On the Application of the Analysis of Probabilities to Determining the Approximate Value of Transcendental Numbers*

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In the first discourse under this title it was shown how the determination of the probability of the encounter of a thin cylinder with the sides of equilateral triangles tiling the plane leads to the approximate value of the ratio of circle to the diameter. We offer now resolution of three new problems related to the same method. The first task will lead us in the same manner to the determination by an arcsine; the second to elliptical functions of the 1st and 2nd kind; finally, a third previously named function, to introduce the logarithmic function into an expression of the probability.

Let the circle be $AQBR$ (diag. 1), and let us assume that at random the thin cylinder is thrown so that its center does not exit the area of circle. Obviously, the length of cylinder is assumed less than the diameter of the circle. The question is, how great is the probability that the cylinder, falling as said, will encounter the circle $AQBR$?

For the resolution of this problem it is necessary at each internal point of the circle $AQBR$, taken as the center of the cylinder, to describe an arc with a radius equal to the half-length of cylinder; this arc in general will cross the circle $AQBR$ at two points; connecting each of them with the center of cylinder, we will obtain the known angle which we indicate by 2ϕ. The ratio of the angle 2ϕ to the entire circle 2π will represent the ratio of the number of cases of encounter to the number of all possible cases, and it will consequently be the measure of the probability of encounter by the cylinder with the circle $AQBR$ under the assumption that the center of the cylinder falls precisely at that point that we are examining. Then, by the rules of integral calculus, from the probability which relates to one point or to the element of area of the circle, we pass to the probability which corresponds to the total area, as that was explained in the first paper.

Let us now turn to the details of the problem solved by us: let us first decompose the circle $AQBR$ into three pieces, signify the two rings: the first by letter ω, and the second by letter $\tilde{\omega}$, and the inner circle by Ω, two concentric circles described with the center C of this circle; the radii CE and CD of these concentric circles we define as follows: in circle $AQBR$, perpendicular to the diameter AB, we contain IK the length

*Paraphrase of the Russian by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 20, 2011

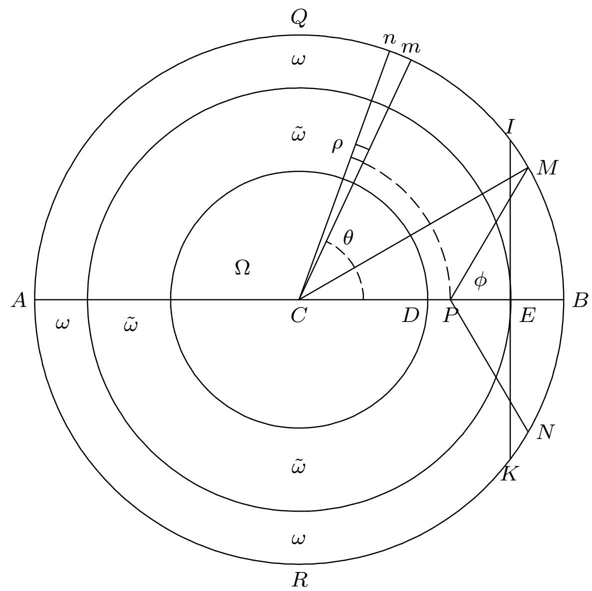

Figure 1:

of this cylinder; the intersection of the lines IK and AB determine the point of E, and consequently the length of a radius of CE. For determining the radius of CD we lay off on BA, from the point B, the half-length of the cylinder, i.e., the line of EI; the extremity of this line determines the point D. With this decomposition of the original circle, it is easy to see 1°, that while the center of the cylinder falls inside circle Ω, then the encounter of the cylinder itself with the circle $AQBR$ is impossible; 2° when the center of the cylinder is located inside the ring $\tilde{\omega}$, then cylinder can meet this circle; 3° when the center of cylinder will fall inside the ring ω, then without fail any position of cylinder would encounter it.

