Practica arithmetice et mensurandi singularis

Chapter LXI, *De Extraordinariis & Ludis*, §§13–17 (f. 143 r. - f. 144 r.)
Last Chapter *On the Error of Fra Luca*, §5 (f. 289 v. – f. 290 r.)

Hieronymus Cardanus (Girolamo Cardano) (1501–1576)*

Milan, 1539

Introduction

Here we present the relevant text of the *Pratica* of Cardano [1] associated with the division of stakes. The text was republished in 1663 in volume four of the collected works of Cardano [2].

It is the text from the collected works (Vol. 4, pp. 112–113) which has been reproduced below rather than the original. This has been done primarily because it is more readable. However, the text of the 1539 edition has been consulted and compared to that of 1663. The several differences from the original are noted and have been restored here.

It is perhaps interesting to compare the two versions. The earlier makes heavy use of contractions and abbreviations. Consider the first sentence of paragraph 13 from the first edition of Cardano.

> "Quãtum ad rõnem ludorũ sciẽdum ẽ q in ludis nõ habet cõliderari nisi terminus ad quẽ & hoc in pgressione diuidendo totũper easdẽ partes." (1539)

Now compare to the same sentence in the collected works.

> "Quantum ad rationem ludorum sciendum est quod in ludis non habet considerari nisi terminus ad quem & hoc in progressione dividendo totum per easdem partes." (1663)

Adjacent to the original we present a translation of it into English. There exists also a translation into German which appeared in the anthology of Ivo Schneider [3]. It has been consulted.

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Text and Translation

Chapter LXI

13. Quantum ad rationem ludorum sciendum est quod in ludis non habet considerari nisi terminus ad quem & hoc in progressione dividendo totum per easdem partes exemplum duo ludunt ad decem unus habet 7. alius 9. quaeritur in casu divisionis non finiendo ludum quantum quisque debet habere subtrahe 7. à 10. remanent 3. subtrahe 9. à 10. remanet 1. progressio 3. est 6. progressio 1. est 1. dabis igitur dividendo totum depositum in 7. partes 6. partes habenti 9. & 1. partem habenti 7. ponamus igitur quod posuissent aureos 7. singuli, tunc totum depositum esset 14. ex quibus 12. contingunt habenti 9. & 2 habenti 7. ludos, quare qui habet 7. perdit $\frac{5}{7}$ capitalis. Aliud exemplum ponamus quod ludus sit ad 10. & unus habeat 3. alius 6. subtrahe fiunt residua 7. & 4. progressio 7. est 28. progressio 4. est 10. igitur totius summae dabo habenti 6. ludos 28. partes, & habenti 3. dabo partes 10. & ita dividam totum depositum in 38. partes, & ille qui habet 3. perdit $\frac{9}{19}$ sui capitalis.

13. What should be known about the reckoning in games is that one takes into consideration with regard to games only the end to which and this in progression[1] dividing the whole by the same parts. For example two gamble to ten. One has 7, the other 9. One now asks how much each should have in the case of division if the game is not finished. Subtract 7 from 10; there remains 3. Subtract 9 from 10; there remains 1. The progression of 3 is 6. The progression of 1 is 1. Therefore by dividing the total into 7 parts you will give 6 parts to the one having 9, and one part to the one having 7. Let us assume that they have staked 7 gold pieces each, then the total stake would be 14, out of which 12 falls to the one having 9, and 2 to the one having 7 games. Hence who has 7, loses $\frac{5}{7}$ of the capital. Another example: let us assume that the game is to 10 and one has 3, the other 6. Subtract. The residuals 7 and 4 are made. The progression of 7 is 28. The progression of 4 is 10. Therefore of the entire sum, I shall give 28 parts to the one having 6 games, and to the one having 3, I shall give 10 parts; & therefore I divide the total stake into 38 parts, and whoever has 3, loses $\frac{9}{19}$ of his capital.

