantinopolitanensibus*, & that one asks of one ignorant if these characters have been cast at random or not; he would answer that it has all appearance that there have been cast at random. But if one put the same question to anyone who would know of the existence of *Constantinople*, & who would know the Latin language, he would respond on the contrary that the odds are total, & that it is even certain, that this arrangement is not the effect of chance. This is what the first ignores, & what the second knows as the arrangement of these characters is such, as it has been, almost surely, the work of one intelligent cause. It is likewise in the game in question. The experience & the knowledge that we have of the laws of nature, teach us that the same events never happen a great number of times in sequence; & it is by virtue of this *acquired knowledge*, that we revoke in doubt the repetition of *heads* or of *tails* a great number of consecutive times. As all is linked in the order of things, we could, if we knew the law of the connection of the causes & of the effects, to divine & predict that which will happen in each toss, if it will be *heads* or *tails*; under the ignorance where we are of the secret of nature, we cannot say precisely if it will be *tails* or *heads*; but as experience has taught us that the same result repeats itself rarely, we can at least, when *heads* has happened many times in sequence, conjecture with probability that *tails* will come. We suppose here that there is no particular reason to deduce from the construction of the piece, in order to have happen *heads* rather than *tails*; because if it were, *heads* arriving many times in sequence, it could render probable that *heads* will happen again.

16. M. de Buffon, in Book IV of his Supplémens à l'Histoire Naturelle, believes that the probability must be regarded as null, when it is equal to that which a man in good health will die in the day, a probability that he evaluates to $\frac{1}{10000}$. Consequently the probability in the problem of Petersburg is null, according to him, after the thirteenth toss; because on the thirteenth toss, the probability is $\frac{1}{2^{13}} = \frac{1}{8192}$, & on the fourteenth toss, it is $< \frac{1}{10000}$. In this case, the wager would be as 6 to 7 écus, & should likewise be further diminished, because

if the probability $\frac{1}{10000}$ must be calculated $= 0$, the probabilities $\frac{1}{8192}$, $\frac{1}{4096}$, &c. must be calculated lower than their value. I claim neither to adopt, nor to reject this hypothesis of M. de Buffon; I remark only that it confirms my doubts on the equality of possibility of all the cases.

17. M. de Buffon says further that having played 2048 times this game of *heads & tails*, this which makes 2048 matches, the 2048 games have produced in total 10057 écus, this which makes, says he, nearly 5 écus for each match, & it is to this sum that he limits the wager. This would suppose that one plays to 10 tosses; & in this case the player would catch up with his stake, & above, from the fourth toss, since if *heads* came only on the fourth toss, he would have 8 écus. But he loses, if *heads* came before. It is to the Mathematicians to judge this result, on which the uncertainty of the theory prohibits me to pronounce.

18. We see now if one would not find a result more conforming to the truth, by supposing that all cases not be equally possible, & that when *tails*, for example, has happened one or many times in sequence, there is place to expect that *heads* will come next, rather than *tails*, at least if the piece has no greater tendency to fall on one side than the other.

19. We suppose therefore that, if *tails* has happened on the first toss, the probability that *heads* will happen on the second is $\frac{1+a}{2}$, instead of $\frac{1}{2}$, a being a very small quantity; that if *tails* has happened the first two tosses, the probability that *heads* will happen the third toss, is $\frac{1+a+b}{2}$; & thus in sequence, in a way that $a+b+c+d$, &c. is never $= 1$, so that the probability never becomes absolute certitude. This posed,

20. The probability that heads will happen on the first toss, is $\frac{1}{2}$.

21. This which will happen only on the second toss; is the product of $\frac{1}{2}$, the probability that *tails* will happen on the first toss, by $\frac{1+a}{2}$, the probability that *heads* will happen in this case on the second toss.

22. The probability that *tails* will happen again on the second toss is $\frac{1-a}{2}$, & consequently the probability that two tosses will happen in sequence is $\frac{1}{2} \times \frac{1-a}{2}$; whence the probability that *heads* will happen only on the third toss, is $\frac{1}{2} \times \frac{1-a}{2} \times \frac{1+a+b}{2}$.

23. By the same reason the probability that *tails* will happen again on the third toss, is $\frac{1}{2} \times \frac{1-a}{2} \times \frac{1-a-b}{2}$, and that *heads* will happen only on the fourth toss, is $\frac{1}{2} \times \frac{1-a}{2} \times \frac{1-a-b}{2} \times \frac{1+a+b+c}{2}$.

