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DOUBTS AND QUESTIONS
ON
THE CALCULUS OF PROBABILITIES

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One complains rather commonly that the formulas of the mathematicians, applied to the objects of nature, are found only too much in error. No person nonetheless has further perceived or has believed to perceive this inconvenient in the *calculus of probabilities*. I have dared first to propose some doubts on some principles which serve as the base in this calculus. Some great geometers have judged these doubts *worthy of attention*; other great geometers have found them *absurd*; for why would I soften the terms of which they avail themselves? The question is to know if they have been wrong to use them, and in this case they could have doubly erred. Their decision, which they have not judged apropos to motivate, has encouraged some mediocre mathematicians, who have hurried to write on this subject, and to attack me without understanding me. I am going to try to explain myself so clearly, that nearly all my readers will be led to judge me.

I will remark first that it would not be astonishing that some formulas where we ourselves propose to calculate the same *incertitude*, can, in certain regards at least, participate in this incertitude, and allow in the mind some clouds on the rigorous truth of the result that they furnish. But I will not at all insist on this reflection, so vague that we can conclude nothing from it. I will not stop myself any longer to show that *the theory of probabilities*, such as it is presented in the books which treat it, is towards some points entirely neither so enlightening nor so complete as we can believe it; this detail can be extended only by the mathematicians; and yet one time I am going to try here to be extended to everyone. I adopt therefore, or rather I admit for good in the mathematical rigor, the *ordinary theory of the probabilities*, and I am going to examine only if the results of this theory, when they could be outside of the reach of geometric abstraction, are not susceptible to restriction, when we apply these results to nature.

In order to explain myself in the most precise manner, here is the point of the difficulty that I propose.

The calculus of probabilities is supported on this proposition, that all the different combinations of one same effect are equally possible. For example, if we toss a coin into the air one hundred times in sequence, we suppose that it is equally possible that *tails* happen one hundred times in sequence, or that *tails* and *heads* are *mixed*, by following moreover among them such particular succession as we would wish among them; for example, *tails* on the first trial, *heads* on the following two trials, *tails* on the fourth, *heads* on the fifth, *tails* on the sixth, on the seventh, etc.

[1]I do not know if these doubts on certain general principles received in the calculus of probabilities are so founded as they appear to me, but I believe at least to have proved that some very able mathematicians have supposed tacitly and without perception of them, in many scholarly researches, of the principles similar to those that I try to establish.

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These two cases are without doubt equally possible, mathematically speaking; this is not thence the point of the difficulty, and the mediocre mathematicians of whom I spoke a little while ago have taken the quite useless effort to write some long dissertations to prove this equal possibility. But the question is to know if these two cases, equally possible *mathematically*, are also *physically* and in the order of things; if it is *physically* also possible that the same effect happen one hundred times in sequence, then it is that this same effect is mixed with the others according to that law which we will wish to indicate. Before making our reflections above, we will propose the following question, well known of the algebraists.

Pierre plays with Paul at *heads* or *tails*, with this condition that if Paul brings forth *tails* at the first trial, he will give an écu to Pierre; if he brings forth *tails* only at the second trial, two écus; if he brings it forth only at the third, four écus; at the fourth, eight écus; at the fifth, sixteen; and thus in sequence until *tails* comes; we demand the expectation of Paul, or that which is the same thing, that which he must give to Pierre before the game begins, in order to play with him at a fair game, or, as we express ourselves ordinarily, for his *stake*.

The known formulas *of the calculus of probabilities* show easily, and all the mathematicians agree to it, that if Pierre and Paul play only to one trial, Paul must give to Pierre a half-écu; if they play only to two trials, two half-écus, or one écu; if they play only to three trials, three half-écus; to four trials, four half-écus, etc. Whence it is evident that if the number of trials is indefinite, as we suppose it here, that is to say if the game must cease only when *tails* will come, that which can mathematically speaking never happen, Paul must give to Pierre an infinity of times one half-écu, that is to say an infinite sum. No mathematician contests this consequence; but there is no one who does not sense and avow that the result of it is absurd, and that there is no player who wished in a fair game to risk five écus alone, and even much less.

