SOLUTION

D'un Problème de Probabilité, relatif au Jeu de rencontre;

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1. *An urn contains m balls marked $a, b, c, d, \ldots,$ that one draws successively each, in order to replace them afterwards. What is the probability that, in two successive drawings, n balls will exit in the same order?*

We suppose that one makes the first drawing, and that in measure as the balls exit from the urn, one writes their names on one same line; we suppose also that the same thing has taken place for the second drawing, and that the second line is written below the first. One will obtain from the drawing, two sequences composed each of m letters, and which will be for example:

1st drawing...	$g,$	$a,$	$h,$	$l,$	$i,$	$c,$	$e,$	$d, \ldots$	(1)
2nd drawing...	$i,$	$l,$	$h,$	$g,$	$b,$	$d,$	$e,$	$f, \ldots$	(2)

I will call a *correspondence* the encounter, at the same rank, of two similar letters: thus, the letters h, e, form two correspondences.

The question reverts then to this one here:

What is the probability that in writing at random the sequences (1) and (2) , composed of the same letters, one will obtain n correspondences?

We write arbitrarily the first line; then, in order to form the second, we begin to make correspond n letters; we must next write the other $m - n$ letters, in a manner that they present no correspondence: I designate for an instant by X_{m-n} the number of solutions of which this question is susceptible.

We will have then, for n *designated* correspondences, X_{m-n} systems. And as the n letters, instead of being designated, are any, the number X_{m-n} must be multiplied by the number of combinations of m letters, taken n by n; a quantity that I will designate by $C_{m,n}$.

Thus, for any arrangement of the letters of the first line, there are $C_{m,n} \times X_{m-n}$ letters of the second, for which n letters correspond. Moreover, the letters of the first line can be disposed in as many ways as indicate the number of permutations of m letters, taken altogether, it follows that the number of chances favorable to the event demanded, is

$$P_m.C_{m,n}.X_{m-n} \tag{3}$$

The total number of chances is evidently $(P_m)^2$: therefore the probability sought has for expression

$$p = \frac{C_{m,n}.X_{m-n}}{P_m}; \tag{4}$$

or else, by putting for $C_{m,n}$ and P_m their known values,

$$p = \frac{X_{m-n}}{1.2.3\dots n.1.2.3\dots(m-n)} \tag{5}$$

2. We determine X_{m-n}.

By replacing $m-n$ by μ, the question can be posed in this manner:

The μ letters a, b, c, d, $\dots h$, i being written on one same line, to find in how many ways one can form a second line of these same letters, with the condition that none of them occupy the same rank in these two lines.

This quantity will be designated by X_μ.

We suppose this operation already effected for the $\mu-1$ letters a, b, c, d, $\dots h$; and we consider any one of these systems:

$$\left.\begin{array}{lllll} 1^{\text{st}}\text{ line}\dots & a, & b, & c, \dots h, \\ 2^{\text{nd}}\text{ line}\dots & g, & d, & a, \dots c. \end{array}\right\} \tag{6}$$

We bring the μ^{th} letter i to the end of each line; then, in the second, we change successively each of the $\mu-1$ letters which enter there, into i, and reciprocally.

It is clear that we will obtain in this way, $\mu-1$ systems, which will make part of the $X\mu$ systems demanded.

We consider next two lines of $\mu-1$ letters, among which there is 1 correspondence; for example:

$$\left.\begin{array}{lllll} 1^{\text{st}}\text{ line}\dots & a, & b, & c, & d, \dots h, \\ 2^{\text{nd}}\text{ line}\dots & a, & f, & d, & b, \dots c. \end{array}\right\} \tag{7}$$

We write the μ^{th} letter i at the end of each line; then, in the second, we change i into a; we will have next one of the arrangements sought. We can make the same thing for each of the $\mu-1$ letters a, b, $c, \dots h$; and as, for one letter which corresponds, there remain $\mu-2$, which one can invert in as many ways as $X_{\mu-2}$ indicates it, it follows that

$$X_\mu = (\mu - 1)(X_{\mu-1} + X_{\mu-2}) \tag{8}$$

It is evident that $X_1 = 0$, and $X_2 = 1$. One has next $X_3 = 2.1 = 2$, $X_4 = 3(2+1) = 9$, $X_5 = 4(9+2) = 44$, etc.