24. Therefore, since one gives (*hyp.*) on the first toss 1 écu to Pierre, on the second 2, on the third 4, &c. under the hypothesis of the problem of Petersburg, the wager of Pierre will be $\frac{1}{2}[1+1+a+(1-a)(1+a+b)+(1-a)(1-a-b)(1+a+b+c)+$&c.$]$.

25. Therefore for a toss, the wager will be $\frac{1}{2}$; for two tosses, the wager will be $\frac{1}{2}(2+a)$; for three tosses, $\frac{1}{2}(3+a-aa+b-ba)$, &c.

26. It is clear that, as one has on the third term $(1-a)(1+a+b) = 1-a^2+b-ba$, this term will be $>$ than the second, if $b > \frac{a+a^2}{1-a}$, that is to say, if $b > a+2a^2+2a^3+$&c. in the contrary case it will be smaller, & equal if $b = \frac{a+a^2}{1-a}$.

The value of the first term of the sequence is $\frac{1}{2}$; & it first increases, as it is evident, its last value is evidently $= 0$, because the factor $1-a-b-c-d-e-f$, &c. to infinity $= 0$. Thus there is a term which is greatest.

27. But as each term is positive (until the last at infinity, which is $= 0$), the sum of the terms always increase.

28. Under the hypothesis that we follow here, the wager, which would be $\frac{1}{2}(1+1+1+$&c.$)$ by supposing all the cases equally possible, becomes (by supposing them unequally possible) $= \frac{1}{2}(1+1+a+(1-a)\times(1+a+b)\cdots+$&c.$)$. The last term of this

sequence, as one comes to observe, is zero, & the sequence $1,\ 1+a$, &c. first increases, that is to say that the second term at least is >1, thus the sum of this last term (if one supposes that the sequence has only a finite number of terms) can be $=$, or $>$, or $<$ than the sum of the first $1+1+1$, &c. according to the law of the quantities $a,\ b$, &c. but this sum is not for this one equal to infinity, when the number of terms $=\infty$.

29. In fact, we suppose that the quantities $a, b,\ c$, &c. are such as one will wish, but subject to the conditions expressed above; it is first clear that $\frac{1+a+b+c\ \&\mathrm{c}}{2}<1$. In second place, that $1-a-b-c-d-e$ &c. is $<1-a-b-c-d$ &c., by subtracting always the last term e.

30. Whence it follows that an arbitrary term $(1-a)(1-a-b)(1-a-b-c)(1-a-b-c-d)(1-a-b-c-d-e)$, is $<(1-a)(1-a-b)(1-a-b-c)(1-a-b-c-d)\times 2$; & the following term $(1-a)(1-a-b)(1-a-b-c)(1-a-b-c-d)^2\times 2$.

31. Let therefore m be the number of terms until $(1-a)(1-a-b)(1-a-b-c)(1-a-b-c-d)(1+a+b+c+d+e)$ exclusively, & M the sum of these terms, it is easy to see that by naming $1\pm\alpha$ the quantity $(1-a)(1-a-b)(1-a-b-c-d)$, the sum total of the series will be $<M+$a geometric progression of which $1\pm\alpha$ is the first term & the second $(1\pm\alpha)\times(1-a-b-c-d)$, the aforesaid sequence being multiplied by 2; that is to say that the sum will be $<M+\frac{2(1\pm\alpha)}{a+b+c+d}$, a quantity which will be always finite, & which could even, in the same way as M, be supposed contained within certain limits, according to the supposition that one will make on the value & the law of the very small quantities $a,\ b, c,\ d,\ e$, &c.

32. For example, suppose for a moment that $a,\ b,\ c,\ d,\ e$, &c. are so small, that one can neglect all the powers of these quantities, beginning with the square, the terms of the sequence will be evidently, by setting apart the constant coefficient $\frac{1}{2}$ which multiplies all,

$$\begin{array}{l} 1, \\ 1+a, \\ 1+b, \\ 1-a+c, \\ 1-2a-b+d, \\ 1-3a-2b-c+e, \\ 1-4a-3b-2c-d+f, \end{array}$$

& thus in sequence.

33. The first vertical column will give m (number of terms); if one takes successively the sum of the exterior terms diagonally by descending from left to right, & if one supposes them equal, one will have $a+b+c+d+e+f=a+a+a+a+a+a=(m-1)a$; & thus in sequence; after which taking back the sum of the remaining vertical terms, & supposing always $a=b=c$, &c. the total sum will be $m+(m-1)a-a(1+2+3+4\cdots+m-3)-a(1+2+3\cdots+m-4)-a(1+2+3\cdots+m-5)$ &c.$-a$. That is to say $m+(m-1)a-a$ multiplied by the sum of the triangular numbers from 1, until the one which is the $(m-3)^{\text{rd}}$, or $-a$ multiplied by the $(m-3)^{\text{rd}}$, pyramidal number.

