On the Application of the Analysis of Probabilities to Determining the Approximate Value of Transcendental Numbers*

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For the determination by means of the analysis of probabilities the value of a transcendental number, we have in mind expressing a transcendental number by the probability of the occurrence of any event, depending on countlessly many cases. Then a great number of trials, randomly selected cases on which the event depends, is produced, and both the number of occurrences of the event and number of all trials are noted. The ratio of the first to the latter will depict, by the well-known theorem of *Jakob Bernoulli* the approximate value of the probability of the said event. Equations found by the ratio of these things *a priori* to express the probability, will give an equa- [458]
tion from which it will be easy to determine the desired transcendental number.

The task, for which we propose here the solution, consists of the determination of ratio of circle to the diameter; although this kind of question is one of the simplest, but along with that all the more difficult of those which have been solved until now. Here is what it is:

Let us assume that a given definite or indefinite quantity of the plane tiled by equilateral triangles; onto this plane is thrown, at random, a very thin cylinder of known length. The question is, how great is the probability that the cylinder will fall at least on one of the sides of the triangles drawn on the plane?

Let us note that the desired probability for the entire system of triangles will be identical with the probability corresponding to one of the component triangles; for this it suffices to determine the latter, and therefore, taking into consideration one of the triangles of plane to determine the probability that a cylinder, falling with its center inside the triangle in question, will cross one or two of its sides, must proceed as follows: about every internal point of the triangle, taken as the center of the cylinder, we describe a whole circle with a radius equal to the half-length of the cylinder; and then determine how great is the angle at which the cylinder will intersect the sides of the triangle. The ratio of the angle to 360° thus found, it is obviously equal to the ratio of the number of cases of encounter to the number of all possible cases, and consequently represents the probability of the encounter, when the center of cylinder will fall at a

*Paraphrased by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. January 5, 2022 given point inside the triangle. Let the probability be $\frac{\phi}{2\pi}$. Let us designate by the characteristic S the quantity pertaining to all points of triangle. It is obvious that $\frac{\phi}{S2\pi}$ [459] will represent the probability that the center of cylinder will fall at a designated point in advance, and that the cylinder itself will meet at least one side of the triangle, and the expression $\frac{S\phi}{S2\pi}$ represents the desired probability. Multiplying both the numerator and denominator of the latter fraction by the element of area $dxdy$, after replacing the summation sign S by the double integral, we obtain

$$\frac{\iint \phi\, dxdy}{\iint 2\pi\, dxdy} = \frac{\iint \phi\, dxdy}{2\pi \iint\, dxdy}$$

If, for brevity, we indicate by A^2 the area of a triangle, then we find the following expression for the probability of encounters of the cylinder with divisions

$$\text{(1)} \qquad \frac{\iint \phi\, dxdy}{2\pi A^2}$$

And so, for resolution of our problem, it is necessary to determine the numerator of previous fraction. Because of the variety of circumstances, which are represented in different parts of the triangle, we must decompose it into several partial figures.

Let ABC (Figure 1) be the triangle in question, L each of its sides, and $2r$ the length of this cylinder.

At the perpendicular distance equal to r, from each side of the triangle and inside of it, we draw three parallel lines; thus they will be composed: 1°. The equilateral triangle of abc, for which let us name the area Ω. 2°. Three trapezoids $abK'K''$, $acKM''$, $bcMM'$; let $\tilde{\omega}$ be the area of each of them. 3°. Three rhombi $AaKK'$, $BbMK''$, $CcM'M''$, and the area of each of them ω. Consequently, the area of the triangle ABC is equal $\frac{\sqrt{3}}{4}L^2$, we will also have $= \Omega + 3\tilde{\omega} + 3\omega$.

Let us note that this decomposition of the triangle is correct only for the case when [460] r is less than the radius of the inscribed circle of triangle ABC. Consequently, we assumed $r < \frac{L}{2\sqrt{3}}$. However, if it is not possible to exclude the case $r = \frac{L}{2\sqrt{3}}$; then it is necessary to take $\Omega = 0$.