One sees therefore that the sequence of terms $X_1,\ X_2,\ X_3, \dots X_{\mu-2},\ X_{\mu-1},\ X_\mu, \dots$ forms a sequence in which *any term is equal to the sum of the two preceding, multiplied by the rank of the term which precedes the one that one seeks*.

3. One can transform formula (8) into another simpler:

First, for the symmetry of the calculation, we set $X_1 = 1$: this value satisfies the general law, because then $X_2 = 1(X_1 + X_0)$,

Next, by changing in the formula above, μ into $\mu - 2,\ \mu - 4,\ \dots$ and supposing μ *even*, we will have

$$\left.\begin{aligned} X_\mu &= (\mu - 1)(X_{\mu-1} + X_{\mu-2}), \\ X_{\mu-2} &= (\mu - 3)(X_{\mu-3} + X_{\mu-4}), \\ &\dots \\ X_4 &= 3(X_3 + X_2) \\ X_2 &= 1(X_1 + 1) \end{aligned}\right\} \tag{9}$$

The sum of all these equations is

$$X_\mu = (\mu - 1)X_{\mu-1} + (\mu - 2)X_{\mu-2} + \cdots + 3X_3 + 2X_2 + 1X_1 + 1. \tag{10}$$

Changing μ into $(\mu - 1)$, we will have, $(\mu - 1)$ being *odd*,

$$X_{\mu-1} = (\mu - 2)X_{\mu-1} + \cdots + 3X_3 + 2X_2 + 1X_1 + 1; \tag{11}$$

whence

$$X_\mu = \mu X_{\mu-1} + 1.$$

μ being *odd*, we will obtain likewise

$$X_\mu = \mu X_{\mu-1} - 1.$$

The general formula is therefore

$$X_\mu = \mu X_{\mu-1} \pm 1, \tag{12}$$

by taking the superior sign if μ is even.

One sees therefore, that, *in order to obtain any term, it suffices to multiply its rank by the preceding term, and to add or to subtract unity*.

The value of X_μ increases very rapidly with μ: one has $X_1 = 0$, $X_2 = 1$, $X_3 = 2$, $X_4 = 9$, $X_5 = 44$, $X_6 = 265$, $X_7 = 1854$, $X_8 = 14{,}833$, $X_9 = 133{,}496$, $X_{10} = 1{,}334{,}961$, $X_{11} = 14{,}684{,}570$, $X_{12} = 176{,}214{,}841$, $X_{13} = 2{,}290{,}792{,}932$, $X_{14} = 32{,}071{,}101{,}049$, $X_{15} = 481{,}066{,}515{,}734$, etc.

4. We determine the general term X_μ solely as function of μ.

By changing in equation (12), μ into $\mu - 1$, $\mu - 2$, $\ldots$ we will obtain the $\mu + 1$ equations,

$$\left.\begin{aligned} X_\mu &= \mu X_{\mu-1} \pm 1, \\ X_{\mu-1} &= (\mu-1)X_{\mu-2} \mp 1, \\ X_{\mu-2} &= (\mu-2)X_{\mu-3} \pm 1, \\ &\ldots \\ X_3 &= 3X_2 - 1, \\ X_2 &= 2X_1 + 1, \\ X_1 &= 1X_0 - 1, \\ X_0 &= 1. \end{aligned}\right\} \tag{13}$$

Multiplying then the 2nd equation by μ, the 3rd by $\mu(\mu-1)$, etc.; it comes, by adding the products:

$$\left.\begin{aligned} X_\mu = {} & \pm 1 \mp \mu \pm \mu(\mu-1) \mp \mu(\mu-1)(\mu-2) \pm \cdots \\ & - \mu(\mu-1)\cdots 3.2 + \mu(\mu-1)\cdots 3.2.1. \end{aligned}\right\} \tag{14}$$

Therefore X_μ *is equal to the difference between the number of permutations of* μ *letters taken in even number, and the one of the permutations of these same letters taken in odd number*.

