Sur la nouvelle méthode d'interpretation comparée la méthode des moindres carrés

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My new method of interpolation, as all those who have been proposed by geometers, is able to be reduced to the resolution of certain linear equations. Besides, the problems which serve to resolve linear equations are of two distinct types. In the one, the number of unknowns is fixed in advance, and the question is to draw from certain exact or approximate equations the values of these unknowns. In other problems, the number of unknowns which the formulas will contain is not fixed in advance, and one has, hence, to determine not only the values of the unknown arranged in a certain order, but still the number of those that one should calculate. We imagine, in order to fix the ideas, that the concern is to construct a series ordered according to the ascendant or descendant powers of a variable, and supposed convergent, in the case where one knows, for diverse values of the variable, the sum of the series. Then, evidently, one should research all at the same time, both the number of terms after which the series will be able to stop without that one has to fear sensible errors, and the values of these same terms. It is to the solution of the problems of the first type that the method of least squares have been generally applied; it is, on the contrary, in order to resolve the second kind of problems, that I have given in 1835 the new method of interpolation.

On the other hand, the values of m unknowns, linked to one another by n linear equations, n being equal or superior to m, are able to be calculated more or less rapidly and with an exactness more or less great. This rapidity, this exactness is able to depend, not only on the number and on the nature of the given equations, but still on the methods employed in order to resolve them.

[101] If one has

$$n = m,$$

that is to say if m unknowns $x, y, z, \ldots, v, w$ are determined by the system of m linear equations

$$\mathscr{A} = 0, \quad \mathscr{B} = 0, \quad \mathscr{C} = 0, \ldots, \quad \mathscr{X} = 0, \tag{1}$$

the values of the unknowns will not depend on the methods employed, which all will lead to the same results, but will be able to be more or less rapid. Then also, one will

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 6, 2010 be able to obtain these values by aid of the general formulas which present them under the form of fractions of which the common denominator is the resultant constructed with the coefficients of the diverse unknowns. But the calculation of the terms contained in the denominator and in the numerator of each fraction will be very painful, if the number m becomes considerable; and one will avoid this calculation if, after having eliminated successively x, then y, then z,..., then v from the given equations, one goes up again from the last of the formulas thus obtained to the first. Moreover, as, in order to eliminate a variable x from a linear function $\mathscr{B}$ by aid of a linear equation $\mathscr{A} = 0$, it suffices to subtract from the function $\mathscr{B}$ the product of $\mathscr{A}$ by the ratio between the coefficients of x in $\mathscr{B}$ and in $\mathscr{A}$, the successive elimination of the variables $x, y, z, \dots, v$ among the equations (1), will reduce the first members of these equations to the *differences of diverse orders* indicated, when one follows the notation that we have adopted, by aid of the characteristic letter Δ. After having thus reduced the functions $\mathscr{B}, \mathscr{C}, \dots, \mathscr{Z}$ to the *differences of the first order* $\Delta\mathscr{A}$, $\Delta\mathscr{B}, \Delta\mathscr{C}, \dots, \Delta\mathscr{Z}$, by eliminating x by aid of the equation

$$\mathscr{A} = 0;$$

then the differences $\Delta\mathscr{C}, \dots, \Delta\mathscr{Z}$ to the *differences of the second order* $\Delta^2\mathscr{C}, \dots, \Delta^2\mathscr{Z}$, by eliminating y; etc., one will be able to substitute in the equations (1) the *final equations*

$$\mathscr{A} = 0, \quad \Delta\mathscr{B} = 0, \quad \Delta^2\mathscr{C} = 0, \dots, \quad \Delta^m\mathscr{Z} = 0, \tag{2}$$

that one will resolve without difficulty by ascending from the last, which will furnish the value of w, in the preceding, which will furnish, the one after the other, the values of the unknowns $v, \dots, z, y, x$.