Many great mathematicians have endeavored to resolve this singular case. But their solutions, which accord themselves not at all, and which are deduced from circumstances strange to the question, prove only how much this question is embarrassing.[1] One among them believes to have solved it by saying that Paul must not give an infinite sum to Pierre, because the wealth of Pierre is not infinite, and that he can neither give nor promise more than he has. But in order to see at what point this solution is illusory, it suffices to consider that, whatever enormous riches which we suppose to Pierre, Paul, unless being mad, would not give to him one thousand écus alone, although he must catch up to these thousand écus and to beyond if *tails* will happen only at the eleventh trial; more than two thousand écus if *tails* will happen only at the twelfth, four thousand écus at the thirteenth, and thus in sequence.

Now if we demand of Paul why he would not give these thousand écus? it is, he will answer, because it is not possible that *tails* will happen only at the eleventh trial. But, we say to him, if *tails* happens only after the eleventh trial, that which can be, you will win wealth beyond your thousand écus; I swear, Paul will reply, that in this case I could win considerably; but it is so little probable that *tails* not happen before the eleventh trial, that the gross sum that I would win beyond this eleventh trial, is not sufficient to engage me to incur this risk.

When Paul would keep himself to this reasoning, it would be already enough to show that the rules of the probabilities are at fault when they propose, in order to find the stake,

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[1] We can see these solutions in the fifth volume of the Mémoires de l'Académie de Pétersbourg, in the compilation of M. Fontaine, etc.

to multiply the expected sum by the probability of the case which must make this sum winning; because, whatever enormity that the expected sum is, the probability of winning it can be so small, that we would be insane to play a fair game. For example, I suppose that out of two thousand tickets of the lottery, all equal, there must be one of them which bears a lot of twenty million; it would be necessary, according to the ordinary rules, to give ten thousand francs for a ticket; and this is assuredly that which no person would dare make: if there will be found some men rich enough or foolish enough for that, we put the lot at two thousand millions, each ticket then will be one million, and I believe that for the trial no person would dare to take it.

However it is quite certain that whatever one would win in this lottery, and that consequently each of the bettors in particular have expectation to win; instead of which in the proposed case, where Paul would be obliged to give to Pierre an infinite sum, Pierre would always be certain to wager, however long that the game endured; so that Pierre will be in the right to complain, if having not fixed the number of trials, and *tails* arriving finally at such trial as we will wish, for example at the twentieth, Paul satisfied himself for his stake to give a sum double or triple, or one hundred times of five hundred twenty-four thousand two hundred eighty-eight écus, a sum which Pierre must on his side give to Paul.

In a word, if the number of trials is not fixed, and if Paul puts into the game, before if begins, such sum as he will wish, put he all the gold or silver which is on the earth, Pierre is right to say to him that he does not put enough, if we deduce it from the received formulas.

Now I demand if it is necessary to go seek very far the reason for this paradox, and if it does not leap to the eyes that this pretended *infinite sum* due by Paul at the beginning of the game, is infinite, in appearance, only because it is supported on a false assumption, namely on the assumption that *tails* can never happen, and that the game can endure eternally?

It is however true, and even evident, that this assumption is possible in mathematical rigor. It is therefore only *physically speaking* that it is false.

It is therefore false, *physically speaking*, that *tails* can never happen.

It is therefore impossible, *physically speaking*, that *heads* happens an infinity of times in sequence.

Therefore, *physically speaking*, *heads* can happen in sequence only a finite number of times.

What is this number? this is that which I at no point undertake to determine. But I am going further, and I demand by what reason *heads* is not known to happen an infinity of times in sequence, *physically speaking*? We can give for it only the following reason: *it is that it is not in nature that an effect is always and constantly the same, as it is not in nature that all men and all trees resemble themselves*.