5. The value of X_μ can be set under the form

$$X_\mu = 1.2.3\cdots(\mu-1)\mu\left[1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3} + \cdots \pm \frac{1}{1.2.3\ldots(\mu-1)\mu}\right] \tag{15}$$

The series between parenthesis has a remarkable analogy with the development of the base of the Napierian logarithms: one knows that this development has for value the limit of $\left(1 + \frac{1}{n}\right)^n$. Likewise, the series above has for value the sum of the $\mu + 1$ first terms of the development of $\left(1 - \frac{1}{n}\right)^n$, after one has made n infinity.

By neglecting the superior powers in the first, one has $\frac{1}{1+\frac{1}{n}} = 1 - \frac{1}{n}$: it follows that by designating as ordinarily by e the base of the Napierian logarithms, the series above is the development of $\frac{1}{e}$, limited to the first $\mu + 1$ first terms. It is that which becomes evident if one takes the relation

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \cdots \tag{16}$$

and if one puts $x = -1$.

It follows also that the *limiting* value of X_μ is

$$\frac{1.2.3.4\ldots(\mu-1)\mu}{e} \tag{17}$$

As the series (15) is very convergent, the value (17) is very near, as soon as μ passes a certain limit, which is not elevated. By making the calculation, one finds that, for $\mu > 13$,

$$\frac{1}{e} = 0.367\,879\,441\,19\ldots \tag{18}$$

Therefore also, for $\mu > 13$,

$$X_\mu = 0.367\,879\,441\,19 \times 1.2.3\ldots(\mu-1)\mu \tag{19}$$

Finally, if one sets for the product of the first μ natural numbers, its approximate value, one will have, very nearly,

$$X_\mu = \frac{\mu^\mu\sqrt{2\pi\mu}}{e^{\mu+1}}\left(1 + \frac{1}{12\mu} + \frac{1}{288\mu^2} + \cdots\right) \tag{20}$$

5. Returning to the problem which makes the object of our note, we will have, by replacing X_{m-n} by its value, in formula (5)

$$p = \frac{1}{1.2.3\ldots(n-1)n}\left[1 - \frac{1}{1} + \frac{1}{1.2} - \cdots \pm \frac{1}{1.2.3\ldots(m-n-1)(m-n)}\right], \tag{21}$$

for the exact expression of the probability. When $m - n$ passes 13, the very near value is

$$p = \frac{0.367\,879\,441\,19}{1.2.3\ldots(n-1)n}. \tag{22}$$

We take for example $m = 20$, $n = 5$, $m - n = 15$; formulas (21) and (22) give equally

$$p = 0.003\,065\,662.$$

6. We seek the probability that, in the two drawings, no letter will exit at the same rank. Setting $n = 0$ in formula (21), there comes for the demanded probability,

$$p' = 1 - \frac{1}{1} + \frac{1}{1.2} - \cdots \pm \frac{1}{1.2.3\ldots(m-1)m} \tag{23}$$

And if m is infinite,

$$p' = \frac{1}{e} \tag{24}$$

Such is the probability that, in two sequences of the same independent events the ones of the others, and in number infinite, no event will arrive in the same order.

If from unity we subtract p', we will obtain, for the probability *of at least* one correspondence in the two successive drawings,

$$p'' = \frac{e-1}{e}. \tag{25}$$

This probability is that of the *game of rencontre*, which consists in this here:

Two players have each a deck of cards, complete; each of them draw successively a card from his deck, until the same card exits at the same time, from the two sides. One of the players wagers that he will have *rencontre*; the other wagers the contrary. By supposing the number of cards infinite, it is clear that the probability of the first is p'', and that of the second, p'.