If one has $n > m$, that is to say if m unknowns $x, y, z, \dots v, w$ are linked among them by n linear equations

$$\varepsilon_1 = 0, \quad \varepsilon_2 = 0 \quad , \dots, \quad \varepsilon_n = 0, \tag{3}$$

[102] n being superior to m, there will happen from two things one: either the equations (3) will be exact, or they will be simply approximates. Under the first hypothesis, all the methods of resolution will lead to the same results, and one will be able to be content to resolve m equations, chosen arbitrarily in the given system, by applying to them the method indicated for the case where one had $n = m$. On the contrary, under the second hypothesis, that is to say when the equations (3) will be simply approximates, the diverse methods of resolution will be able to differ among them under the double relation of the brevity of the calculation and of the exactness of the results obtained. Then also, in order to construct the *final equations*, analogous to formulas (2), one will be able to employ two distinct processes. The first, that one is able to name *indirect*, consists in substituting in the n given equations m equations of the form (1), by taking for $\mathscr{A}, \mathscr{B}, \mathscr{C}, \dots, \mathscr{Z}$, m linear functions of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, and to deduce next from the equations (1) the equations (2), by eliminating one after the other the unknowns $x, y, z, \dots, v$. The second process, that one is able to name *direct*, consists in deducing directly the final equations from the given equations, without passing through the equations (1). When one has recourse to this last process, it is not necessary to fix a priori, and since the commencement of the operation, the values attributed to the diverse systems of factors by which one must multiply $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ in order to obtain the

functions $\mathscr{A},\mathscr{B},\mathscr{C},\dots,\mathscr{Z}$. In fact, let

$$\lambda_1,\lambda_2,\dots,\lambda_n;\quad \mu_1,\mu_2,\dots,\mu_n;\quad \nu_1,\nu_2,\dots,\nu_n;\dots$$

be these same factors, so that one has

$$\mathscr{A}=\lambda_1\varepsilon_1+\lambda_2\varepsilon_2+\cdots+\lambda_n\varepsilon_n,\quad \mathscr{B}=\mu_1\varepsilon_1+\mu_2\varepsilon_2+\cdots+\mu_n\varepsilon_n,$$
$$\mathscr{C}=\nu_1\varepsilon_1+\nu_2\varepsilon_2+\cdots+\nu_n\varepsilon_n,$$

One will have further

$$\Delta\mathscr{B}=\mu_1\Delta\varepsilon_1+\mu_2\Delta\varepsilon_2+\cdots+\mu_n\Delta\varepsilon_n,\quad \Delta^2\mathscr{C}=\nu_1\Delta^2\varepsilon_1+\nu_2\Delta^2\varepsilon_2+\cdots+\nu_n\Delta^2\varepsilon_n,\dots$$

Hence, in order to obtain $\Delta\mathscr{B}$, it will not be necessary to begin by constructing $\mathscr{B}$, by assigning immediately to the factors $\mu_1,\mu_2,\dots,\mu_n$ some determined values; it will suffice to reduce, by eliminating x by aid of the equation

$$\mathscr{A}=0,$$

the functions $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n$ to the differences of first order $\Delta\varepsilon_1,\Delta\varepsilon_2,\dots,\Delta\varepsilon_n$, next to add to one another these differences respectively multiplied by any factors $\mu_1,\mu_2,\dots,\mu_n$ which will be able to depend, if one wishes, [103] on these same differences, that is to say on the coefficients that they contain. Similarly, in order to obtain $\Delta^2\mathscr{C}$, it will not be necessary to begin by constructing $\mathscr{C}$, by assigning a priori to the factors $\nu_1,\nu_2,\dots,\nu_n$ some determined values; it will suffice to reduce, by eliminating y by aid of the equation

$$\Delta\mathscr{B}=0,$$

the differences of the first order $\Delta\varepsilon_1,\Delta\varepsilon_2,\dots,\Delta\varepsilon_n$ to the differences of second order $\Delta^2\varepsilon_1,\Delta^2\varepsilon_2,\dots,\Delta^2\varepsilon_n$, next to add to one another these differences of second order respectively multiplied by any factors $\nu_1,\nu_2,\dots,\nu_n$ which will be able to depend, if one wishes, on coefficients contained in these same differences; etc.

Before going further, we will make an important remark. In order that one is able to draw successively from equations (2), and ascending from the last to the first, the values of the unknowns $w,\dots,z,y,x$, it is necessary that the coefficients of x in the first, of y in the second, of z in the third, ..., of w in the last, not vanish. Besides, each of these coefficients being represented by the sum of many terms, one will have not at all to fear that it vanishes, if each of these terms is positive. Now, this is that which will arrive always, if, ε designating any one of the functions $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n$, the factor λ, or μ, or ν,... which, in the sum represented by $\mathscr{A}$, or by $\Delta\mathscr{B}$, or by $\Delta^2\mathscr{C}$, ..., precedes the function ε, or $\Delta\varepsilon$, or $\Delta^2\varepsilon,\dots$, is always a quantity affected of the same sign as the coefficient of the first of the unknowns contained in this same function. Henceforth, we will suppose this condition always fulfilled in the final equations formed by the direct process; and consequently, these equations will furnish always for the unknowns some finite values, which will be exact if the equations (3) are exact themselves.