I demand next if is it possible, *physically speaking*, that the same effect happen a very great number of times in sequence, ten thousand times, for example, when it is only that this effect happen an infinity of times in sequence? For example, is it possible, *physically speaking, that if one casts a coin in the air ten thousand times in sequence, there comes in sequence ten thousand times* heads *or* tails? On this I call to all players. Let Pierre and Paul play together at *heads* or *tails*, let it be Pierre who casts, and let *heads* happen only ten times in sequence, this will already be much, Paul exclaims infallibly, on the tenth trial, that the thing is not natural, and that surely the coin has been prepared in a manner to bring forth *heads* always. Paul supposes therefore that it is not in nature that an ordinary coin, fabricated and cast into the air without fraud, falls ten times in sequence on the same side. If we do not find ten times enough, we set it at twenty; there will result always that there is no player at all who makes tacitly this assumption, that one same effect is not known to happen in sequence a certain number of times.

There is some time that having had occasion to reason on this matter with a wise geometer, the following reflections came to me again, in support of those which I have already exhibited. We know that the mean length of the life of men, to count from the moment of birth, is around 27 years, that is that 100 infants, for example, coming at the same time into the world, will live only around 27 years taking one thing with the other; we have recognized likewise that the duration of the successive generations for the community of men is around 32 years, that is to say that 20 successive generations more or less, must give only around 20 times 32 years; finally we have proved by all the lists of the duration of the reigns in each part of Europe, that the mean duration of each reign is around 20 to 22 years, so that 15, 20, 30, 50 successive kings and more, reign only around 20 to 22 years taking one thing with the other. We can therefore wager, not only with advantage, but at a sure game, that 100 infants born at the same time will live only around 27 years taking one thing with the other; that 20 generations will endure no longer than 640 years or about; that 20 successive kings will reign only around 420 years more or less. Therefore a combination which will make the 100 infants live 60 years taking one thing with the other, which will make the 20 generations endure 80 years each, which will make 20 successive kings reign 70 years taking one thing with the other, will be illusory, and outside the *physically* possible combinations. However, to hold it to the mathematical order, this combination will be evidently as possible as any other. Because if two kings in sequence, for example, have reigned 60 years, there will be no mathematical reason that their successor not reign as much; the one here dies, there will be no longer be any mathematical reason that the following was not in the same case, and thus in sequence. Whence there results that there are some combinations which we must exclude, although *mathematically* possible, when these combinations are contrary to the constant order observed in nature. Now it is contrary to this order that the same effect happen 100 times, 50 times in sequence. Therefore the combination where we suppose that *tails* or *heads* happen 100 or 50 times in sequence, is absolutely to reject, although *mathematically* as possible as those where *heads* and *tails* are mixed.

Another reflection; because the more we think on this matter, the more it furnishes to it. There is no banker at all of Pharaon who does not enrich himself in this occupation; why? It is that the banker having the advantage in this game, because the number of cases which are winning is greater than the number of cases which are losing, there happens at the end of a certain time that there are more times of winning than losing. Therefore at the end of a certain time there has happened more cases favorable to the banker than unfavorable cases. Therefore since there are, as the calculus proves it and as we suppose it, more cases favorable to the banker than cases unfavorable, it is clear that at the end of a certain time, the sequence of events has in effect brought forth more often that which ought more often to happen. Therefore the combinations which contain more of the unfavorable cases than of the favorable, are, at the end of a certain time, less possible *physically* than the others, and perhaps even must be rejected, although *mathematically* all the combinations are equally possible. Therefore, in general, the more the number of favorable cases is great in any game, the more at the end of a certain time the gain is certain; and we can add even that this time will be so much less long as the number of favorable cases is greater. Therefore if Pierre and Paul are supposed to play at *heads* and *tails* during a year, for example, the one who will wager that *tails* or *heads* will not happen consecutively during an entire year, during one month even, will be *physically*, that is, *absolutely certain to win and to win* *much*. Therefore it is necessary to reject all the combinations which would give *heads* and *tails* a too great number of times in sequence.

Thence, and from that which we have said above, there results again another consequence; it is that if we suppose the time a little long, the combinations of *heads* and *tails* will happen in a manner that at the end of this time there will be very nearly as many of the one as of the other; so that if the coin is marked with 1 on the side of *heads* and with 2 on the side of *tails*, there will happen at the end of 100 times, or more, that the sum of the numbers which will come will be very nearly equal to 50 times 2 and 50 times 1, that is to say to 150; a new reason in order to reject from the number of these *physically* possible combinations, those which contain the same case a too great number of times in sequence.