On this basis, let us proceed to determinate the probability of an encounter of the cylinder with the circumference. Let $2l = IK$ be the length of cylinder, $r = CB$ the radius of the circle $AQBR$; obviously there will be: $CD = R - l$, $CE = \sqrt{r^2 - l^2}$. Let us represent by z the desired probability, and by n the number of cases of the encounter of cylinder with the circle when its center will fall inside the middle ring $\tilde{\omega}$; obviously we will obtain

$$z = \frac{n + \pi l^2 .2\pi}{\pi r^2 .2\pi} = \frac{n + 2\pi^2 l^2}{2\pi^2 r^2}, \tag{1}$$

since for the circle Ω there will be no cases of encounter; and as far as the ring ω is concerned, the number of encounters will be instead the number of encounters, consequently this latter is equal to area πl^2 of the ring, multiplied by 2π. The denominator $2\pi^2 r^2$ represented by the area of this circle, multiplied by the entire circumference 2π, i.e., the number of all possible cases in the casting of the cylinder at random, assuming yet that its center is not to leave the area of this circle.

And so now the problem is to the determination of the value n. For this, let us take, where it is inside the ring $\tilde{\omega}$, the area μ, formed by two adjacent radii Cm, Cn and by two infinitely close concentric arcs; μ will represent the element of area of this ring. Let ρ be the distance of the element μ from the center C, and θ the angle of BCm; we will obtain $\mu = \rho d\rho d\theta$. If, for convenience, we transfer the area μ onto the line CB at P, so that $CP = \rho$, and from the point P with the radius l, equal of the half-length of the cylinder, let us describe the arc, that intersects this circle at the points M and N; the angle MPN will be the one that is in the region that there is an encounter of the cylinder with the circumference according to the assumption that the center of cylinder is located at P; let us describe this angle by 2ϕ. If the element $\rho d\rho d\theta$ is multiplied by 2ϕ and then we take integral of the product from $\theta = 0$ to $\theta = 2\pi$, and $\rho = CD = r - l$ to $\rho = CD = \sqrt{r^2 - l^2}$, then we will obtain the value of n; and thus

$$n = 2\int_0^{2\pi}\int_{r-l}^{\sqrt{r^2-l^2}} \phi\rho d\phi d\theta.$$

Let us note that θ does not depend on ϕ, nor on ρ; consequently

$$n = 4\pi\int_{r-l}^{\sqrt{r^2-l^2}} \phi\rho d\rho.$$

Integrating by parts, we obtain

$$\int \phi\rho d\rho = \frac{\rho^2\phi}{2} - \frac{1}{2}\int \rho^2 d\phi,$$

and moreover, as with $\rho = r - l$ there will be $\phi = 0$, and with $\rho = \sqrt{r^2 - l^2}$, $\phi = \frac{\pi}{2}$, then we will obtain

$$\int_{r-l}^{\sqrt{r^2-l^2}} \phi\rho d\rho = \frac{r^2 - l^2}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \rho^2 d\phi,$$

Whence

$$n = (r^2 - l^2)\pi^2 - 2\pi\int_0^{\frac{\pi}{2}} \rho^2 d\phi. \tag{2}$$

But based on the triangle CMP we obtain the equation

$$r^2 = \rho^2 + l^2 + 2l\rho\cos\phi,$$

from which we deduce

$$\begin{aligned} \rho^2 &= r^2 - l^2 - 2l\rho\cos\phi \\ \rho &= -l\cos\phi + \sqrt{r^2 - l^2\sin^2\phi}; \end{aligned}$$

Consequently

$$\int_0^{\frac{\pi}{2}} \rho^2 d\phi = \int_0^{\frac{\pi}{2}} (r^2 - l^2) d\phi + 2l^2 \int_0^{\frac{\pi}{2}} \cos^2\phi . d\phi - 2l \int_0^{\frac{\pi}{2}} \sqrt{r^2 - l^2 \sin^2\phi} . \cos\phi d\phi.$$

