SUR LE SYSTÈME DES VALEURS QU'IL FAUT ATTRIBUER À DEUX ÉLÉMENS DÉTERMINÉS PAR UN GRAND NOMBRE D'OBSERVATIONS POUR QUE LA PLUS GRANDE DE TOUTES LES ERREURS, ABSTRACTION FAITE DU SIGNE, DEVIENNE UN MINIMUM.*†

Augustin Cauchy

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Oeuvres de Cauchy, Ser. II, 2, pp. 312–322

We suppose that one has already an approximate value of the two elements that one considers. We designate by the variable x the correction which must affect the first element, and by the variable y the correction that it is necessary to bring to the second. Among the diverse hypotheses that one will be able to make on the values of x and of y, one alone will satisfy to the first of the given observations; and, for each other hypothesis, the error of this observation will be designated by a function of x and of y, in which, seeing the smallness supposed of the corrections to make, one will be able to neglect the powers of the variables superior to the first. In general, whatever be the number of given observations, their respective errors will be able to be represented by as many polynomials of the first degree in x and y. Let

$$\begin{aligned} &a_1+b_1x+c_1y=e_1,\\ &a_2+b_2x+c_2y=e_2,\\ &\dots\dots\dots\dots\dots\\ &a_n+b_nx+c_ny=e_n \end{aligned}$$

these same polynomials (n being the number of given observations). The question will be reduced to determine for x and y a system of values such that the greatest of the

*"Onthe system of values that it is necessary to attribute to two elements determined by a great number of observations in order that the greatest of all the errors, setting aside the sign, becomes a minimum."

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polynomials that one considers, setting aside the sign, becomes a minimum; and if one makes

$$-a_1 - b_1 x - c_1 y = e_{n+1},$$
$$-a_2 - b_2 x - c_2 y = e_{n+2},$$
$$\dots\dots\dots\dots\dots\dots$$
$$-a_n - b_n x - c_n y = e_{2n}$$

it is evident that it will suffice to seek the system of the values of x and y, for which the one of the polynomials $e_1, e_2, \dots, e_n, e_{n+1}, e_{n+2}, \dots, e_{2n}$, which will have the greatest positive value, will become a minimum.

The method[1] that we have proposed for the solution of the analogous problem relative to any number of elements, is reduced in the present case to that which follows.

1° One will begin by supposing in all the elements at the same time one of the variables null, for example, $y = 0$, and one will determine the other variable x in a manner that the greatest of the polynomials, which will have a positive value, is a minimum. Let α be the value of x thus determined. For the system of values

$$x = \alpha, \quad y = 0$$

two polynomials e_p, e_q will become superior to all the others, and thereafter the system of which there is question will satisfy the equation

$$e_p = e_q$$

it is besides easy to prove that, in the two polynomials e_p, e_q, the coefficients of x will be necessarily of contrary signs.

2° One will examine if, in order to make diminish the common value of the two polynomials e_p, e_q, it is necessary to make increase or diminish y.

3° We suppose that in order to make diminish the common value of the two polynomials e_p, e_q one is obliged to make y increase, one will seek among all the polynomials remaining a third polynomial such, that by equating this last polynomial to the first two, one obtains for y the smallest positive value possible. Let e_r be the third polynomial of which there is question. The double equation

$$e_p = e_q = e_r$$

will determine for x and y a new system of values that I will represent by

$$x = \alpha_1, \quad y = \beta_1;$$

and this system will be able to be the one which must resolve the proposed question.

It will resolve it effectively, if for some values of y superior to β_1 the polynomial e_r equal to the one of the polynomials e_p, e_q where the coefficient of x has a contrary

[1] This method was the object of the Memoir which I have presented to the Institute 28 February 1814, and for which Messers Laplace and Poisson have been named commissioners. It is likewise at the demand of these two commissioners that the present extract has been written.

sign, becomes superior to the value common to the three polynomials e_p, e_q, e_r corresponding to the system

$$x = \alpha_1, \quad y = \beta_1.$$

In the contrary case, let e_q be the one of the two polynomials e_p, e_q where the coefficient of x is the sign opposed to the coefficient of the same variable in e_r: one will seek a new polynomial e_s such that the double equation

$$e_p = e_r = e_s$$

determines the smallest positive value possible of $y - \beta_1$. Then one will obtain a new system of values of x and of y, which I will designate by

$$x = \alpha_2, \quad y = \beta_2,$$

and which will be able to resolve the question proposed in many cases.

By continuing likewise, one will test successively many systems of values of x and of y. For each of these systems three polynomials at least will become at the same time positive, equals among them and superior to all the others. The number of the tests will never be able to surpass therefore the number of systems which enjoy this remarkable property. The question is now to determine the limit of this last number.