14. Ratio autem demonstrativa super hoc est quod si facta divisione iterum ludus esset inchoandus, partes haberent deponere idem quod receperunt stante conditione. & sit in exemplo primo quod quis dicat volo ludere, hac conditione ut tu non possis vincere nisi vincas 3. sine intermissione, & si ego vinco unum volo vincere. & deponat ille qui vult vincere 3. ludos aureos 2. quantum habet deponere alius dico quod deponet 12. ratio nam si ad unum ludum haberent ludere

14. But the demonstrating rule concerning this is: If a game must be started again after division happened, the parties would have to be put the same as what they have received under the existing condition. And it is in the first example, that to which one says, “I wish to play with the condition, that you cannot win except by winning 3 times without pause, and, if I win 1, I will win.” And he who wishes to win 3 games must wager 2 gold pieces. How much must the other stake be? I say

sufficeret ponere 2. & si duos, haberet ponere triplum, ratio quia vincendo simpliciter 2.[2] ludos vinceret 4. sed hic stat cum periculo perdendi secundum victo primo, igitur lucrari debet triplum, & si ad 3. sexcuplum, quia duplicatur difficultas, igitur haberet ponere 12. & iam accepit 12. & ille 2. igitur divisio fuit conveniunter facta: & hoc ubi separatio esset de voluntate partium, aliter si sit causa habentis plus dividitur per aequalia si causa habentis minus perdit[3] totum.

that he will stake 2. For the reasoning: If they must play to one game, it is sufficient to stake 2, and if two, he would have to stake the triple. The reason because by winning simply 1 game, he would win 4, but here he stands with danger of losing the second with the first win; therefore he must win the triple, and if to 3 six times, because the difficulty is duplicated; therefore he would have to stake 12. Now also he already has received 12 and that other 2; therefore the division has been accomplished with the agreement made: & this where the division must be made with the consent of the parties. Otherwise, if it is caused by whoever has the more, it is divided into equals; if it was caused by whoever has the fewer, he loses all.

15. Duo ludebant unus ponebat 4. contra 5. alius 13. contra 16. quaeritur quis meliore posuit conditione, hoc fit per regulam trium: ducendo 5. in 13. fit 65. divide per 4. exit $16\frac{1}{4}$ & contra $16\frac{1}{4}$ debuit ponere ille qui posuit 13. cum igitur posuerit contra 16. posuit deteriore conditione quam ille qui posuit 4. contra 5. si vis scire quantum pro 100. dic si 13 capitale producit $\frac{1}{4}$, quid producet 100. & producet $1\frac{12}{13}$, & tanto deteriore conditione posuit addit postmodum Frater Lucas quod hoc est veluti in transmutationibus & bene dixit.

15. Two play. One staked 4 against 5. The other 13 against 16. It is asked which one has staked under the better condition. This is done through the rule of three: Multiplying 5 into 13, it makes 65. Divide by 4. $16\frac{1}{4}$ exits. And against $16\frac{1}{4}$ this one must put that who has staked 13. When therefore he should have staked against 16, he has staked the poorer condition than that one who staked 4 against 5. If you wish to know how much for 100, say if 13 capital produces $\frac{1}{4}$, what will 100 produce. And it will produce $1\frac{12}{13}$ & by so much he stakes with more unfavorable condition. Later Brother Luca adds because this is just as in transmutation & he has said well.

16. Quidam vult ludere ad primum pro se, & vult ponere 12. contra 1. quaeritur ad quot debet ludere socius, quaeras progressionem de 12. pro summa per Rem nam capio rem & divido per aequalia fit $\frac{1}{2}$ co. adde ad eam $\frac{1}{2}$ per regulam fit $\frac{1}{2}$ co. p̃. $\frac{1}{2}$, duc in 1 co. fit $\frac{1}{2}$ ce. p̃. $\frac{1}{2}$ co. aequalia 12. igitur 1 ce. p̃. 1 co. aequalia 24. quare res valet ℞ $24\frac{1}{4}$ m̃. $\frac{1}{2}$, &

16. Someone wants to play first of all for himself and wants to stake 12 against 1. One asks, to how many must a partner play. You look for the progression of 12 for the sum through the unknown. For I take the unknown and divide it into equal parts, it makes $\frac{1}{2}n$. Add $\frac{1}{2}$ to it according to the rule, it makes $\frac{1}{2}n + \frac{1}{2}$; multiply by $1n$, it makes $\frac{1}{2}n^2 + \frac{1}{2}n = 12$; therefore

hic est maior terminus igitur cum ℞ $24\frac{1}{4}$ m̃. $\frac{1}{2}$ sit maior 4. & minor 5. dices quod ludendo ad 4. luderet meliore conditione quàm ille qui ludit ad 1. & ludendo ad 5. luderet deteriore conditione quàm socius.

there is $n^2 + n = 24$. Therefore the unknown is worth $\sqrt{24\frac{1}{4}} - \frac{1}{2}$, and this is the greater part. Therefore when $\sqrt{24\frac{1}{4}} - \frac{1}{2}$ is greater than 4 and smaller than 5, you will say that the one who plays to 4 plays under a more favorable condition than the one who plays to 1, and that the one who plays to 5 plays under a more unfavorable requirement than the partner.