Here is another question, which is the next of those which just concern us. If an effect has happened many times in sequence, for example, if *tails* happens three times in sequence, is it equally probable that *heads* or *tails* will happen at the fourth trial? It is certain that if we admit the preceding reflections, we must wager for *heads*, and it is in effect in this way that wealth of the players use it. The difficulty is knowing how much the odds are that *heads* will happen rather than *tails*; and it is on what the calculus has not taken enough.

That which we just said is based on the assumption that *tails* has not happened in sequence a very great number of times: because it would be more probable that this is the effect of some particular cause in the construction of the coin, and for when there will be advantage to wager that *tails* would happen next. Whatever it be, I imagine that there is no wise player at all who must in this case be embarrassed to know if he will wager *heads* or *tails*, while at the beginning of the game, he will say, without hesitation, *heads* or *tails* indifferently.

I demand therefore in consequence:

1°. If among the different combinations which a game admits, must we not exclude those where the same effect would happen a great number of times in sequence, at least when we will wish to apply the calculus to nature?

2°. Suppose that we must exclude the combinations where the same effect will happen, for example, 20 times in sequence; on what standing will we consider the combinations where the same effect will happen 19 times, 18 times in sequence, etc.? It seems to me little consequent to regard them as also possible as those where the effects would be mixed. Because if it is also possible, for example, that *heads* happen 19 times in sequence, as it is that *tails* happen on the first trial, *heads* next, next *tails* two times if we wish, and thus of the rest, by mixing *heads* and *tails* together without making them happen a long time in sequence the one or the other; I demand why we would exclude absolutely, as should never arrive in nature, the case where *heads* would come twenty times in sequence? How could it be that *tails* can happen 19 times in sequence, as well as any other trial, and that *tails* not happen 20 times in sequence?

For me, I see with this only one reasonable response: it is that the probability of a combination where the same effect is supposed to happen many times in sequence, is so much smaller, all things equal besides, as this number of times is greater, so that when it is very great, the probability is absolutely null or as null, and that when it is small enough, the probability is only small or point diminished by this consideration.

To assign the law of this diminution, it is this that neither me, nor a person, I believe, can make: but I think to have said enough in order to convince my readers that the principles *of the calculus of probabilities* could well have need of some restrictions when we will wish to consider them *physically*.

In order to strengthen the preceding reflections, permit me to add this here.

I suppose that one thousand characters that we would find arranged on a table, form a language and a sense; I demand who is the man who will not wager everything that this arrangement is not the effect of chance? However it is from the last evidence that this arrangement of words which gives a sense, is quite possible also, *mathematically speaking*, as another arrangement of characters, which would form no sense at all. Why does the first appear to us to have incontestably a cause, and not the second? if this is not because we suppose tacitly that it has neither order, nor regularity, in the things where chance alone presides; or at least when we perceive in some thing order, regularity, a kind of design and project, there is much greater odds that this thing is not the effect of chance, than if we perceived neither design nor regularity.

In order to expand my idea with yet more clearness and precision, I suppose that we find on a table some printed characters arranged in this way:

C o n s t a n t i n o p o l i t a n e n s i b u s,
or a a b c e i i i i l n n n n n n o o o o p s s s s t t t t u,
or n b s a e p t o l n o i a u o s t n i s n i c t n,

these three arrangements contain absolutely the same letters: in the first arrangement they form a known word; in the second they form no word at all, but the letters are disposed according to their alphabetical order, and the same letters are found as many times in sequence as they are found in turn in the twenty-five characters which form the word *Constantinopolitanensibus*; finally, in the third arrangement, the characters are pell-mell, without order, and at random. Now it is first certain that, *mathematically speaking*, these three arrangements are equally possible. It is not less that all sane men who would cast a glance on the table where these three arrangements are supposed to be found, will not doubt, or at least will wager everything that the first is not the effect of chance, and that he will scarcely be less led to wager that the second arrangement is not no longer. Therefore this sane man does not regard in some manner the three arrangements as equally possible, *physically speaking*, although the *mathematical* possibility is equal and the same for all three.

