SUR UN INSTRUMENT DESTINÉ À FACILITER L'APPLICATION NUMÉRIQUE DE LA MÉTHODE DES MOINDRES CARRÉS, ET À CONTRÔLER LES RÉSULTATS OBTENUS PAR CETTE MÉTHODE*

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Calculators know by experience how much, in the greater part of the cases, the numerical application of the *method of least squares* is laborious and painful. In order to pass from the equations of condition to the final equations of the problem, one must form the squares and the products of different groups of numbers, and to sum them next, this which gives place generally to some quite prolix calculations. And this is not all: after having found the sought numerical result, one finds oneself most often in the apprehension of having committed some error, too great in order to be able to be neglected.

These considerations have led me to seek, if it would not be possible to resolve *graphically*, by aid of an instrument of simple construction, the two principal numerical questions of the method of least squares, namely 1° *the formation of the squares of a sequence of numbers and the summation of these squares*: 2° *the formation of the products of the two factors, and the summation of a series of these products*. The instrument that I have the honor to set under the eyes of the Académie, attains to a certain point the end that I myself have proposed. It resolves with swiftness and with a sufficient precision, at least for the control of the direct calculations, the two problems mentioned. When this part of the work, which gives the most opportunity to errors, is controlled by aid of the instrument, one continues the calculation with more confidence in order to resolve the final equations, deduce the value of the weights of the result and of the other elements that one considers in the application of the method of least squares to the calculation of the observations.

The fundamental idea of the instrument of which there is concern is extremely simple; the description that we are going to give of it and the indication of the manner to serve oneself with it, will show that it is based on the single proposition of *Pythagoras*.

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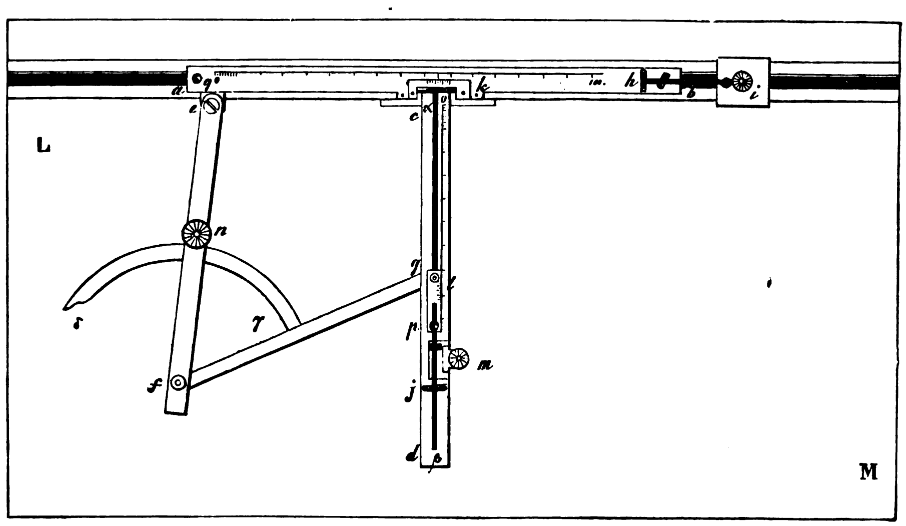

Figure 1: The Instrument

Figure 1[1] represents the instrument, that one could call *équerre sommatrice* [summing square] in regard to its form and to its destination. Its two principal pieces are the two rules in brass ab and cd; the first is around 10 English inches long, the second of 8 inches. The rule cd is fixed, and reposes on a plate of wood LM; as for ab, it is able to glide freely the length of a prismatic filament, to right and to left, in a direction perpendicular to cd; but one is able also to render it immobile by aid of the *set screw* i. Each of these two rules is equipped with a scale; the first ab is divided into 165 equal parts, and the second cd into 110 parts. By means of the two sliding scales k and l, of which the first is fixed to the upper extremity of the rule cd, and the other mobile in the groove $\alpha\beta$, each of the said divisions will be subdivided into *ten* parts. Thus, one will be able to take on the scale of ab all the whole numbers from 1 to 1650, and on that of cd, all the numbers no superior to 1100. The two *micrometrical screws* h and j, of which the first is moved with the rule ab, and the second with the sliding scale l, are destined to communicate small movements which define the indications of the sliding scales. The set screw i, destined in first place to fix the rule ab, serves at the same time as fulcrum for the working of the micrometrical screw h; the screw m has the same destination relative to the second micrometrical screw j.