One has

$$p'' = 0.632\ldots \qquad p' = 0.368\ldots;$$

and as these values are quite near when m is greater than 13, it follows that one can regard them as exact, even for a deck of 32 cards.[1]

[1] The problem of which I occupy myself here, had been proposed to me, it is more than two years, at the École Polytechnique. It is only after having sent the solution to M. Liouville, that I

7. Problem (1) presents a rather remarkable circumstance: the value (22) contains in the denominator only the variable n. As for the numerator, one comes to see that as soon as $m-n$ passes 13, it remains, very nearly, constant. Therefore also, the probability demanded is, nearly rigorously, independent of the number of balls that the urn contains: it depends solely on the quantity of balls which must exit at the same ranks, in the two drawings.

8. In formula (22), we make n vary from 0 to m, and we add all the results: the sum is evidently unity, which is the symbol of certitude. Therefore

$$1=\sum_{0}^{m}\frac{1-\frac{1}{1}+\frac{1}{1.2}-\cdots\pm\frac{1}{1.2.3\ldots(m-n)}}{1.2.3\ldots n} \tag{26}$$

This equation can be put under the form

$$\begin{aligned}1=1&\left(1+\frac{1}{1}+\frac{1}{1.2}+\cdots+\frac{1}{1.2.3\ldots m}\right)-\frac{1}{1}\left(1+\frac{1}{1}+\frac{1}{1.2}+\cdots+\frac{1}{1.2\ldots(m-1)}\right)\\&+\frac{1}{1.2}\left(1+\frac{1}{1}+\frac{1}{1.2}+\cdots+\frac{1}{1.2\ldots(m-2)}\right)-\cdots\\&\mp\frac{1}{1.2.3.4\ldots(m-1)}\left(1+\frac{1}{1}\right)\pm\frac{1}{1.2.3\ldots(m-1)m}\end{aligned}$$

Multiplying the two members by $1.2.3\ldots(m-1)m$, it becomes

$$\begin{aligned}1.2.3\ldots(m-1)m=&[1+m+m(m-1)+\cdots+m(m-1)(m-2)\ldots3.2.1]\\&-\frac{m}{1}[1+(m-1)+(m-1)(m-2)+\cdots+(m-1)(m-2)\ldots3.2.1]\\&+\frac{m}{1}\frac{m-1}{2}[1+(m-2)+(m-2)(m-3)+\cdots+(m-2)(m-3)\ldots3.2.1]\\&\ldots\\&\mp\frac{m}{1}\frac{m-1}{2}\cdots\frac{2}{m-1}[1+1]\pm1.\end{aligned} \tag{27}$$

If one represents by S_n the sum of the numbers of permutations of n letters, taken 0 by 0, 1 by 1, 2 by 2,… n by n, this equation can be set under the form

have learned that Euler had occupied himself with the problem of rencontre, which is, as one sees it, a very particular case of mine.

One will find the solution of Euler in the *Mémoires de l'Académie de Berlin*, year 1751. One can consult also the *Calcul des Probabilités* of Laplace, p. 217, and Tome XII of the *Annales de Mathématiques*. I have had knowledge of all this only since a short time.

$$1.2.3\ldots(m-1)m = S_n - C_{m,1}.S_{m-1} + C_{m,2}.S_{m-2} - \cdots \mp C_{m,1}.S_1 \pm 1. \quad (28)$$

Equation (27) expresses a theorem on the numbers, equation (28) a theorem on the combinations.

9. I will suppose now that instead of extracting all the balls from the urn, one draws from it only a number t. Problem I is changed into this other, more general:

What is the probability that, in two consecutive drawings from an urn containing m balls marked a, b, c, $\ldots h$, i, of which there exit of them t at each drawing, n letters will exit in the same order?

By following the same march as previously, one sees that, after having made correspond n letters in a system of two lines, there remains to place in each of them, $t-n$ other letters, taken among the $m-n$ remaining; and that, with the condition that it presents no longer any correspondence. We suppose for an instant that this operation has been effected in all the possible ways, and we designate by $Y_{m-n,\,t-n}$ the number of systems thus obtained.