We are astonished that the moon turns about its axis in a time precisely equal to the one that it expends to turn about the earth, and we seek what is the cause of it? If the ratio of the two times was the one of two numbers taken at random, for example of 21 to 33, we would no longer be surprised, and we would not seek cause; however the ratio of equality is evidently as possible, *mathematically speaking*, as the one of 21 to 33; why therefore seek a cause in the first and not in the second?

A great geometer, Daniel Bernoulli, has given to us a scholarly memoir where he seeks by what reason *the orbits of the planets are contained in a very small zone parallel to the ecliptic, and which is only the seventeenth part of the sphere;* he calculates how much are the odds that the five planets, *Saturn, Jupiter, Mars, Venus* and *Mercury*, cast at random about the sun, would deviate themselves so little from the plane where the sixth planet turns, which is the *Earth*; he finds that there are odds more than 1400000 against one that the thing would not happen so; whence he concludes that this effect is not at all due to chance, and consequently he seeks in it and determines of it good or bad the cause. Now I say that, *mathematically speaking*, it was equally possible, either that the five planets deviate themselves as little as they are from the plane of the ecliptic, or that they take any other arrangement, which they would have much more deviation, and dispersed as the comets under all possible angles with the ecliptic; however no person is informed to demand why the comets are not limited in their inclination, and we demand why the planets have them? What can be the reason? otherwise again one time because we regard as very likely, and nearly as evident that one combination where it seems from the regularity and a kind of design, is not the effect of chance, although, *mathematically speaking*, it is also possible that any other combination where we would see neither order nor any singularity, and in which, by this reason, we would not think to seek a cause.

If we will cast five times in sequence a die with seventeen faces, and if all these five times *sonnez*[2] happens, Bernoulli could prove that it had precisely the same odds to make as in the case of the planets, that *sonnez* would not happen thus. Now, I demand of him if he would seek a cause in this event, or if he would not seek it: If he seeks not at all, and if he regards it as an effect of chance, why does he seek a cause in the arrangement of the planets, which is precisely in the same case? And if he seeks a cause in the trial of the die, as he must do in order to be consequent, why would he not seek a cause in any other particular combination, where the die with seventeen faces, cast five times in sequence, would produce some different numbers, without order and without sequence, for example 3 on the first trial, 7 on the second, 1 on the third, etc.? However there would be odds as great that this combination would not happen, as there would be odds that *sonnez* would not happen five times in sequence with a die of seventeen faces. Therefore Bernoulli would regard tacitly this last combination of *sonnez* five times in sequence, as being less possible than the other. He would suppose therefore that it is not in nature that the same effect happen seventeen times in sequence, mainly when the total combination of the effects indicates that the number of possible cases is equal to 17 multiplied four times in sequence with itself?

We go further, always according to the calculation of Bernoulli. If the planets were all in the same plane, and if we applied to that case there the reasonings of the author, we would find that there are odds infinity against one, that this arrangement would not happen, and we would conclude with him that the odds is infinite that this arrangement is produced by a particular cause and not fortune; that is to say, that it is *impossible* that this arrangement is the effect of chance; *because to wager the infinite that a thing is not, it is assured that it is impossible*. However any other particular and arbitrary arrangement as we will wish to imagine (for example *Mercury* at 20 degrees inclination, *Venus* at 15, *Mars* at 52, *Jupiter* at 40, *Saturn* at 83) is unique, as the one of the arrangement of the planets in the same plane; there are odds likewise of infinity against one that this case will not happen; why therefore does Bernoulli seek a cause in the first case, when he would not at all seek it in the second, if it is not by the reason which we have said?

That which there is of the singular, this is what this great geometer of whom I speak, has found *ridiculous*, at least that which one assures me, my reasonings on *the calculus of probabilities*. For complete response, I pray only he agree with himself, and to make us understand quite clearly why he would not seek a cause in certain combinations, while he seeks it in others, which, *mathematically speaking*, are equally possible?

