NOTE
SUR UNE FORMULE DE COMBINAISONS

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Having remarked, in one of the last numbers of this Journal (p. 169), the enunciation of an interesting problem of probabilities, I sought to comprehend the calculations of the author, and to remake the calculations. But I myself perceive well that it is infiltrated with errors, which, in being accumulated, render false the principal formula. As it has not been possible by me to grasp perfectly the march followed by the author, I have sought to replace his formula by another.[1]

Let us recall first two formulas frequently employed in the theory of combinations.

1. In representing by $C_{m,p}$ the number of combinations of m letters, taken p by p, the formula of the binomial gives

$$(1+x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{1.2}x^2 + \cdots + C_{m,i}x^i + \cdots ,$$
$$(1+x)^{m'} = 1 + \frac{m'}{1}x + \frac{m'(m'-1)}{1.2}x^2 + \cdots + C_{m',i'}x^{i'} + \cdots ,$$

If one multiplies the two developments, and if one orders the result with respect to x, the term containing x^p, in $(1+x)^{m+m'}$, will be equal to the sum of the products two by two of the terms such that $C_{m,i}x^i$, $C_{m',i'}x^{i'}$, in which the sum $i+i'$ of the exponents is equal to p. Therefore

$$C_{m+m',p} = \sum C_{m,i} \times C_{m',i'},$$

or

$$C_{m+m',p} = \sum_0^p C_{m,i} \times C_{m',p-i}, \tag{1}$$

This formula supposes $p < m$, $p < m'$.

[1]My Note has been read to the Société Philomatique in the month of July last. Since, and unless it has been known from this lecture, Mr. the captain Coste has rectified the errors of which I speak above.

2. The formula of the binomial gives also

$$\frac{1}{(1-x)^m} = 1 + \frac{m}{1}x + \frac{m}{1}\frac{m+1}{2}x^2 + \cdots + C_{m+i-1,i}x^i + \cdots,$$
$$\frac{1}{(1-x)^{m'}} = 1 + \frac{m'}{1}x + \frac{m'}{1}\frac{m'+1}{2}x^2 + \cdots + C_{m'+i-1,i}x^i + \cdots$$

One will have therefore, by making the product, and by supposing $i + i' = p$,

$$C_{m+m'+p-1,p} = \sum_0^p C_{m+i-1,m-1} \times C_{m'+p-i-1,m'-1}. \tag{2}$$

3. By taking successively $m' = 1, 2, 3, \ldots$, one obtains, by aid of this last formula, these which follow:

$$\left\{\begin{array}{ll} C_{m+p,p} & = \sum_0^p C_{m+i-1,m-1}, \\ C_{m+p+1,p} & = \sum_0^p C_{m+i-1,m-1} \times C_{p-i+1,1}, \\ C_{m+p+2,p} & = \sum_0^p C_{m+i-1,m-1} \times C_{p-i+2,2}, \\ C_{m+p+3,p} & = \sum_0^p C_{m+i-1,m-1} \times C_{p-i+3,3}, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right. \tag{3}$$

4. We propose actually to transform the quantity

$$A = C_{m,m-p} \times C_{m'+1,p'} + C_{m-1,m-p} \times C_{m'+2,p'} + \cdots + C_{m-p,m-p} \times C_{m'+p+1,p'}$$

or

$$A = \sum_0^p C_{m-i,m-p} \times C_{m'+i+1,p'}. \tag{4}$$

In order to comprehend the end of the sought transformation, it is necessary to suppose that m, m' and p are great numbers, and that p' is a small number. It is quite evident that the numerical calculation of A will be impractical; now we wish to replace this quantity by another equivalent, but composed of $p' + 1$ terms only.

By aid of the formulas (3), the factor $C_{m+i+1,p'}$ is able to be transformed into a sum. We suppose successively $i = 0, 1, 2, \dots, p$; we will have

$$C_{m'+1,p'} = \sum_0^{p'} C_{m'-l,m'-p'},$$
$$C_{m'+2,p'} = \sum_0^{p'} C_{m'-l,m'-p'} \times C_{l+1,1},$$
$$C_{m'+3,p'} = \sum_0^{p'} C_{m'-l,m'-p'} \times C_{l+2,2},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$C_{m'+p+1,p'} = \sum_0^{p'} C_{m'-l,m'-p'} \times C_{l+p,p},$$

We substitute these values into formula (4); they will become

$$A = C_{m,m-p} \sum_0^{p'} C_{m'-l,m'-p'} + C_{m-1,m-p} \sum_0^{p'} C_{m'-l,m'-p'} \times C_{l+1,1} + \cdots$$
$$+ C_{m-p,m-p} \sum_0^{p'} C_{m'-l,m'-p'} \times C_{l+p,p}$$

With a little attention, one will see that this quantity is able to be put under the form

$$A = \sum_0^{p'} (C_{m,m-p} + C_{m-1,m-p} \times C_{l+1,1} + \cdots + C_{m-p,m-p} \times C_{l+p,p}) C_{m'-l,m'-p'}.$$

According to equation (2), the quantity within parenthesis is equivalent to $C_{m+l+1,p}$. Therefore

$$A = \sum_0^{p'} C_{m'-i,m'-p'} \times C_{m+i+1,p}. \tag{5}$$

Such is the formula to which we wished to arrive.