Actually, the n corresponding letters being able to be any, and being able to occupy t places, it follows that the number above must be multiplied by $C_{m,n}.P_{t,n}$. The chances favorable to the event demanded are therefore in number $C_{m,n}.P_{t,n}.Y_{m-n,\,t-n}$. The number of possible chances is $(P_{m,t})^2$. The probability sought has therefore for expression

$$p = \frac{C_{m,n}.P_{t,n}.Y_{m-n,\,t-n}}{(P_{m,t})^2}. \quad (29)$$

10. We determine $Y_{m-n,\,t-n}$.

By replacing $m-n$ by μ and $t-n$ by α, the question reverts to this here:

In how many ways can one form two lines composed of α letters, taken among μ given letters, with the condition that no letter occupies the same rank in the two lines? This number will be represented by $Y_{\mu,\,\alpha}$.

Let the μ letters be a, b, c, $d\ldots g$, h. We consider any of the systems of two lines formed only by $\alpha-1$ letters, a system which has no correspondence, and which will be, in order to fix the ideas:

$$\left.\begin{matrix} a, & f, & i, & b,\ldots e, \\ g, & i, & a, & h,\ldots d. \end{matrix}\right\} \quad (30)$$

At the end of each of these two lines, we bring any one of the $\mu-(\alpha-1)$ letters which do not enter there: for example, g for the first, and c for the second.

We will have then two lines of α letters; namely

$$\left.\begin{array}{cccccc} a, & f, & i, & b, \ldots e, & g, \\ g, & i, & a, & h, \ldots d, & c. \end{array}\right\} \tag{31}$$

It is clear that this system will be one of those demanded, except the case where the two letters introduced will be similar: we will return to this circumstance.

By not taking account of it, we see that, for a system of $(\alpha - 1)$ letters, we obtain from it $(\mu - \alpha + 1)^2$ of α letters. And as the number of the systems of $\alpha - 1$ letters is represented by $Y_{\mu,\alpha-1}$, the one of the systems of α letters will be by $(\mu - \alpha + 1)^2.Y_{\mu,\alpha-1}$, of which it is necessary actually to subtract the number of systems composed of lines terminated by the same letter.

Now, if we have placed one same letter a at the end of two lines of $\alpha - 1$ letters, it is because it does not enter it yet: these two lines can therefore be considered as composing one of the systems of $\alpha - 1$ letters, taken only among the $\mu - 1$ other letters b, c, $d, \ldots h$. Therefore, among the systems obtained a little while ago, there are $Y_{\mu-1,\,\alpha-1}$ terminated by a, a, as many by b, b, etc.; in all, $\mu.Y_{\mu-1,\,\alpha-1}$ systems to reject. We have therefore

$$Y_{\mu,\alpha} = (\mu - \alpha + 1)^2 Y_{\mu,\,\alpha-1} - \mu.Y_{\mu-1,\,\alpha-1} \tag{32}$$

11. Before going further, we remark that, through the symmetry of the calculations, one can suppose $Y_{\mu,0} = Y_{\mu-1,0} = 1$: because then, by making $\alpha = 1$ in the formula, there comes

$$Y_{\mu,1} = \mu^2 - \mu = \mu(\mu - 1).$$

It is evident indeed that, if each line contains only one letter, the number of systems is equal to the number of permutations of μ letters, taken 2 by 2.

Now, we change α into $\alpha - 1$, $\alpha - 2$, $\ldots 3$, 2, 1, we will obtain the α equations

$$\left.\begin{aligned} Y_{\mu,\alpha} &= (\mu - \alpha + 1)^2.Y_{\mu,\alpha-1} - \mu.Y_{\mu-1,\alpha-1}, \\ Y_{\mu,\alpha-1} &= (\mu - \alpha + 2)^2.Y_{\mu,\alpha-2} - \mu.Y_{\mu-1,\alpha-2}, \\ Y_{\mu,\alpha-2} &= (\mu - \alpha + 3)^2.Y_{\mu,\alpha-3} - \mu.Y_{\mu-1,\alpha-3}, \\ &\cdots \\ Y_{\mu,2} &= (\mu - 1)^2.Y_{\mu,\alpha} - \mu.Y_{\mu-1,1}, \\ Y_{\mu,1} &= \mu^2.1 - \mu.1, \end{aligned}\right\} \tag{33}$$