However, since

$$\begin{aligned} \int_0^{\frac{\pi}{2}} (r^2 - l^2) d\phi &= (r^2 - l^2)\frac{\pi}{2}, \\ \int_0^{\frac{\pi}{2}} \cos^2\phi . d\phi &= \frac{1}{4}\pi, \\ \int_0^{\frac{\pi}{2}} \sqrt{r^2 - l^2 \sin^2\phi} . \cos\phi d\phi &= \frac{\sqrt{r^2 - l^2}}{2} + \frac{r^2}{2l} \arcsin\left(\frac{l}{r}\right), \end{aligned}$$

then we find

$$\int_0^{\frac{\pi}{2}} \rho^2 d\phi = (r^2 - l^2)\frac{\pi}{2} + l^2 \cdot \frac{\pi}{2} - l\sqrt{r^2 - l^2} - r^2 . \arcsin\left(\frac{l}{r}\right).$$

Substituting this last value into equation (2), we obtain

$$n = \pi \left[2l\sqrt{r^2 - l^2} - l^2\pi + 2r^2 . \arcsin\left(\frac{l}{r}\right) \right],$$

and consequently, by equation (1),

$$z = \frac{\pi l^2 + 2l\sqrt{r^2 - l^2} + 2r^2 . \arcsin\left(\frac{l}{r}\right)}{2\pi r^2}. \tag{3}$$

That expression of the probability that the cylinder, cast at random inside the circle, will fall on the circle is the expression above with the number π, it itself contains an arcsine, equal to the ratio of the length of the cylinder to the diameter of circle, i.e., the arc of BI.

[522] The second problem, to which we offer here the solution, is as the following: An arc QBR of indeterminate measure (diag. 2), is described from the center C, and able to be contained in a plane, a cylinder is thrown onto it. Through the center B of this arc and the center C a straight line BA is prolonged. Point A can be located on one or on the other side of center C; let us assume that it lies on the left side, as it is represented in the drawing. Let us imagine that the center of cylinder runs freely along the line AB, and the cylinder itself, at the same time, turns freely about its center in the plane of arc QBR. The question is, how great is the probability that the two movements of the said cylinder, independently of one another, stopping, will intersect this arc QBR?

Let the radius be $CB = r$, half the length of the cylinder $PM = l$, $AC = a$. If as in the previous problem, we will contain the cylinder IK, perpendicular to CB, in arc QRB, then we determine point E; it is obvious that while the center of the cylinder is found on the line EB, then with all its positions, it is necessary the cylinder encounter this arc between the points I and K.

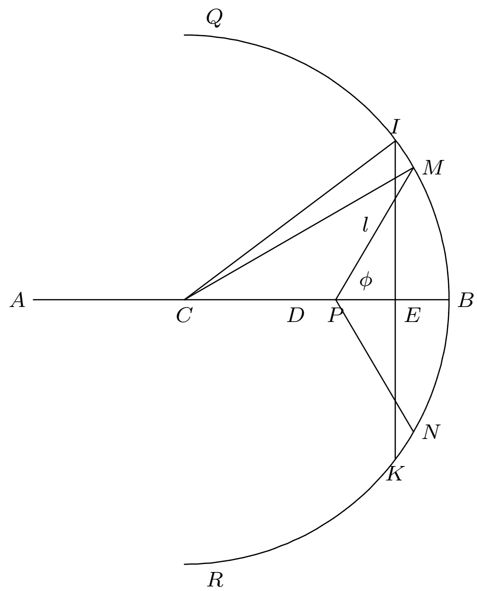

Figure 2:

Then, from the point B we lay off in the direction BA the half-length of the cylinder, and thus determine point D; an encounter of the cylinder with the arc will be possible at a point on the segment ED; let us represent by n the number of cases with which this encounter will occur. Finally, it is obvious that the cylinder, describing with its center the line AB, can not meet the arc QBR being before what position. And so, after denoting by z the desired probability, and after noting that $AB = a + r$, $EB = r - \sqrt{r^2 - l^2}$, we will obtain