5. By bringing together the values (4) and (5), one has this very symmetric equation,

$$\sum_0^{p} C_{m-i,m-p} \times C_{m'+i+1,p'} = \sum_0^{p'} C_{m'-i,m'-p'} \times C_{m+i+1,p}. \tag{6}$$

6. As application and verification, we take $m=10$, $p=7$, $m'=10$, $p'=4$; we must have

$$\sum_0^7 C_{10-i,3}\times C_{11+i,4}=\sum_0^4 C_{10-i,6}\times C_{11+i,7}.$$

The first member, being developed, becomes

$$\begin{aligned}&\frac{10.9.8}{1.2.3}\cdot\frac{11.10.9.8}{1.2.3.4}+\frac{9.8.7}{1.2.3}\cdot\frac{12.11.10.9}{1.2.3.4}+\frac{8.7.6}{1.2.3}\cdot\frac{13.12.11.10}{1.2.3.4}\\ &+\frac{7.6.5}{1.2.3}\cdot\frac{14.13.12.11}{1.2.3.4}+\frac{6.5.4}{1.2.3}\cdot\frac{15.14.13.12}{1.2.3.4}+\frac{5.4.3}{1.2.3}\cdot\frac{16.15.14.13}{1.2.3.4}\\ &+\frac{4.3.2}{1.2.3}\cdot\frac{17.16.15.14}{1.2.3.4}+\frac{18.17.16.15}{1.2.3.4}=120.330+84.495\\ &+56.715+35.1001+20.1365+10.1820+4.2380+3060\\ &=39600+41580+40040+35035+27300+18200\\ &+9520+3060=214335\end{aligned}$$

The second member has for value[2]

$$\begin{aligned}&\frac{10.9.8.7}{1.2.3.4}\cdot\frac{11.10.9.8}{1.2.3.4}+\frac{9.8.7}{1.2.3}\cdot\frac{12.11.10.9.8.}{1.2.3.4.5}+\frac{8.7}{1.2}\cdot\frac{12.11.10.9.8}{1.2.3.4.5.6}\\ &+7\cdot\frac{14.13.12.11.10.9.8}{1.2.3.4.5.6.7}+\frac{15.14.13.12.11.10.9.8}{1.2.3.4.5.6.7.8}=210.330\\ &+84.792+28.1716+7.3432+6435=69300+66528\\ &+48048+24024+6435=214335.\end{aligned}$$

7. Formula (6) demonstrates a remarkable property of the integer negative poseres of a binomial $(1-x)$.

We replace $m-p+1$ by q, and $m'-p'+1$ by q'. That formula will be able first to be put under the form

$$\sum_0^p C_{p+q-1+i,p-i}\times C_{p'+q'+i,q'+i}=\sum_0^{p'} C_{p'+q'-1-i,p'-i}\times C_{p+q+i,q+i}.$$

[2] The formula cited at the beginning of this Note would give, instead of the second member that we just calculated, the quantity

$$\begin{aligned}&\frac{11.10.9.8}{1.2.3.4}\left[\frac{8.9.10.11}{1.2.3.4}+\frac{12}{5}\cdot\frac{8.9.10}{1.2.3}+\frac{12.13}{5.6}\cdot\frac{8.9}{1.2}+\frac{12.13.14}{5.6.7}\cdot 8+\frac{12.13.14.15}{5.6.7.8}\right]\\ =&330\left[330+288+\frac{936}{5}+\frac{416}{5}+\frac{39}{2}\right]=330.618+66.1352+165.39\\ =&103940+89232+9435=199607.\end{aligned}$$

We put next

$$
\begin{aligned}
\frac{1}{(1-x)^q} &= 1 + a_1 x + a_2 x^2 + \cdots + a_p x^p + \cdots\ , \\
\frac{1}{(1-x)^{p'+1}} &= 1 + b_1 x + b_2 x^2 + \cdots + b_{q'} x^{q'} + \cdots\ , \\
\frac{1}{(1-x)^{q'}} &= 1 + c_1 x + c_2 x^2 + \cdots + c_{p'} x^{p'} + \cdots\ , \\
\frac{1}{(1-x)^{p'+1}} &= 1 + d_1 x + d_2 x^2 + \cdots + d_q x^q + \cdots\ ,
\end{aligned}
$$

We will have, by means of the equation above

$$
(7) \qquad \left\{ \begin{aligned} a_p b_{q'} + a_{p-1} b_{q'+1} + \cdots + a_1 b_{p+q'-1} + b_{p+q'} &= c_{p'} d_q + c_{p'-1} d_{q+1} \\ &+ c_1 d_{p'+q-1} + d_{p'+q}. \end{aligned} \right.
$$

This equation expresses the enunciated property.

8. In order to arrive to a symmetric result, we have transformed the factor $C_{m'+i+1,p'}$ from formula (4). Now, as this factor is able, in different ways, to be equal to a sum of products, formula (6) is able equally to be replaced by other formulas that we will not indicate here, because they are more complicated than this last.