We multiply the second equation by $(\mu-\alpha+1)^2$, the third by $(\mu-\alpha+1)^2.(\mu-\alpha+2)^2$, etc., next we add. There comes

$$\left.\begin{aligned}&Y_{\mu,\alpha}=(\mu.\mu-1.\mu-2\ldots\mu-\alpha+1)^2-\mu.[Y_{\mu-1,\alpha-1}\\&+(\mu-\alpha+1)^2Y_{\mu-1,\alpha-2}+(\mu-\alpha+1)^2.(\mu-\alpha+2)^2Y_{\mu-1,\alpha-3}\\&+\cdots+(\mu-\alpha+1)^2.(\mu-\alpha+2)^2\cdots(\mu-1)^2]\end{aligned}\right\}\quad(34)$$

This equation in finite differences, is more complicated than equation (32); but it is going to lead us easily to the general expression of $Y_{\mu,\alpha}$.

For this, we set successively $\alpha=1,\ 2,\ 3,\ \ldots$ in this equation, and in that which one deduces from it by changing μ into $\mu-1$; we will obtain:

for $\alpha=1,\quad Y_{\mu,1}=\mu^2-\mu,\quad Y_{\mu,1}=\mu(\mu-1)$

and $Y_{\mu-1,1}=(\mu-1)[(\mu-1)-1]$,

$$\begin{aligned}\alpha=2,\quad Y_{\mu,2}&=\mu^2(\mu-1)^2-\mu[(\mu-1)^2-(\mu-1)+(\mu-1)^2]\\&=\mu^2(\mu-1)^2-\mu[2(\mu-1)^2-(\mu-1)]\end{aligned}$$

or $\quad Y_{\mu,2}=\mu(\mu-1)[\mu(\mu-1)-2(\mu-1)+1]$,

$$Y_{\mu\text{-}1,2}=(\mu-1)(\mu-2)[(\mu-1)(\mu-2)-2(\mu-2)+1],$$

$$\begin{aligned}\alpha=3,\quad Y_{\mu,3}&=\mu^2(\mu-1)^2(\mu-2)^2-\mu[(\mu-1)^2(\mu-2)^2\\&-2(\mu-1)(\mu-2)^2+(\mu-1)(\mu-2)+(\mu-1)^2(\mu-2)\\&-(\mu-1)(\mu-2)^2+(\mu-2)^2(\mu-1)^2]\\&=\mu^2(\mu-1)^2(\mu-2)^2-\mu[3(\mu-1)^2(\mu-2)^2\\&-3(\mu-1)(\mu-2)^2+(\mu-1)(\mu-2)\\Y_{\mu,3}&=\mu(\mu-1)(\mu-2)[\mu(\mu-1)(\mu-2)-3(\mu-1)(\mu-2)\\&\qquad+3(\mu-2)-1],\end{aligned}$$

$$\begin{aligned}\alpha=4,\quad Y_{\mu,4}&=\mu(\mu-1)(\mu-2)(\mu-3)[\mu(\mu-1)(\mu-2)(\mu-3)\\&-4(\mu-1)(\mu-2)(\mu-3)+6(\mu-2)(\mu-3)-4(\mu-3)+1]\end{aligned}$$

etc.