$$z = \frac{n + 2\pi(r - \sqrt{r^2 - l^2}}{2\pi(a + r)}. \tag{4}$$

[523] In order to find n, we take any point on the line DE, for example, P; let us represent by ρ the variable distance of the point P from A, so that $AP = \rho$; the element of this line will be $d\rho$. From the point P, with the radius of the semi-cylinder l, we intersect this arc at the points M and N; let ϕ be the angle BPM, and consequently 2ϕ the angle MPN; $2\phi d\phi$ will represent the element quantity n. Integrating this expression from $\rho = AD = a + r - l$ to $\rho = AE = a + \sqrt{r^2 - l^2}$, finds the value of n. And thus

$$n = 2\int_{a+r-l}^{a+\sqrt{r^2-l^2}} \phi\, d\phi.$$

But

$$\int \phi\, d\phi = \phi\rho - \int \rho\, d\phi,$$

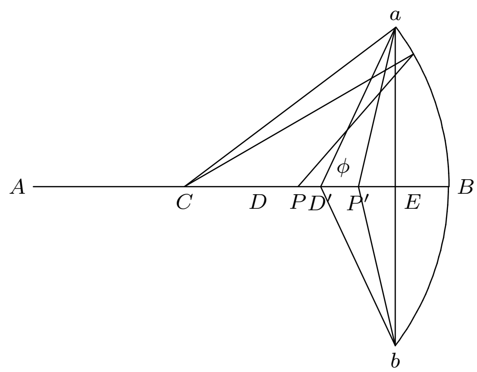

Figure 3:

and moreover as $\rho = a + r - l$ there will be $\phi = 0$, and when $\rho = a + \sqrt{r^2 - l^2}$, $\phi = \frac{\pi}{2}$, then we find

$$n = \pi(a + \sqrt{r^2 - l^2}) - 2\int_0^{\frac{\pi}{2}} \rho\, d\phi.$$

To exclude the value of ρ we note that we have in the triangle CMP

$$r^2 = (\rho - a)^2 + l^2 + 2l(\rho - a)\cos\phi,$$

whence

$$\rho = a - l\cos\phi + \sqrt{r^2 - l^2 \sin^2\phi};$$

consequently

$$\int_0^{\frac{\pi}{2}} \rho\, d\phi = a \cdot \frac{\pi}{2} - l + r\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{l^2}{r^2}\sin^2\phi}.d\phi.$$

And thus

$$n = \pi(a + \sqrt{r^2 - l^2}) - a\pi + 2l - 2r\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{l^2}{r^2}\sin^2\phi}d\phi.$$

[524] Substituting this value into equation (4), we obtain

(5) $$z = \frac{2\pi r + 2l - \pi\sqrt{r^2 - l^2} - 2r\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{l^2}{r^2}\sin^2\phi}d\phi}{2\pi(a + r)}.$$

We see from this expression of the probability that the resolved problem leads us to the integral $\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{l^2}{r^2}\sin^2\phi}d\phi.$, which, as is known, relates to elliptical functions of the 1st and 2nd kinds.

If arc QBR was a specific value, for example it would equal aBb (diag. 3), and the cylinder could not fit in it, then in such cases it would be necessary to proceed as follows: since we assume that a radius of the arc CB divides the arc aBb in half, then after connecting by a straight line points a and b, the line of aEb will be less than the length of the cylinder. Consequently, when the center of cylinder is found beyond the line of EB, then the cylinder, at least its position, will meet the arc aBb. If from point a or b, with a radius l equal of the half-length of the cylinder, we intersect the line CB with the arc, then let us determine the point D', and throughout the line $D'e$, the angle at which the regions of which the encounter will occur, will be determined by the straight lines drawn from a and b to the center of the cylinder; for example, when the center of cylinder is located at P', then the angle about which we speak will be $aP'b$. Let this angle $aP'b = 2\phi'$. Finally, after laying off from the point B through the line BA, the length of the semi-cylinder l, we define the point D, and with the center of the cylinder on the line DD', the calculation will remain the same as in the previous cases. Let n be the number of cases in which there is an encounter, when the center of the cylinder describes the segment DD'; m is the same, relative to the straight line $D'e$; for eB this number is obviously equal to the product $\overline{eB} \times 2\pi$.