In order to arrive to it, it is necessary to observe that, if one gives to the two variables x and y some determined values, one will be able to form relatively to the system of these values, three different hypotheses. In fact it will be able to be made; 1° that for the system of which the question is one polynomial alone becomes superior to all the others; 2° that two polynomials e_p, e_q become equal between them and superior to all the others; 3° that at least three polynomials e_p, e_q, e_r are equal among them and superior to all the others. If the first hypothesis takes place, it will yet subsist, when one will make x and y vary separately between certain limits. If the second hypothesis takes place, it will yet subsist, when one will make x and y vary between certain limits, in a manner however that the equation $e_p = e_q$ is always satisfied. But if the third hypothesis takes place, it will subsist uniquely for the system of values of x and of y determined by the double equation

$$e_p = e_q = e_r.$$

According as one or the other of these three cases will take place, I will say that the given system is of the first, of the second or of the third order. This put, theorems 4, 5, 9 and 10 of the Memoir presented to the Institute, will suffice in order to determine the limit of the number of tests that one will be obliged to make, in the case where one considers only two elements. We are going to reduce these four theorems to that which they must be in particular case of which there is question.

THEOREM IV. — *If one passes successively in review all the possible systems of values of x and y, one will find that the systems of the first order have the systems of the second order for respective limits, and that these have themselves the systems of the third order for limits.*

Demonstration. — As for each system of values of x and of y it is necessary that at least one polynomial surpasses all the others, the diverse systems of values of x and of y will be found apportioned by groups, if I am able to express myself thus, among the diverse given polynomials. In some of these groups the values of the variables will remain always finite, in others they will be able to be extended to infinity. Moreover, as one will not be able to exit from a group without passing into another, one will encounter necessarily in this passage from the systems for which the two polynomials at the same time will become superior to all the others. Thus the systems of the second order will serve as respective limits to the different groups among which the systems of the first order will be found partitioned.

We consider now any system of the second order, for example, one of those for which the two polynomials e_p, e_q become at the same time equals among them and superior to all the others. If one made x and y vary at the same time, but in a manner to leave the equation $e_p = e_q$ always subsist, one will obtain, at least between certain limits, anew systems of the second order similar to the one that one considers, and for each of these systems the common value of the two polynomials e_p, e_q will be superior to that of all the other polynomials. But if one makes increase or decrease one of the variables, y for example, in a continued manner, a moment will arrive where the two polynomials e_p, e_q are found equal to a third. Thus the sequence of the systems of the third order which correspond to one same equation between two given polynomials, will have in general for limits two combinations of the third order, one of these limits being relative to the constant values of y, and the other to some decreasing values of the same variable. It nevertheless is able to happen that one of these two limits extends to infinity.

Remark. — It is easy to give to the preceding theorem a geometric interpretation. In fact, we imagine that the diverse polynomials

$$e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}, \ldots, e_{2n},$$

all of the first degree in x and y, represent the ordinates of as many planes different from one another, and that one has only regard to the portion of each in these planes which, for certain values of x and of y, become superior to all the others. The portions of the diverse planes which enjoy this property will form a convex polyhedron open in its superior part; and, if through any point of the plane of x, y one erects an ordinate, this ordinate will encounter a face, an edge, or a vertex of the polyhedron, according as the system of values of x and of y which determines the foot of the ordinate will be of the first, of the second or of the third order. This put, the preceding theorem is reduced to saying that the projections of the faces of the polyhedron have for limits the projections of the edges, and that those here have themselves the projections of the vertices for limits.

THEOREM V. — *If to the number of groups formed by the systems of the first order one adds the number of the systems of the third order, the sum will surpass by one unit the number of the sequences formed by the diverse systems of the second order.*

Demonstration. — It follows from the preceding theorem: 1° that the groups formed by the diverse systems of the first order have for limits the systems of the second order; 2° that the systems of the second order which serve as limits to a like group of systems of the first order, are partitioned into many sequences, of which each has itself for limits two systems of the third order, at least however, if one of these limits does not extend toward infinity. If therefore one increases by one unit the number of systems of the third order in order to take place some limits which diverge toward infinity, one will find placed in some circumstances completely similar to that which would take place if the systems of the first and of the second order would be able to be extended only to some finite values of x and y. Let now

M_1 be the number of the groups formed by the systems of the first order;

M_2 be the number of the sequences formed by the systems of the second order;

M_3 the number of the systems of the third order;

M_3+1 will be this last number increased by unity; and, in order to demonstrate the theorem enunciated above, it will suffice to see that one has

$$M_1+M_3=M_2+1. \tag{3}$$

One arrives easily there as it follows.