We imagine now that, for brevity, one designates, by aid of the characteristic S, by the notation $S\lambda\varepsilon$, or $S\mu\Delta\varepsilon$, or $S\nu\Delta^2\varepsilon$, ... the sum of the products of the form $\lambda_l\varepsilon_l$, or $\mu_l\Delta\varepsilon_l$, or $\nu_l\Delta^2\varepsilon_l$, l being any of the numbers $1,2,3,\dots,n$; one will have

$$\mathscr{A}=S\lambda\varepsilon,\quad \Delta\mathscr{B}=S\mu\Delta\varepsilon,\quad \Delta^2\mathscr{C}=S\nu\Delta^2\varepsilon,\dots \tag{4}$$

Let besides α be the ratio between the coefficients of x in the functions ε and $\mathscr{A}$, β the ratio between the coefficients of y in the functions $\Delta\varepsilon$ and $\Delta\mathscr{B}$, γ the ratio between the coefficients of z in the functions $\Delta^2\varepsilon$ and $\Delta^2\mathscr{C}$, ... One will have

$$\Delta\varepsilon = \varepsilon - \alpha\mathscr{A}, \quad \Delta^2\varepsilon = \Delta\varepsilon - \beta\mathscr{B}, \ldots, \tag{5}$$

[104] or, that which returns to the same,

$$\Delta\varepsilon = \varepsilon - \alpha S\lambda\varepsilon, \quad \Delta^2\varepsilon = \Delta\varepsilon - \beta S\mu\Delta\varepsilon, \ldots, \tag{6}$$

This is not all: the equations (3) being linear with respect to x, y, z, ..., w, each of these equations will be able to be presented under the form

$$ax + by + cz + \cdots + hw = k,$$

or, that which returns to the same, under the form

$$\varepsilon = 0, \tag{7}$$

the value of ε being

$$\varepsilon = k - ax - by - cz - \cdots - hw, \tag{8}$$

and $a, b, c, \ldots, h, k$ being some constants which will receive, in the function ε_1, certain values $a_1, b_1, c_1, \ldots, h_1, k_1$; in the function ε_2, some other values a_2, b_2, $c_2, \ldots$, h_2, k_2, etc.; finally, in the function ε_n, some other values a_n, b_n, $c_n, \ldots$, h_n, k_n. This put, the first of the formulas (4) will give

$$\mathscr{A} = S\lambda k - xS\lambda a - yS\lambda b - \cdots - wS\lambda h, \tag{9}$$

and, hence, the ratio α between the coefficients of x in the functions ε and $\mathscr{A}$ will be determined by the formula

$$\alpha = \frac{a}{S\lambda a}. \tag{10}$$

Moreover, the first of the equations (6) joined to the formula (8) will give

$$\Delta\varepsilon = \Delta k - x\Delta a - y\Delta b - \cdots - w\Delta h, \tag{11}$$

the values of Δk, Δa, Δb, ..., Δh being determined by some formulas similar to the first of the equations (6) and that one deduces from it by substituting in the letter ε one of the letters k, a, b, ..., h, so that one will have, for example,

$$\Delta k = h - \alpha S\lambda k. \tag{12}$$

One will establish in the same manner the formulas

$$\Delta\mathscr{B} = S\mu\Delta k - yS\mu\Delta b - zS\mu\Delta c - \cdots - wS\mu\Delta h, \tag{13}$$

$$\beta = \frac{\Delta b}{S\mu\Delta b}, \tag{14}$$

$$\Delta^2\varepsilon = \Delta^2 k - y\Delta^2 b - z\Delta^2 c - \cdots - w\Delta^2 h, \tag{15}$$

the values of $\Delta^2 k$, $\Delta^2 b$, $\Delta^2 c$, $\ldots$,$\Delta^2 h$ being determined by some formulas [105] similar to the second of the equations (6), so that one will have, for example,