17. Quidam pauper ibat ad domum divitis singulo die ut luderet aureum unum, hoc modo, quod cum pauper perdebat aureum cessabat à ludo, si vincebat continuabat ad singulos ludos, & ille semper deponebat quantum habebat pauper usque ad 4. ludos, deinde cessabant, & sit exemplum primo ludo dives deponebat aureum, si vincebat finiebatur ludus pro illa die, si perdebat pauper habebat 2. aureos, unde in secundo ludo deponebat dives aureos 2. si vincebat adhuc finitus erat ludus, si perdebat pauper habebat 4. aureos, unde dives deponebat aureos etiam ipse 4. & ita in quarto ludo deponebat 8 si igitur dives vincebat, pauper amittebat 7. iam lucratos, & unum de suis aureis si vicisset tunc auferebat 16. aureos, 15. videlicet superlucratos, quaeritur igitur continuando pluribus mensibus hoc modo, pari extente fortunâ & scientiâ ludi, quis ludit meliore conditione, & quantum pro 100. clara est responsio progressio de 4. est 10. igitur non deberet dives ponere nisi 10. aureos, & iam perdit 15. igitur peiore conditione ludit dives quam pauper, & quia 5. est medietas 10. igitur conditio est deterior 50. pro 100. continuando igitur pauper multum lucrabitur, ita quod in anno lucrabitur 182. aureos, quia dimidium depositi, quod si fortuna sit dispar etiam longe melius quia omnis proportio addita maiori, & minori aequaliter, auget magis supra maiorem quam supra

17. A certain poor man walked around every day to the house of a rich man in order to gamble for a gold coin in the manner that if the poor person lost the gold coin, he gave up the game; if he won, he continued to a single game. That one staked always so much as the poor person had up to 4 games, afterwards they stopped. And let be for example with the first game the rich man staked one gold coin, if he won the game he was finished for that day, if he lost the poor man had 2 gold coins, whence in the second game the rich man staked 2 gold coins. If he won at this point the game was ended; if he lost the poor man had 4 gold coins, whence the poor man staked in addition 4 gold coins. And thus in the fourth game he staked 8. If therefore the rich man won, the poor man lost the 7 now gained, & one of his gold coins. If he had won at that time he carried away 16 gold coins, 15 evidently gained besides. Therefore it is asked by continuing for several months in this manner, with equal luck and skill of the game, who wagers with the better condition, & how much percent? The answer is clear. The progression of 4 is 10. Therefore the rich man must stake only 10 gold coins, & now he loses 15. Therefore the rich man gambles with worse condition than the poor man, and, because 5 is the half of 10, therefore the condition is more than 50 percent worse. Therefore with continuation the poor per-

minorem, & ita remotis fraudibus, & scientia aequali existente, impossibile quasi esset pauperem non vincere, verum pauperes multum[4] aliquando impedit timor, aut laetitia, divides autem non cum tanto affectu ludunt, & ideo securius, &c.

son will win much, so that he will win in a year 182 gold coins, because it is the half of the stake, but if fortune is unequal yet by far better because every proportion increased to the greater, & to the lesser equally, increases to a greater extent for the greater than for the lesser, & thus with fraud removed, & equal skill existing, it is nearly impossible the poor man to not win, most certainly sometimes fear or joy hinder poor men but rich men do not play with so much passion, & therefore more untroubled, etc.

1 Cardano means by *progressio* the summing of the natural numbers up to a certain point. The *progressio* of n is thus $1 + 2 + \cdots + n = n(n + 1)/2$.

2 *Opera Omnia* has 1 rather than 2.

3 *Opera Omnia* has perditit rather than perdit.

4 *Opera Omnia* omits multum.