Moreover, let us put $AP = \rho$, $D'P' = x$, $\overline{ae} = h$, $\overline{CB} = r$, $\overline{DB} = \overline{aD'} = l$, $\overline{AC} = a$; there will be $\overline{AD} = a + r - l$, $\overline{AD'} = a + \sqrt{r^2 - h^2} - \sqrt{l^2 - h^2}$, $\overline{D'e} = \sqrt{l^2 - h^2}$, $\overline{eB} = r - \sqrt{r^2 - h^2}$. After denoting, as before, by z the desired probability, we obtain

$$z = \frac{n + m + 2\pi(r - \sqrt{r^2 - h^2}}{2\pi(a + r)}. \tag{6}$$

From the above, it is easy to see that

$$\begin{aligned} n &= 2\int_{a+r-l}^{a+\sqrt{r^2-h^2}-\sqrt{l^2-h^2}} \phi\, d\phi, \\ m &= 2\int_0^{sqrtl^2-h^2} \phi'\, d\phi, \end{aligned}$$

or

$$\begin{aligned} n &= 2\phi\rho - 2\int \rho\, d\phi \\ m &= 2\phi' x - 2\int x, d\phi \end{aligned}$$

observing also that the limits with respect to ϕ will be $\phi = 0$ and $\phi = \arcsin\frac{h}{l}$, and in the reasoning ϕ', $\phi' = \arcsin\frac{h}{i}$ and $\phi' = \frac{\pi}{2}$.

But it is deduced from the triangles CMP and $aP'e$

$$r^2 = (\rho - a)^2 + l^2 + 2l(\rho - a)\cos\phi,$$

$$\tan\phi' = \frac{h}{\sqrt{l^2 - h^2 - x}},$$

or

$$\rho = a - cos\phi + \sqrt{r^2 - l^2 \sin^2 \phi},$$

$$x = \sqrt{l^2 - h^2} - \frac{h}{\tan \phi'};$$

Consequently

$$n = 2(a+\sqrt{r^2 - h^2}-\sqrt{l^2 - h^2}) \arcsin \frac{h}{i} - 2\int_0^{\arcsin \frac{h}{i}} (a - l \cos \phi + \sqrt{r^2 - l^2 \sin^2 \phi}) d\phi,$$

$$m = \pi \sqrt{l^2 - h^2} - 2\int_{\arcsin \frac{h}{l}}^{\frac{\pi}{2}} (\sqrt{l^2 - h^2} - \frac{h}{\tan \phi'}) d\phi'.$$

[526] After assuming for brevity

$$\int_0^{\arcsin \frac{h}{l}} \sqrt{1 - \frac{l^2}{r^2} \sin^2 \phi}.d\phi = \Delta,$$

we obtain

$$n = 2(\sqrt{r^2 - h^2} - \sqrt{l^2 - h^2}) \arcsin \frac{h}{l} + 2h - 2r\Delta. \tag{7}$$

After finding the integrals which determine value m, we will obtain after the reductions

$$m = 2\sqrt{l^2 - h^2}. \arcsin \frac{h}{i} - 2h \log \left(\frac{h}{i}\right). \tag{8}$$

Substituting the last values (7) and (8) into equation (6), we obtain finally the following expression for the probability z:

$$z = \frac{2h + 2r(r - \sqrt{r^2 - l^2} + 2\sqrt{r^2 - h^2}. \arcsin \frac{h}{l} - 2r\Delta - 2h \log \left(\frac{h}{l}\right)}{2\pi(a + r)}, \tag{9}$$

which turns into (5), when we put $h = 1$.

And so we see the last question to lead us to an expression that consists in itself, besides circular and elliptical functions, also there is the logarithmic $\log \left(\frac{h}{l}\right)$. Diversifying problems, we obtained another formula with different transcendental numbers; producing then a great number of trials required by the terms and conditions of the problem, and after computing the number of encounters of the cylinder with the perimeter of the figure in question, so the number of all tests, lets us determine the ratio of these two numbers. The equation is the ratio of the expression found for the probability, we obtain an equation, from which in consequence of a corollary of the theorem of Jakob Bernoulli, it will be possible to derive the approximate value of transcendental numbers, in the expression of the probability, as it was explained in the argument.