$$\Delta^2 k = \Delta k - \beta S\mu\Delta k, \ldots \tag{16}$$

By continuing thus, one will arrive definitely to the equations

$$\Delta^m\varepsilon = \Delta^m k - w\Delta^m h, \tag{17}$$

$$\Delta^m\mathscr{X} = S\varsigma\Delta^m k - wS\varsigma\Delta^m h; \tag{18}$$

and if from formula (17) one eliminates w by aid of the equation $\Delta^m\mathscr{X} = 0$, one will obtain a new formula, namely

$$\Delta^{m+1}\varepsilon = \Delta^{m+1}k, \tag{19}$$

which, joined to the diverse formulas already found, will furnish the constant values of the expressions of the form $\Delta^{m+1}\varepsilon$, that is to say of the differences

$$\Delta^{m+1}\varepsilon_1, \quad \Delta^{m+1}\varepsilon_2, \ldots, \quad \Delta^{m+1}\varepsilon_n. \tag{20}$$

These values, by virtue of formula (19), will be precisely those of the differences

$$\Delta^{m+1}k_1, \quad \Delta^{m+1}k_2, \ldots, \quad \Delta^{m+1}k_n. \tag{21}$$

Therefore these last as the previous will be reduced to zero, if one has $n = m$, or if the equations (3) are exact; and if, n being superior to m, the equations (3) will be only approximate, to some quantities which should be in general so much smaller (setting aside the signs) as the approximation will be greater.

We consider now in a special manner the case where the number m of unknowns is not given a priori. We suppose, in order to fix the ideas, that these unknowns are the coefficients contained in the diverse terms of a convergent series, of which k represents the sum, and that, hence, the constants

$$k_1, k_2, \ldots, k_n$$

express n values of this same sum determined directly, to the aid of a certain number of experiences or of observations. Generally these values, which will be able to be, for example, some angles measured by aid of instruments more or less perfect, will not be exact, but blemished by certain errors which will include the observations of which there is question. This put, we imagine that one employs, for the formation of the final equations, from which one must draw the values of the unknowns, [106] the direct process, which furnishes with these equations the diverse values of Δk, $\Delta^2 k$, $\Delta^3 k$, … In order that the values of $\Delta^{m+1}k$ become comparable to the errors of observation, it will be generally necessary that the whole number m acquire a value sufficiently great, and such that one is able, without sensible error, to be limited to conserve in the development of k into series the first m terms. Reciprocally, when, m coming to increase, the diverse values of $\Delta^{m+1}k$ will be become comparable to the errors of observation, the problem of the development of k into series will be able to be considered as resolved. Because, by attributing to the coefficients of the conserved terms the values given by the calculation, and to the coefficients of the terms neglected with the insensible values, one will obtain a series of which the sum k will have for particular values of the quantities very little different from $k_1, k_2, \ldots, k_n$, the differences being represented by the diverse values of $\Delta^{m+1}k$, and being able to be in consequence attributed to the errors of observation.

In summary, *if, in the development of a function k into a convergent series, of which each term contains an unknown coefficient, one wishes to determine at the same time both the number m of the terms after which one is able to arrest the series, without having to fear sensible errors, and the coefficients contained in these same terms, one should, in adopting the direct process for the formation of the final equations, to pay especially his attention on the values of the differences of the diverse orders*

$$\Delta k, \Delta^2 k, \Delta^3 k, \ldots$$

The number m will have effectively acquired the value that it is proper to attribute to it, when the diverse numerical values of $\Delta^{m+1}k$ *will be become rather small in order to be comparable to the errors of observation that include the diverse values of k.*

It is easy now to compare between them the two methods that Mr. Bienaymé has put in presence with one another, namely: the method of least squares and the new method of interpolation.

The end ordinarily assigned to the method of least squares consists in deducing from approximative equations the values of the unknowns of which the number is fixed in advance. On the contrary, the special end assigned to the new method of interpolation, in the Memoir of 1835, is to determine into a convergent series, proper to represent the development of a function, not the unknown coefficients of certain terms of which the number will be fixed in advance, but *the coefficients of the terms which one is able* [107] *to neglect without having to fear that there results a sensible error in the values of the function* (see the lithographed Memoir of 1835, page 3).