Before we address the solution of the problem, let us write for convenience some values for which we will have need. Here they are (on Figure 1):

$$\overline{AB} = L, \quad \overline{ab} = L - 2\sqrt{3}.r \quad \overline{bH} = r$$
$$\overline{bM} = \frac{2r}{\sqrt{3}} = l, \quad \overline{MH} = \frac{r}{\sqrt{3}} = \frac{1}{2}l, \quad \text{angle}(ABC) = 60^\circ$$
$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}, \quad \sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}.$$

Let P be the numerator of the fraction (1); the ratio $\frac{P}{2\pi.\frac{\sqrt{3}}{4}L^2} = \frac{2P}{\sqrt{3}\pi L^2}$ represents the probability of the encounter of the length of the cylinder meeting one of the sides of triangle.

Let us now turn to the determination of the value of P, and note first that while the center of cylinder is located inside the triangle of Ω, then the cylinder itself cannot fall on any one of the sides AB, BC, CA. Consequently, the area of Ω will not be the case of encounter. Let us represent by m the number of cases in which the cylinder, falling

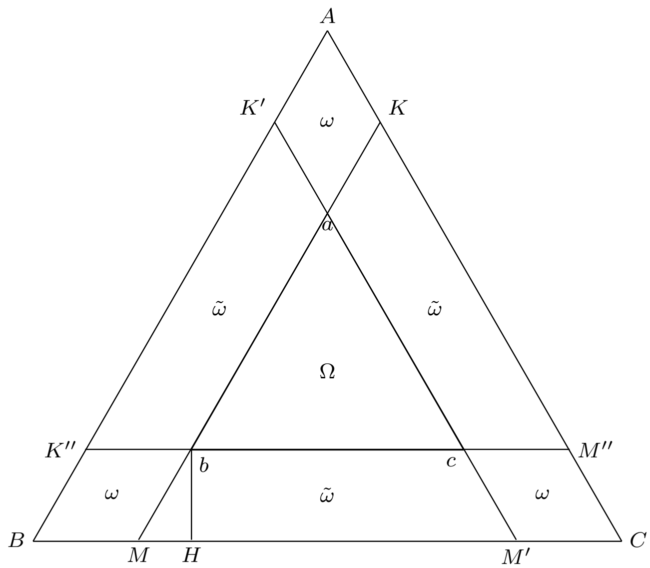

Figure 1: ⟨Partitions of the equilateral triangle⟩

with its center inside the area $\tilde{\omega}$, will cross one of the sides AB, BC, CA, and by n with the same reasoning in each of the areas ω. We will obtain $P = 3m + 3n$, and consequently, the desired probability, which we represent by z, will be determined by the formula

(2) $$z = \frac{2\sqrt{3}(m+n)}{\pi L^2}.$$

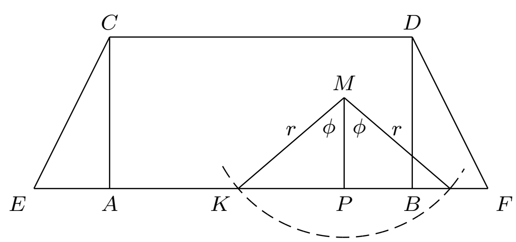

Figure 2: ⟨Analysis within rectangle of a trapezoid⟩.

The problem now consists in the determination the value of m and n. Let us study first the former. Let $ECDF$ (Figure 2) be the trapezoid $\tilde{\omega}$; let us decompose it into the rectangle $ABCD$ and two right triangles ACE and BDF. Let μ be the number of cases of encounter, when the center of cylinder is located inside the rectangle, and λ [461]

the same in relation to each of two triangles. Consequently

$$m = \mu + 2\lambda. \tag{3}$$

In order to find μ, let us take inside the rectangle of $ABCD$ an arbitrary point M, and let us represent the rectangular coordinates of AP and PM by x and y. Let us assume that the center of the cylinder $2r$ falls at this point, and that the cylinder is turning about it, describing the ends of its entire circumference; let ϕ be the angle, comprehended between the axis of cylinder with the perpendicular MP at the instant when the cylinder touches one end of the line of AB at the point K. Angle 2ϕ represents the part of the circle, described by each end of the cylinder, in which this latter will fall beyond the side of AB. And so, 4ϕ is a value of the angle, with which will occur the encounter of the cylinder with the side of AB, assuming its center at the point M, and the double integral $4\iint \phi\, dxdy$, taken between proper limits, expresses the value, which we represented above by μ.