The law is actually evident, and we are right to set, *save verification*

$$\left.\begin{aligned}&Y_{\mu,\alpha}=\mu(\mu-1)\cdots(\mu-\alpha+1)[\mu(\mu-1)\cdots(\mu-\alpha+1)\\&-\frac{\alpha}{1}(\mu-1)(\mu-2)\cdots(\mu-\alpha+1)+\frac{\alpha}{1}\frac{\alpha-1}{2}(\mu-2)\cdots(\mu-\alpha+1)\\&\qquad-\cdots\pm\frac{\alpha}{1}(\mu-\alpha+1)\mp1].\end{aligned}\right\}\quad(35)$$

By employing the same relations as above, this formula can be set under the simplest form

$$\left.\begin{aligned} Y_{\mu,\alpha} = P_{\mu,\alpha}[P_{\mu,\alpha} - C_{\alpha,1}.P_{\mu-1,\alpha-1} + C_{\alpha,2}.P_{\mu-2,\alpha-2} \\ - \cdots \pm C_{\alpha,1}.P_{\mu-\alpha+1,1} \mp 1]. \end{aligned}\right\} \tag{36}$$

The integral of equation (32) have been obtained by way of induction, it is essential to verify it. For this, we change first α into $\alpha - 1$ in (36), then μ into $\mu - 1$ and α into $\alpha - 1$; we will have

$$\begin{aligned} Y_{\mu,\alpha-1} &= P_{\mu,\alpha-1}[P_{\mu,\alpha-1} - C_{\alpha-1,1}.P_{\mu-1,\alpha-2} + C_{\alpha-1,2}.P_{\mu-2,\alpha-3} \\ &\qquad - \cdots \mp C_{\alpha-1,1}.P_{\mu-\alpha+2,1} \pm 1]. \\ Y_{\mu-1,\alpha-1} &= P_{\mu-1,\alpha-1}[P_{\mu-1,\alpha-1} - C_{\alpha-1,1}.P_{\mu-2,\alpha-2} + C_{\alpha-1,2}.P_{\mu-3,\alpha-3} \\ &\qquad - \cdots \pm C_{\alpha-1,1}.P_{\mu-\alpha+1,1} \mp 1]. \end{aligned}$$

We multiply the first of these equations by $(\mu - \alpha + 1)^2$, next we subtract from it the second multiplied by μ. In noting that one has in general, $P_{m,n} = (m - n + 1).P_{m,n-1}$ and $P_{m,n} = m.P_{m-1,n-1}$, we will obtain first

$$\begin{aligned} (\mu - \alpha + 1)^2.Y_{\mu,\alpha-1} - \mu.Y_{\mu-1,\alpha-1} = P_{\mu,\alpha}[P_{\mu,\alpha} - (C_{\alpha-1,1} + 1).P_{\mu-1,\alpha-1} \\ + (C_{\alpha-1,2} + C_{\alpha-1,1}).P_{\mu-2,\alpha-2} - \cdots \pm (1 + C_{\alpha-1,1})P_{\mu-\alpha+1,1} \mp 1]. \end{aligned}$$

But one knows also that $C_{m,p} + C_{m,p-1} = C_{m+1,p}$; therefore the second member becomes

$$= P_{\mu,\alpha}[P_{\mu,\alpha} - C_{\alpha,1}.P_{\mu-1,\alpha-1} + C_{\alpha,2}.P_{\mu-2,\alpha-2} - \cdots \pm C_{\alpha,1}.P_{\mu-\alpha+1,1} \mp 1] :$$

an expression identical with that which we have found for $Y_{\mu,\alpha}$.

12. If in formula (35), we set $\mu - \alpha = \delta$, the development will become

$$\begin{aligned} Y_{\mu,\alpha} = [\mu(\mu-1)(\mu-2\ldots(\delta+1)]^2 \Bigg[1 - \frac{1}{1}\left(1 - \frac{\delta}{\mu}\right) + \frac{1}{1.2}\left(1 - \frac{\delta}{\mu}\right)\left(1 - \frac{\delta}{\mu-1}\right) \\ - \cdots \pm \frac{1}{1.2.3\ldots(\alpha-1)}\left(1 - \frac{\delta}{\mu}\right)\left(1 - \frac{\delta}{\mu-1}\right)\cdots\left(1 - \frac{\delta}{\mu+2}\right) \\ \mp \frac{1}{1.2.3\ldots(\alpha-1)\alpha}\left(1 - \frac{\delta}{\mu}\right)\left(1 - \frac{\delta}{\mu-1}\right)\cdots\left(1 - \frac{\delta}{\mu+1}\right)\Bigg] \end{aligned}$$

Comparing this expression with formula (15), one sees that if $\alpha = \mu$,

$$Y_{\mu,\mu} = 1.2.3\ldots(\mu-1).\mu X_\mu \tag{38}$$

this which is evident besides.