On the Error of Fra Luca

Et erravit ludorum determinatione errore, manifestissimo, & a puero etiam cognoscibili, dum alios arguit & suam laudat exquisitam opinionem; unde ludentibus ad 6 & habenti 5, alteri 2, dat post multas superfluas supputationes partes 5, & 2, ita quod totam summam dividit in 7.

And he has erred in the determination of the games most evidently, & even recognizable by a boy, while he accuses others & praises his opinion as exquisite; whence with playing to 6 & to the one who has 5 to the other 2, after many superfluous computations, he gives 5 & 2 parts, thus with respect to which he divides the total sum into 7 [parts].

Ponamus igitur quod duo ludant ad 19 & unus habeat 18, alius tantum 9, dabit[1] igitur primo $\frac{2}{3}$ totius summe, & secundo $\frac{1}{3}$, sit igitur depositum aurei 12 summa[2] amborum erit 24 e[3] quibus 16 primo, & 8 secundo, contingent: non igitur ille qui habet 18 ludos lucratus est nisi aureos 4, & ex adversario, qui sunt tertia pars depositi, & tam ad complendum non deest nisi unus ludus, secundo autem desunt 10, hoc autem est absurdissimum

Let us assume therefore that two gambled to 19 and one would have 18, the other only 9. Then he gives to the first $\frac{2}{3}$ of the whole sum and to the second $\frac{1}{3}$. Therefore let the stake be 12 gold coins; the sum of both will be 24, from which 16 befalls to the first and 8 to the second: therefore that one who has won 18 games has only 4 gold coins from his adversary, which is to a third part of the stake, & nonetheless he lacks only one game to completion, while the second lacks 10. But this is most absurd.

praeterea illam partem quisque debet assumere, quam aequa ratione deponere posset ea conditione, sed habens 18 cum

Furthermore each ought to take that part, that he is able to stake with equal reckoning under that condition; but the one habente 9 potest eundo ad 19 deponere 10, contra 1 imo 20 contra unum: igitur in divisione debet habere partes 20, & ille tantum unam,

having 18 with the one having 9 is able by advancing to 19 to wager 10 against 1, yes even 20 to 1: therefore he must have with the division 20 parts & the other only one.

Tertio si ludimus ad 19, & unus habeat 2, alter nullum, per suam rationem qui habet 2 debet acquirere totum depositum, patet ex suo computo, hoc autem quale sit inconveniens non est dubitandum, cum ex tam modica superatione, cum tanta remotione a fine debeat acquirere tantum, quantum si lucratus fuisset 19 ludos: secundo quia ad deterius ille non potest venire qui perdit depositum, sed dato quod haberet 18 ludos primus, & secundus nullos, adhuc non debentur omnes, quia ultimus esset superfluus, quanto igitur minus debet habere totum per duos tantum acquisitos.

Thirdly, if we wager to 19, and one has 2, the other none, by his reasoning whoever has 2 must acquire the total stake, it is clear from his computation, but this kind of condition is unsuitable is not to be disputed, since with such a modest lead and at such distance from the goal he may acquire as much, as if he would have won 19 games: because to the second who loses the stake is not able to come to worse; but I give who had the first 18 games, & the second none, yet all is not owed, because the last would be superfluous, therefore by how much less must hold the total obtained through two only,

Quarto ad principale si unus habeat 3 alius 1 eundo ad 13, primo contingent partes 3[4] secundo 1, & si primus haberet 12 secundus 9 darentur primo $\frac{4}{7}$, & secundo $\frac{3}{7}$, & ita multo deterior esset conditio primi in secundo casu, quam in primo, quod est absurdissimum, cum in secundo casu non continget primo perdere in sex vicibus semel, & in primo non sit magna disparitas & hoc iam declaravimus in capitulo sexagesimo primo.

Fourthly to the main point: if one shows 3, the other 1 in going to 13, 3 parts would be due to the first, 1 to the second. & if the first had 12, the second 9, to the first $\frac{4}{7}$ and to the second $\frac{3}{7}$ would be given, & thus with much worse condition would be of the first in the second case than in the first, which is most absurd, since in the second case the first will not happen to lose once in six turns & in the first there is no great disparity. & this I have already explained in Chapter Sixty-one.

1 *Opera Omnia* has habet rather than dabit.