To determine this integral, we note that $y = r\cos\phi$, and consequently $dy = -r\sin\phi.d\phi$ the limits relative to ϕ will obviously be $\frac{\pi}{2}$ and 0, and the limits relative to x will be evidently 0 and $AB = L - 2\sqrt{3}.r$. And so, we will obtain

$$\mu = 4r\int_0^{L-2\sqrt{3}.r}\int_0^{\frac{\pi}{2}} \phi\sin\phi.dxd\phi = 4r(L - 2\sqrt{3}.r). \tag{4}$$

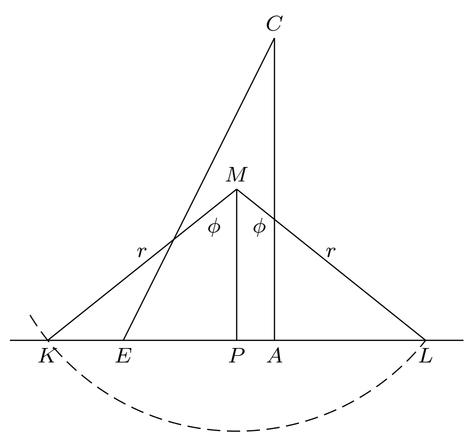

Figure 3: ⟨Analysis within triangle of a trapezoid⟩.

The value of λ is determined similarly: only the upper limit with respect to y will be variable. Let us indicate by $x = EP$, $y = PM$ (Figure 3) the coordinates of the points of M, at which we assume the center of cylinder, and render through 2ϕ the angle comprehended between the two extreme positions of the cylinder, so that $\overline{MK} = \overline{ML} = r$. It is obvious that value λ will be expressed by the double integral $4\iint \phi\, dxdy$, undertaken between the limits that encompass the entire triangle of ACE;

consequently, the limits in reasoning on the variable y will be 0 and $\sqrt{3}.x$, since the equation of the straight line of EC, by reason that the angle CEA, being equal to 60°, is $y = \sqrt{3}.x$; relative to the abscissas of x, the integral must be taken from $x = 0$ to $x = EA = \frac{r}{\sqrt{3}}$ and thus

$$\lambda = 4\int_0^{\frac{r}{\sqrt{3}}}\int_0^{\sqrt{3}.x} \phi\, dxdy.$$

However, since $y = r\cos\phi$, then $\phi = \arccos\frac{y}{r}$, this is why

$$\lambda = 4\int_0^{\frac{r}{\sqrt{3}}}\int_0^{\sqrt{3}.x} \arccos\frac{y}{r}\, dxdy = \frac{4r^2}{\sqrt{3}}\left(1 - \frac{\pi}{8}\right).$$

Consequently, in view of formulas (3) and (4), we will obtain

$$m = 4rL - \frac{16r^2}{\sqrt{3}} - \frac{\pi r^2}{\sqrt{3}}. \tag{5}$$

The determination of the value of n is somewhat more complex than the previous of m. For convenience we will increase the dimension of rhombus ω. Let $ABCD$ (Figure 4) be this rhombus. From the point A, with a radius r of the semi-cylinder (by construction equal of the height of the rhombus), let the circular arc dcb be described. It is obvious that until the center of cylinder is located inside the sector of $Abcd$, then the cylinder with all of its positions, certainly will fall on one side of AB, AD, even on both sides. Consequently we can assume that the number of connections belonging to this occasion, will be equal to entire circle, multiplied by the area of the sector; this product is equal to $\frac{\pi^2 r^2}{3}$. If we represent by p the integral $\iint \phi\, dxdy$, common to all [463]
the remaining length of the rhombus figure $dDCBbc$, then obviously we will obtain