We have already remarked that to each system of the first order corresponded a polynomial superior to all the others; to each system of the second order, two polynomials superior to all the others; and to each system of the third order, three or a greater number of polynomials superior to all the others. This put, it will be easy to see that, if the systems of the first order which correspond to the polynomial e_p are not able to be extended to some infinite values of x and of y, the sequences of systems of the second order corresponding to this same polynomial will be in number equal to the one of the systems of the third order which serve as limit to them. For each sequence of systems of the second order will have necessarily for limits two systems of the third order, and reciprocally each of these last will serve as limits to two sequences of systems of the second. Let now e_q be a polynomial which, conjointly with the polynomial e_p correspond to a sequence of systems of the second order; and we suppose again that the systems of the first order which correspond to the polynomial e_q is not able to be extended to infinity, the system of the third order which will correspond at the same time to the two polynomials e_p, e_q will be in number two. Thence the number of sequences of systems of the second order, which will correspond to the polynomial e_q without corresponding to the polynomial e_p, will surpass by one unit the number of the systems of the third order, which will correspond to the first polynomial without corresponding to the second: whence it is easy to conclude that the number of the sequences of systems of the second order which will correspond to one of the polynomials e_p, e_q, will surpass by one unit the number of the systems of the third order corresponding to these same polynomials. In general we designate under the name of contiguous systems of the first order, those which have for common limit a like sequence of systems of the second order; and let e_p, e_q, e_r, e_s, e_t, ... be a sequence of polynomials corresponding to some systems of the first order, all contiguous to one another, and in which the values of the variables are not able to be extended to infinity. One will show, by some reasonings similar to the previous: 1° that the number of the sequences of systems of the second order corresponding to one of the three polynomials e_p, e_q, e_r, surpasses by two units the number of the systems of the third order which correspond to them; 2° that the number of the sequences of systems of the second order which correspond to one of the four polynomials e_p, e_q, e_r, e_s, surpass by three units the number of the systems of the third order corresponding to these same polynomials, etc. If therefore one designates:

— by N_1 the number of the polynomials $e_p, e_q, e_r, e_s, e_t, \ldots$;

— by N_2 the number of the sequences of systems of the second order which correspond to one of them;

— by N_3 the number of the systems of the third order which correspond to one of these same polynomials, one will have generally

$$N_2 = N_3 + N_1 - 1,$$

or

$$N_1 + N_3 = N_2 + 1. \tag{6}$$

Besides, if one supposes that the sequence $e_p, e_q, e_r, e_s, e_t, \ldots$ contains all the given polynomials, with the exception of a single one, and if one wishes to pass from the hypothesis where some systems of the first and of the second order are extended to infinity, to that in which all the systems would be able to be extended only to some finite values of x and of y, it will be necessary to make

$$N_1 = M_1 - 1,$$
$$N_2 = M_2,$$
$$N_3 = M_3 + 1.$$

This put, equation (6) will become

$$M_1 + M_3 = M_2 + 1.$$

QED

Remark. — The preceding theorem is able to be interpreted, in Geometry, in the following manner.

In a polyhedron open in its superior part, the sum made of the number of faces and of the number of vertices surpasses by one unit the number of edges.

In order to deduce this proposition from the theorem of Euler, it suffices to consider a closed polyhedron, and to imagine that in this polyhedron the diverse edges which unite in one same vertex taken in the superior part, depart the one from the other and diverge toward infinity.

THEOREM IX. — *Each system of the third order serves as limit at least to three sequences of systems of the second order.*

Demonstration. — In fact, each system of the third order corresponds at least to three polynomials $e_p, e_q, e_r, \ldots$ Moreover, among the sequences of systems of the second order which correspond to one of these polynomials, there are always necessarily two which have for common limit the system of the third order that one considers; and reciprocally the sequences of systems of the second order, which have this last for limit, correspond always to two of the polynomials of which there is concern. Hence the number of these sequences is always equal to the one of the polynomials $e_p, e_q, e_r, \ldots$it is thus at least equal to 3.

Geometric interpretation. — At least three edges of a polyhedron are reunited always at each of its vertices.

Corollary. — Let always M_3 be the number of the systems of the third order, and M_2 the number of the sequences formed by the systems of the second order. Since each system of the third order serves as limit at least to three sequences of systems of the second order, and since each sequence has for limits a single one or at most two systems of the third order, one will have necessarily

$$3M_3 < 2M_2.$$

This inequality, joined to equation (3) suffices, as one is going to see, in order to determine a limit of the number of tests which the proposed method requires.

THEOREM X. — *The number of tests that the proposed method requires never surpasses the double of the number of polynomials which can become superior to all the others.*

Demonstration. — In effect, the number of tests which the proposed method requires never surpasses the number of the systems of the third order designated above by M_2. Besides the number of the polynomials which are able to become superior to all the others, is equal to the number of the systems of the first order designated by M_1. It will suffice therefore to show that one has always

$$M_2 < 2M_1.$$

Now one has, by virtue of equation (3),

$$M_2 + 1 = M_3 + M_1, \tag{3}$$

and, by virtue of the theorem

$$M_3 + \frac{1}{2}M_3 < M_2.$$

By adding, member to member, this last inequality to equation (3), one will have

$$1 + \frac{1}{2}M_3 < M_1,$$

and hence

$$M_3 < 2(M_1 - 1) < 3M_1.$$

QED

Geometric interpretation. — In a polyhedron open at its superior part, the number of vertices is not able to surpass the double of the number of the faces.

Corollary. — As the number of polynomials which are able to become superior to all the others is at most equal to the number of the polynomials which one considers, that is to say, to the double of the number of observations, the number of tests that the proposed method requires is never able to surpass the quadruple of the number of observations. Thus the limit of the number of tests is simply proportional to the number of given observations.