2 *Opera Omnia* has summae rather than summa.

3 *Opera Omnia* omits e.

4 *Opera Omnia* omits 3

Cardano's Solution and Commentary

§§13–14 present Cardano's method for the division of stakes. He states that the number of games yet required to achieve a win is what determines the division. Suppose then Players A and B play to n points. Let Player A have a points and Player B have b. They lack $n - a$ and $n - b$ respectively of the goal.

Cardano next computes the progression of each difference. That is, he computes in the first case $p_n(a) = 1 + 2 + \cdots + (n-a) = (n-a)(n-a+1)/2$. Define $p_n(b)$ analogously. The division is completed by awarding to Player A the fraction $P(A) = \frac{p_n(b)}{p_n(a)+p_n(b))}$ of the total stake and similarly to Player B the fraction $P(B) = \frac{p_n(a)}{p_n(a)+p_n(b))}$.

If each player stakes the same amount, then the portion gained or lost of his stake by Player A is $2 \times P(A) - 1$ and by Player B is $2 \times P(B) - 1$.

He next justifies the use of the progression with Cardano reasoning that winning each successive game increases in difficulty. That is, the second win is twice as difficult as the first; the third is three times as difficult at the first. Therefore, in a fair game the one who needs to achieve 2 wins, if successful, should receive three times the one who would need to achieve but 1. Similarly, the one who needs to achieve 3 wins, if successful, should receive six times the one who needs to achieve but 1.

§15 illustrates how to compare wagers so as to determine which is more favorable to a gambler in a given game.

In §16 he seeks a solution to the problem $1 + 2 + \cdots + n = 12$ which, of course, leads to the quadratic equation given. Cardano solves by completing the square. For we note that $n^2 + n = 24$ gives $n^2 + n + \frac{1}{4} = (n + \frac{1}{2}) = 24\frac{1}{4}$. To Cardano then, a fair game would require a stake of approximately 4.42.

In §17 Cardano analyzes a game in which the poor man wins if he can achieve 4 successive wins and the rich man wins if he achieves 1. Following his procedure, the progression of 4 is 10 and the progression of 1 is 1. Therefore, the rich man should stake 10 and the poor man 1. However, the rich man actually pays 15. Rather than being a fair game, it favors the poor man so that he stands to gain, on average, $1/2$ gold coin each day or, 182 per year.

In the last section, Cardano criticizes Pacioli for the absurdity of his method. First of all, he notes that if one player has gained one point, but the other none, the entire stake would go to the first player under the method of Pacioli, even though the player had not won the game. Secondly, since Pacioli has no regard for the nearness of a player to a goal, his apportioning of the stake can lead to a situation in which a player more distant from a goal receives a greater share of the stake than one nearer.

Let's consider the examples presented by Cardano.

Example 1. Suppose the goal is 19 points. Player A has 18 and Player B has 9. Pacioli would award $2/3$ to A and $1/3$ to B. Since $19 - 18 = 1$ and $19 - 9 = 10$, we compute the progressions of 1 and 10 which are 1 and 55 respectively. Therefore, according to Cardano, A should receive $55/56$ and B $1/56$.

Example 2. Suppose the goal is 13 points. First, let A have 3 points and B have 1. Pacioli would award $3/4$ to the first and $1/4$ to the second. If A had 12 and B had 9, then A would receive $4/7$ and B $3/7$. Here then A is closer to the goal and yet receiving less.

Consider now the method of Cardano. In the first case, A would receive $78/133 \approx 0.59$ and B would receive the $55/133 \approx 0.41$. In the second case, A would receive $10/11 \approx 0.91$ and B would receive $1/11 \approx 0.09$.

References

[1] Gerolamo Cardano. *Practica Arithmetice et Mensurandi singularis*. Bernardini Calusci, Milan, 1539. Reprinted in *Opera Omnia*, Vol. 4, 1663.

[2] Gerolamo Cardano. *Opera Omnia*. cura Caroli Sponii, Lyon, 1663. Reprinted by Johnson Reprint Corporation, New York. 1967.

[3] Ivo Schneider. *Die Entwicklung der Wahrscheinlichkeitstheorie von den Anfängen bis 1933*. Wissenschaftliche Buchgesellschaft, Darmstadt, 1988.