$$n = \frac{\pi^2 r^2}{3} + p, \tag{6}$$

And the problem will be reduced thus to the determination of the value of p. To this end, for which there is any point M taken inside the figure $dDCBbc$, with a radius of r, we describe part of a circle, which intersects the sides AB and AD, or of their continuations, at some points 1,2, 3, 4. It is clear that turning the cylinder about the point of M in question, it will meet both sides AB and AD, or only one of them, when there are the inside angles $(1M2)$ and $(3M4)$, precisely: on the sides in the space of angle $(4ME)$ and equal to it and opposite to the vertex $(1ME')$, and one only, as will be in the angles $(EM2)$, $(4MF')$, $(3ME)$, $(FM1)$; outside of these angles an encounter is impossible. Hence it is easy to conclude that the encounter will occur if the cylinder falls within the angle $(FM2)$ and its opposite vertex $(FM3)$, i.e., in the space double the angle $(FM2)$. Now beyond the limits, the cylinder cannot fall beyond the sides of AB, AD of the rhombus. Let $(FM2) = \theta$, and let us determine this angle. From the point M let us drop the perpendiculars of Mh and Mj to the sides AB and AD of the rhombus; let angle $(1M2) = 2\phi$, and $(3M4) = 2\phi'$. Since angle $\theta = 180° - (2M3)$, and angle $(hMj) = 2\phi$, by the property of the rhombus, it is equal to 120°, then there exists $\theta = 60° + \phi + \phi' = \frac{\pi}{3} + \phi + \phi$'. Let AX,

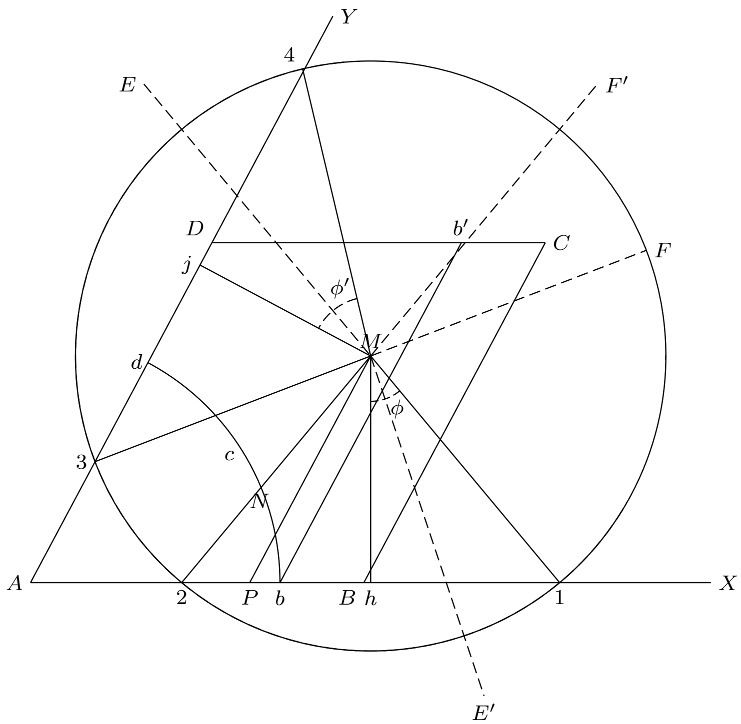

Figure 4: ⟨Analysis of a rhombus⟩

AY be coordinate axes, and $AP = x$, $PM = y$; the element of area there will be $\sin.60°.dxdy = \alpha dxdy$, understanding by α the irrational number $\frac{\sqrt{3}}{2}$. Consequently we will obtain

$$p = 2\alpha \iint \left(\frac{\pi}{3} + \phi + \phi'\right) dxdy.$$

If we express ϕ and ϕ' by means of y and x, and decompose the last integral into two others, one relative to the figure $Ddcbb'D$, and the other relative to the parallelogram of $bBb'C$, then the first integral should be taken from $y =$ the ordinate of the circle of PN to $y =$ the constant of the line AD, and from $x = 0$ to $x = r$; the second, from $y = 0$ to $y =$ the constant of the line AD, and from $x = r$ to $x = AB$. We will describe through y' the ordinates of PN of the circle, and let us note that the line $AB = AD = \frac{2r}{\sqrt{3}} = l : 1$; as in addition we have