In the method of least squares, the diverse systems of factors are determined a priori, and each of them is confounded with the system of coefficients of one same unknown. On the contrary, in the new method of interpolation, the calculator, eliminating one after another the diverse unknowns, in an originally fixed order, and adopting, for the formation of the final equations, that which we have named the direct process, determines successively the diverse systems of factors in measure that the calculation advances, and reduces each factor to ± 1, the sign being the one of the coefficient of the unknown which must be eliminated the first. Moreover, by naming k the constant to which any one of the given equations reduces a linear function of the unknowns, the calculator stops the calculation at the moment where the number m of these unknowns become considerable enough in order that the diverse numerical values of $\Delta^{m+1}k$ are comparable to the errors of which the value of k is susceptible. Thus, that which distinguishes especially the new method of interpolation, it is: 1 ° *the use of factors of which each is reduced, excepting the sign, to* ± 1*, the sign being chosen as one wishes to say it; 2 ° the use of the differences of the form* $\Delta^{m+1}k$ *in order to determine the number m of the unknowns which must be admitted in the calculation.* We remark besides that by following the new method, one will never have to fear to obtain for the unknowns some infinite values, as that would be able to happen, if, in reducing the diverse unknowns to ± 1, one determined the signs otherwise than it has been said.

It is true that by following the method of least squares, one would be able to use, for the formation of the final equations, the direct process, as Laplace has done in the first supplement to the *Calcul des Probabilités*. But then likewise, in order to render the method applicable to the numerical determination of the coefficients which render the development of a function in convergent series, and of the number m of terms which must be conserved in this development, it would be necessary to lend to the new method of interpolation the rule which makes the principal merit of it, that which rests on the consideration of the diverse values of $\Delta^{m+1}k$.

I will say more. Will it suffice to reconcile thus, as much as possible, the method of least squares with the method of interpolation, in order to assure, in all points and in all cases, the superiority of the first? Not at all, and some very simple reflections will put the reader in range to form himself an opinion in this regard.

[108] First, after the indicated modification, the method of least squares will be far from being superior to the new method, under the relation of the brevity of the calculations. On the contrary, the new method will conserve over the other an incontestable advantage, since it will reduce the diverse factors introduced in the final equations to unity.

Will the method of least squares be, under the relation of precision, always superior to the other? But, in the special case where the number n of equations is equal to the number m of unknowns, all the methods furnish the same results, and then the better is evidently that which requires less calculation.

If now the number n of equations become notably superior to the number m of unknowns which must remain in the calculation, there will happen two things the one: either the given values of the function of which there is concern to obtain the development into series will be blemished by grave errors, and then no method will be able to guarantee the precision of the values found for the unknowns; or the given values of the function will be very nearly exact, and, in this case, especially if the number n of the unknowns becomes considerable, the two methods will furnish generally some results little different. There is more: being given the values of the unknown, such as the new method of interpolation furnishes them, it will suffice generally, in order to obtain those that the method of least squares furnished, to add to the first some very small corrections, and that, for this motive, it will be easy to calculate. Mr. Bienaymé says that this process tends to nothing less than to *double the so painful work of elimination*. But, in the Memoir lithographed in 1835, in order to render manifest the advantages of the new method, I have made to the theory of the dispersion of light an application that the Journal of Mr. Liouville has not reproduced, and I have also obtained a development of which the diverse values were precisely those of the developed function. Will one say that then the method of correction recalled above double the work and increases tedious calculation? Far from it, it proves, without calculation, that the method of least squares, rendered applicable by aid of a loan made to the new method, would have led the calculator to the same result, but more laboriously, and requiring more work from him.

It is true that the calculations of Laplace assigns to the method of least squares an important property, that to furnish, as Mr. Bienaymé remarks it, the most probable results. But this property subsists, as I will explain in another article,[1] only under certain conditions; [109] and then even as these conditions are fulfilled, it is able to be made that, in order to obtain the most probable results, the shortest way is to join to the new method, the method of correction of which I have spoken.

"**M. Augustin Cauchy** presented further to the Academy:

1° A *Mémoire sur les variations des constantes arbitraires que comprennent les intégrals des équations différentielles considérées dans l'article précedent, et sur les avantages qu'offre l'emploi des clefs algébriques pour déterminer complétement ces variations, lorsque la fonction dont les équations différentielles referment les dérivées se réduit à une fonction des deux sommes*

$$x^2+y^2+z^2+\ldots, \quad u^2+v^2+w^2+\ldots$$

2° A *Mémoire sur le calcul des probabilités.*

The results obtained in these two Memoirs will be developed in a later session."

[1] *Translator's note:* Mémoire sur les coefficients limitateurs ou restricteurs.