The series within parenthesis is very convergent: because its terms decrease more rapidly than those of the development of e^{-1}. If α, μ and δ are large numbers, one could replace this development by the one here:

$$1 - \frac{1}{1}\left(1 - \frac{\delta}{\mu}\right) + \frac{1}{1.2}\left(1 - \frac{\delta}{\mu}\right)^2 - \frac{1}{1.2.3}\left(1 - \frac{\delta}{\mu}\right)^3 + \cdots \tag{38}$$

which has for value

$$e^{-\left(1 - \frac{\delta}{\mu}\right)} = \frac{1}{e^{\frac{\delta}{\mu}}}.$$

It would be perhaps rather difficult to determine *a priori* the degree of approximation that one will be able to obtain, expecting that the more one advances in the series (37) and (38), the more the terms of the same order differ. In the cases where the series (38) will be able to be employed with advantage, one will have therefore for an approximate value of $Y_{\mu,\alpha}$,

$$Y'_{\mu,\alpha} = \frac{[\mu(\mu-1)(\mu-2)\cdots(\mu-\alpha+1)]^2}{e^{\frac{\alpha}{\mu}}} \tag{39}$$

13. By setting in formula (29), the values of the letters which enter, there comes

$$\left.\begin{aligned} p = \frac{t(t-1)(t-2)\cdots(t-n+1)}{1.2.3\ldots n \times m(m-1)\ldots(m-n+1)} &\left[1 - \frac{1}{1}\left(1 - \frac{\delta}{\mu}\right)\right. \\ &\left. + \frac{1}{1.2}\left(1 - \frac{\delta}{\mu}\right)\left(1 - \frac{\delta}{\mu - 1}\right) - \cdots\right] \end{aligned}\right\} \tag{40}$$

or else

$$\left.\begin{aligned} p = \frac{t(t-1)(t-2)\cdots(t-n+1)}{1.2.3\ldots n \times m(m-1)\ldots(m-n+1)} &\left[1 - \frac{1}{1}\frac{t-n}{m-n}\right. \\ &\left. + \frac{1}{1.2}\frac{t-n}{m-n}.\frac{t-n-1}{m-n-1} - \cdots\right] \end{aligned}\right\} \tag{41}$$

The series within parenthesis has $\alpha + 1 = t - n + 1$ terms: the last has for expression

$$\frac{1}{1.2.3\ldots(t-n)} \cdot \frac{t-n}{m-n} \cdot \frac{t-n-1}{m-n-1} \cdots \frac{1}{m-t+1} = \frac{1}{(m-n)(m-n-1)\cdots(m-t+1)}.$$

14. If one supposes $t = n$, the probability becomes

$$p' = \frac{1}{m(m-1)\cdots(m-n+1)}. \tag{42}$$

It is evident indeed, that if all the letters that one draws from the urn become exited in the same order in the two drawings, the probability has for expression $\frac{P_{m,n}}{(P_{m,n})^2}$.

Finally, if we suppose $n = 0$, the value of p becomes

$$\frac{1}{m(m-1)\ldots(t+1)}\Bigg[1 - \frac{1}{1}\frac{t}{m} + \frac{1}{1.2}\frac{t}{m}\cdot\frac{t-1}{m-1} - \cdots \pm \frac{1}{1.2.3\ldots t}\frac{t}{m}\frac{t-1}{m-1} \cdots \frac{2}{m-t+2}\cdot\frac{1}{m-t+1}\Bigg]$$

This is the probability of the *game of rencontre*, by supposing that one stops this game at the t^{th} trial.