$$x = l\cos\phi' \qquad y = l\cos\phi,$$

and consequently that

$$\phi' = \arccos\frac{x}{l}, \qquad \phi = \arccos\frac{y}{l},$$

and we obtain

$$
\begin{aligned}
p &= 2\alpha \int_0^r \int_{y'}^l \left(\frac{\pi}{3} + \arccos\frac{y}{l} + \arccos\frac{x}{l}\right) dxdy \\
&\quad +2\alpha \int_r^l \int_0^l \left(\frac{\pi}{3} + \arccos\frac{y}{l} + \arccos\frac{x}{l}\right) dxdy
\end{aligned}
$$

For brevity let I_1 be the first, and I_2 the second of the integrals, occurring in the value of p; there will be

$$p = 2\alpha I_1 + 2\alpha I_2.$$

Integrating with respect to the variable y, we obtain

$$I_1 = \int_0^r \left[\frac{\pi}{3}(l-y') + \sqrt{l^2-y'^2} + l.\arccos\frac{x}{l} - y'(\arccos\frac{y'}{l} + \arccos\frac{x}{l})\right] dx$$

$$I_2 = \int_r^l \left[\frac{\pi}{3}l + l + l.\arccos\frac{x}{l}\right] dx.$$

The integral I_1 is involved with the variable x still the ordinate y' of the circle; however, since this latter equation is $x^2 - y'^2 + xy' = r^2$, from which we will obtain $y' = \frac{\sqrt{4r^2-5x^2-x}}{2}$, and also $x = \frac{\sqrt{4r^2-5y'^2-y'}}{2}$ whence we conclude $\sqrt{4r^2-y'^2} = 2x + y'$ or $\sqrt{l^2-y'^2} = \frac{1}{\sqrt{3}}(2x+y')$. Noting in addition that $\int_0^r y'dx = \frac{\pi r^2}{6\alpha}$, we find

$$
(7) \qquad \left\{
\begin{array}{c}
\int_0^r \frac{\pi}{3}(l-y')dx = \frac{\pi}{3}lr - \frac{\pi^2 r^2}{18} \\
\int_0^r \sqrt{l^2-y'^2}.dx = \frac{1}{\sqrt{3}}\int_0^r (2x+y')dx = \frac{r^2}{\sqrt{3}} + \frac{\pi r^2}{6\alpha\sqrt{3}} \\
\int_0^r l.\arccos\frac{x}{l}.dx = l(l - \sqrt{l^2-r^2} + r\arccos\frac{r}{l}).
\end{array}
\right.
$$

To determine the last integral included in the value of I_1, it is

$$\int_0^r y'(\arccos\frac{y'}{l} + \arccos\frac{x}{l})dx,$$

let us note that the sum $\arccos\frac{y'}{l} + \arccos\frac{x}{l} = 120° = \frac{2\pi}{3}$. It is very easy to prove this equality by means of the analysis; but even easier by simple geometric construction. Indeed we have already seen that in general