I would add yet a reflection which seems to me to the advantage of the thesis which I support: it is that it was perhaps more possible, *physically speaking*, that the planets are found all in the same plane, that it is only one same effect happens one hundred times in sequence; because it is perhaps more possible that a single cast, a single impulse produces immediately on different bodies an effect which is the same, that it is only a body, launched successively at random one hundred times in sequence, takes the same situation by falling again: thus the reasoning that Bernoulli deduces from his calculus could be false, that perhaps ours would yet be correct. This could lead me to some other reflections on certain

[2] *Translator's note*: *sonnez*, the double six throw made with a pair of dice. In this context, it would appear that the meaning seems to be to make the same outcome 5 times.

cases which we regard as similar in *the calculus of probabilities*, and which, *physically speaking*, could well not be; but I will end here these doubts, by cautioning that if I am quite lengthy in giving them for some demonstrations, I will not cease any longer to believe them founded, as much as we will oppose only some *purely mathematical* considerations, or some responses that I know before that one has made them to me; in a word, as much as we will not resolve in a clear and precise manner the question which I have proposed on the game of *heads* and *tails*, and which we ourselves will believe by right to seek a cause to the symmetric and regular effects.

Perhaps one will say to me, for last resource, that if we seek a cause in the symmetric and regular effects, it is not that *absolutely speaking*, they could not be the effect of chance, but only because this is not possible. Here is all that which I see that one agrees with me. I will conclude from it first that if the regular effects due to chance are not absolutely impossible, *physically speaking*, they are at least much more likely the effect of an intelligent and regular cause, than the non-symmetric and irregular effects; I will conclude from it, in second place, that if there is in rigor, and even *physically speaking*, any combination which is not possible, the physical possibility of all these combinations, as much as we will suppose the pure effect of chance, will not be equal, although their mathematical possibility is absolutely the same. This will suffice in order to respond to all the difficulties proposed above, and among others to resolve the proposed question on the game of *heads* and *tails*. Because as soon as we will suppose that all the combinations are not equally possible, without even any regard as rigorously impossible in nature, we will find that Paul can not be obliged to give to Pierre an infinite sum. This is that which it would be very easy to prove *mathematically*; this is likewise of what a mediocre calculator could easily assure himself. But this calculus would be difficult to make understood to the community of our readers. I will suppress it therefore as being able to permit no objection, and I will await that some geometers, who merit that I read them or that I respond to them, combat or support the new views that I propose on *the calculus of probabilities*.

P.S. In finishing this writing, I fall by chance on the article *Fatalité* in the *dictionnaire Encyclopédique*, an article which we will recognize easily for the work of a man[3] of spirit and of philosophy; and here is that which I find there, apropos of taking *good luck* or *bad luck* in the game. "Either it is necessary to have regard to the past trials in order to estimate the next trial, or it is necessary to consider the next trial, independently of the trials already played; *these two opinions have their partisans*. In the first case, the analysis of chances leads me to think that if the preceding trials have been favorable to me, the next trial will be contrary to me; but if I have won so many trials, there are odds so much that I will lose the one that I come to play, *and vice versa*. I could never say therefore: I am in bad luck, and I will not risk that trial there; because I could say it only after the past trials which have been contrary to me; but these past trials must rather make me hope that the following trial will be favorable to me. In the second case, that is to say, *if we regard the next trial as completely isolated from the preceding trials*, we have no reason at all to estimate that the next trial will be favorable rather than contrary, or contrary rather than favorable; thus we cannot regulate its behavior in the game, according to the opinion of destiny, of good luck, or of bad luck."

From this passage I deduce two consequences. The first, that, according to the author of this excellent article, we can be divided on the question, *if it is equally probable that an effect happen or not happen, when it is already happened many times in sequence*.

[3] *Translator's note*: André Morellet (1727-1819) is the author of the article *Fatalité* in the Encyclopedia of Diderot.

Now it suffices to me that it is regarded as doubtful, in order to permit me to believe that the object of the preceding writing is not so strange as some clever mathematicians have imagined it. The second consequence, this is that the analysis of chances, such as the author of the article imagines, *gives less probability to the combinations which contain the successive repetition of the same effect, than to the combinations where this effect is mixed with others*. Now this is only to be said of the analysis of chances considered physically; because to consider it on the mathematical side alone, all the combinations, as we have said, are equally possible. I believe therefore to be able to regard the author of the article *Fatalité* as partisan of my opinion that I have tried to establish; and a partisan of this merit persuades me anew that this opinion is not an absurdity.