34. Now the n^{th} pyramidal number being $\frac{n(n+1)(n+2)}{2\cdot 3}$, the $(m-3)^{\text{rd}}$ is $(m-3)(m-2)\times(m-1)\times\frac{1}{2\cdot 3}$. Thus the sum approached, but not rigorously correct, will be $m+(m-1)a-\frac{a(m-3)(m-2)(m-1)}{1\cdot 2\cdot 3}=m+\frac{a}{2\cdot 3}(-m^3+6m^2-5m)$, which is $<m$, if $m>5$. It is clear also that $\frac{2\cdot 3(m-1)-(m-3)(m-2)(m-1)}{2\cdot 3}=\frac{[6-(m-3)(m-2)](m-1)}{2\cdot 3}=\frac{(-mm+5m)(m-1)}{2\cdot 3}=\frac{-m(m-1)(m-5)}{2\cdot 3}$. Therefore the stake is $<\frac{1}{2}m$ if $m>5$, is $=\frac{1}{2}m$ if $m=5$, & $>\frac{1}{2}m$ if $m<5$.

35. One sees further that if m is very great, it is necessary to suppose $am^2 < 6$; so that the wager does not become negative, this which would not be able to take place in each case; but this is only a trial of imperfect calculus in order to show that the stake, under our hypothesis, is not longer infinite as in the problem of Petersburg.

36. More generally, let $1 - a - b - c - d - e$ &c. $= \frac{1}{1+\rho z}$, z being $= m - 1$, such that way $a + b + c + d$ &c. $= \frac{\rho z}{1+\rho z}$, ρ being a very small number, so that when $z = 1$, $a + b + c + d$ &c. which is reduced therefore to a, be very small.

37. One sees easily that when $z = 0$, or $m = 1$, $1 - a - b - c - d$, &c. $= 1$, & that, when z or $m = \infty$, one has $1 + a + b + c + d + d$, &c.$= 1 + \frac{\rho z}{1+\rho z} = 2$, as this must be.

38. It is clear that any term of which the rank is m, will be $= \frac{1}{1+\rho} \times \frac{1}{1+2\rho} \times \cdots \frac{1}{1+(m-2)\rho} \times \frac{1+\rho(m-1)}{1+(m-2)\rho} = \frac{1+2\rho(m-1)}{(1+\rho)(1+2\rho)\cdots(1+m\rho-\rho)}$, & that the difference of two consecutive terms is $\frac{1}{(1+\rho)(1+2\rho)\cdots(1+m\rho-\rho)} \times \left(1 + 2\rho(m-1) - \frac{1+2\rho m}{1+\rho m}\right)$. Now this last factor is $= -2\rho + \rho m - 2\rho\rho m + 2\rho\rho m^2$, the quantity evidently positive, especially when m is very great.

39. Thus, under this hypothesis, the terms are decreasing, by beginning from the second to the third term, since $m = 1$ renders the difference negative, & since $m = 2$ renders it positive.

39. In general, if one takes ω & ρ for any quantities, ρ being always very small, & $1 + \omega + \rho < 2$, the ratio of any term to the preceding is $\frac{(1-\omega)(1+\omega+\rho)}{1+\omega}$; now ρ is $< 1 - \omega$, since $1 + \omega + \rho < 2$; it is therefore $\rho = \rho'(1 - \omega)$, ρ' being a fraction, one will have the ratio in question $= \frac{1-\omega^2+\rho'-2\rho'\omega+\rho'\omega^2}{1+\omega}$, this quantity will be $<$, or $=$, or $> 1 + \omega$ if $-w^2 + \rho' - 2\rho'\omega + \rho'\omega^2 \overset{<}{\underset{>}{=}} \omega$, that is to say, if one has $\rho' \overset{<}{\underset{>}{=}} \frac{\omega+\omega^2}{(1-\omega)^2}$.

40. Since it is necessary that ρ' be a fraction; it is clear that in the case where $\rho' > \frac{\omega+\omega^2}{(1-\omega)^2}$, this last quantity must be a fraction, whence one deduces $\omega + \omega^2 < (1 - \omega)^2$, or $3\omega < 1$.

41. It is clear further, that the quantity $\frac{1-\omega^2+\rho'(1-\omega)^2}{1+\omega}$ is as much greater, as ω is less, in a manner that its greater value or rather the limit of these greater values is found by supposing $\rho' = 1$ & $\omega = 0$, this which gives 2 for the limit.