$$x = l\cos\phi', \qquad y = l\cos\phi;$$

consequently the circle will also be

$$x = l\cos\phi', \qquad y = l\cos\phi;$$

whence

$$\arccos\frac{x}{l} = \phi', \qquad \arccos\frac{y'}{l} = \phi;$$

and thus

$$\arccos\frac{y'}{l} + \arccos\frac{x}{l} = \phi + \phi'.$$

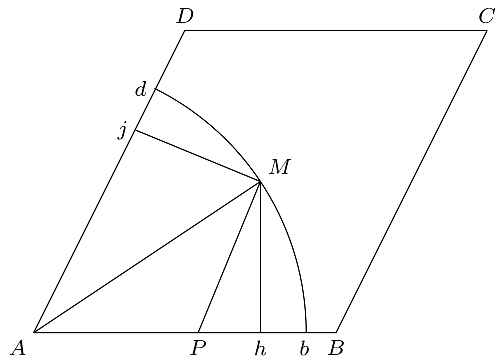

Figure 5: ⟨Further analysis of a rhombus⟩

But it is clear that when the center of the cylinder will be located in whatever point M of the circular arc dcb (Figure 5), then for the reason MA is the equal of r, the angles ϕ and ϕ' will be adjacent, i.e., $(hMA) = \phi$, $(AMj) = \phi'$, which is why $(hMj) = \phi + \phi'$; but on the other hand, it is obvious that angle $(hMj) = 120°$; consequently

$$\arccos\frac{y'}{l} + \arccos\frac{x}{l} = 120° = \frac{2\pi}{3}.$$

And thus

$$(8) \qquad \int_0^r y'\left(\arccos\frac{y'}{l} + \arccos\frac{x}{l}\right)dx = \frac{2\pi}{3}\int_0^r y' dx = \frac{2\pi}{3}\cdot\frac{\pi r^2}{6\alpha} = \frac{\pi^2 r^2}{9\alpha}.$$

Combining integrals (7) and (8), and substituting value l equal to $\frac{2r}{\sqrt{3}}$, and for α the number $\frac{\sqrt{3}}{2}$, we find a reduction to

$$2\alpha I_1 = r^2\left[2 + \frac{2}{\sqrt{3}} + \left(1 + \frac{1}{3\sqrt{3}}\right)\pi - \frac{1}{3}\pi^2\right].$$

With regard to the integral I_2, we obtain without any difficulty

$$2\alpha I_2 = r^2\left[2(\sqrt{3}-1) + \left(\frac{4}{3\sqrt{3}} - 1\right)\pi\right].$$

Consequently, by the formula $p = 2\alpha I_1 + 2\alpha I_2$, there is

$$p = r^2\left(\frac{8}{\sqrt{3}} - 1 + \frac{5}{3\sqrt{3}}\pi - \frac{1}{2}\pi^2\right),$$

and finally, yet equation (6),

$$n = r^2\left(\frac{8}{\sqrt{3}} - 1 + \frac{5}{3\sqrt{3}}\pi\right).$$

If we now substitute into equation (2) the value m given by equation (5) and the value for p now obtained, then we find finally:

$$z = \frac{3\sqrt{3}.rL - r^2(16 + 2\sqrt{3} - \frac{1}{3}\pi)}{\pi L^2}. \tag{9}$$

Here is the expression of the probability that a cylinder, cast at random onto the plane divided into equilateral triangles, will fall at least onto one of their sides; it is obvious that the contrary probability will be $1 - z$. For example, if for the length of cylinder was accepted the maximum value $\frac{L}{\sqrt{3}}$, which is mentioned above, then we would find

$$z = \frac{16 - \sqrt{3} + \frac{3}{2}\pi}{6\pi} = \text{near } \frac{33}{38}$$

and consequently the contrary probability $= \frac{5}{38}$.

At the beginning of our reasoning we said that it expounded a method to lead to the determination of the approximate values of transcendental numbers. Actually, it is possible to use formula (9) derived above to find the approximate value of π, i.e., the ratio of the circle to its diameter. To do this, one has only to draw on the plane equilateral triangles contacting one another, and, at random, throw down a thin cylinder. After repeating this step a great number of times (for example, 100000), and after noting how often the cylinder fell on at least one of the drawn divisions, this last number, divided by 100000, according to Jakob Bernoulli's theorem, will very closely represent the probability of z. Equating this ratio to the second part of formula (9), we will obtain the equation from which it is easy to derive the value of π from the side L of triangle and length $2r$ of the cylinder, which are assumed to be known. However one should not lose sight of the fact that equation (9) is derived with the condition $r < \frac{L}{2\sqrt{3}}$.

We intend to yet return to this kind of rather curious problems, and frequently they are of no small difficulty. For this reason present discussion is now noted as our *first*.