42. On the other hand, ρ' which must be < 1, must be > 0, this is why the limit of the smaller values of the ratio of any one term to the preceding, is $\frac{1-\omega^2}{1+\omega} = 1 - \omega$.

43. If the quantities a, b, c, d, e, &c. (art. 19 & following) are always decreasing in the number of $m - 1$, it is easy to see that by naming ϖ their sum, one will have $\rho < \frac{\varpi}{m-1}$. Let therefore $\rho = \frac{\rho'\varpi}{m-1}$, ρ' being < 1, & one will have $\frac{(1-\varpi)(1+\varpi+\rho)}{1+\varpi} = \frac{1-\varpi^2+\frac{\rho'\varpi(1-\varpi)}{m-1}}{1+\varpi}$, & this quantity will be $\overset{<}{\underset{>}{=}} 1$ according as $\frac{\rho'\varpi(1-\varpi)}{m-1}$ will be $\overset{<}{\underset{>}{=}} \varpi + \varpi^2$.

44. Therefore since ρ' must be < 1, it will be necessary, if one wishes that the first of these quantities surpasses or equals the second, that $(\varpi + \varpi^2) \times \frac{(m-1)}{\varpi(1-\varpi)}$ or $\frac{(1+\varpi)(m-1)}{1-\varpi}$, be a fraction, this which is impossible. Therefore the first quantity would not be able to be greater than the second, nor to be equal to it; therefore the terms are decreasing (farther from the second) in the series which expresses the wager. I say farther from the second; because it is clear by art. 24, that the second will be always $>$ than the first. In it here is enough to show that the terms of the wager are decreasing from the third toss, until the last. We have proved besides (art. 31) that the total wager, sum of these terms, is finite, even

by supposing the number of tosses infinite. Thus the result of the solution that we give here of the problem of Petersburg, is not subject to the insoluble difficulty of the ordinary solutions.

45. If $\frac{1+\pi}{2}$ is the probability that *heads* will come rather than tails, & $\frac{1+a'}{2}$ the probability that it will happen if *tails* has appeared n tosses in sequence, one can ask what will be the total probability which results from these two here?

46. Will one add together the probabilities $\frac{1+\pi}{2}$ & $\frac{1+a'}{2}$? But their sum $\frac{2+\pi+a'}{2}$ will be greater than unity, this which cannot be.

Will one be satisfied to add to the probability $\frac{1}{2}$ what will be that of *heads* in the case of $\pi = 0$ & of $a' = 0$, the increases $\frac{a'}{2}$ & $\frac{\pi}{2}$ that it receives by the two given assumptions? But the sum $\frac{1+a'+\pi}{2}$, could again be greater than unity.

47. Finally, will one add together the probabilities $\frac{1+\pi}{2}$ & $\frac{1+a'}{2}$, which must give *heads*, in order to divide them next by the sum $\frac{1+\pi}{2}$, $\frac{1+a'}{2}$, $\frac{1-\pi}{2}$, $\frac{1-a'}{2}$ of the probabilities which must give *heads* or *tails*, that is to say by 2? In this case, one will have $\frac{2+\pi+a'}{4}$, which is $<$ than the greater of the two quantities $\frac{1+\pi}{2}$, $\frac{1+a'}{2}$; instead it must be greater than the greatest of these quantities.

48. One could say that the probability $\frac{1+a'}{2}$ that *heads* will come, supposed above in the case where *tails* has already come many times in sequence, is not the same when $\frac{1+\pi}{2}$ is the probability that *heads* must fall rather than *tails*, & when this probability is simply $\frac{1}{2}$, that is to say that *heads* can happen the more so than *tails*. This observation can be very just. But it seems at least that the probability must be, in the case in question here, $> \frac{1+\pi}{2}$ which will be the probability of *heads* on the first toss. Now I ask, according to the principles exposed above, how must the probability $\frac{1+\pi}{2}$ be increased after *tails* has fallen in sequence a certain number of times?

49. Under the ordinary theory, & in speaking of all the principles admitted to the present by the Geometers on the indifference of the similar events successive or nonsuccessive, when an event, for example, *heads*, has happened many times in sequence, & when one has moreover no reason to believe that it must happen in that way, it is clear that there is some probability that *heads* had greater tendency to come than *tails*. But how to calculate this probability after the successive & supposed fall of *heads* a certain number of times in sequence? This question has relation to the research on the probability of causes by the events, of which many wise Geometers have themselves occupied. See in the *Philosophical Transactions* of 1763 & 1764, the researches of Messrs. Bayes & Price on this subject, & those of M. de la Place in Volume VI of *Mémoires des Savans Etrangers*, presented to the Academy of Sciences, & printed in 1774.