Beyond the two rules ab and cd, there are two others, ef and fg, each around 7 inches long, joined between them by means of a hinge at f. The first of these rules ef moves freely about an axis at e, perpendicular to a metallic blade adapted to the rule ab; this axis is found placed exactly under the *zero* of the scale ab. The extremity g of the second rule fg moves likewise about an axis fixed at the lower part of the sliding scale l, and corresponding to its *zero*. Finally, the metallic arc $\gamma\delta$ serves to fix, with the need, the angle gfe by means of the set screw n. When the arc $\gamma\delta$ is fixed in this manner,

[1]The linear dimensions of figure 1 are the *third* of those of the instrument.

the distance eg remains invariable, that which is a necessary condition for the working of the instrument. The system of the two rules ef and fg repose on the wooden board LM, and the second among them, fg, is able to glide freely under the rule cd during the movement of the axis with g in the groove $\alpha\beta$.

Such is the quite simple construction of the *summing square*. In order to make use of it, we suppose that one wishes to add the series of squares

$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_s^2.$$

One will begin by loosening the two set screws i and m, and one moves to *zero* the sliding gauge of the scale cd by being aided for this with the knob p. Next, by means of the knob q, one will make the rule ab glide until the *zero* of the sliding scale k indicates the number a_1, that to what one will arrive in a more precise manner by making use of the micrometrical screw h, for the working of which it will be necessary to tighten the screw i. This set screw i, in each case, must be tightened in order to fix the position of the rule ab. After this one will make glide in the groove $\alpha\beta$, by being aided by the knob p, the sliding scale l, in a manner to this that the *zero* which it bears, indicates the second number a_2, this to what one will arrive with more precision by means of the micrometrical screw j, that one makes act after having tightened the set screw m. It is evident besides that the movement indicated from the knob p will not suffer obstacle seeing the disposition of the two rules ef and fg, which are able to turn freely about the three axes at e, f, g. This done, one will fix the angle efg by aid of the set screw n, and one will relax the set screw i. Then by being aided of the knob q and of the extremity f, one makes glide, by going up again, the knob q the length of the groove $\alpha\beta$ to the zero of the scale, and one tightens the set screw i in order to fix ab. The sliding scale k will indicate evidently the length $\sqrt{a_1^2 + a_2^2}$. If, now, after having relaxed the screw n, one bears on the scale of the rule cd the length a_3, and if one operates exactly as one just did, the second indication of the sliding scale will be $\sqrt{a_1^2 + a_2^2 + a_3^2}$, and so forth until the last, equal to $\sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_s^2}$. By squaring the number of the last indication, one will have the sought sum $a_1^2 + a_2^2 + a_3^2 + \cdots + a_s^2$.

The sum of the products of two factors such as

$$a_1 h_1 + a_2 h_2 + a_3 h_3 + \cdots + a_s h_3$$

will be able to be obtained by means of the same instrument in the following manner: by supposing that one has

$$a_1 > h_1, \quad a_2 > h_2, \quad a_3 > h_3 \ldots a_s > h_s,$$

one will commence by calculating the half-sums and the half-differences

$$\frac{a_1 + h_1}{2}, \quad \frac{a_2 + h_2}{2} \cdots \frac{a_s + h_s}{2}$$

$$\frac{a_1 - h_1}{2}, \quad \frac{a_2 - h_2}{2} \cdots \frac{a_s - h_s}{2}$$

after which, by observing that

$$\begin{aligned} a_1 h_1 &= \left(\frac{a_1+h_1}{2}\right)^2 - \left(\frac{a_1-h_1}{2}\right)^2 \\ a_2 h_2 &= \left(\frac{a_2+h_2}{2}\right)^2 - \left(\frac{a_2-h_2}{2}\right)^2 \\ &\cdots\cdots\cdots\cdots \\ a_s h_s &= \left(\frac{a_s+h_s}{2}\right)^2 - \left(\frac{a_s-h_s}{2}\right)^2 \end{aligned}$$

one will have, by making the sum of these equations

$$S\left(a_s h_s\right) = S\left(\frac{a_s+h_s}{2}\right)^2 - S\left(\frac{a_s-h_s}{2}\right)^2.$$

Each of the two sums

$$S\left(\frac{a_s+h_s}{2}\right)^2 \text{ and } S\left(\frac{a_s-h_s}{2}\right)^2.$$

will be obtained, as we just demonstrated, by the aid of the instrument and by an elevation to the square. The difference of the two squares thus obtained, will represent the sought sum $a_1h_1 + a_2h_2 + a_3h_3 + \cdots + a_sh_3$.

One sees by that which precedes, how much the summing square is able to be useful in the numerical calculation of the most advantageous results. In truth, the instrument such as the one that I had constructed by the Engineer Mr. Albrecht, is able to serve only to the summation of squares of numbers expressed by means of four digits; in much of the cases this limit will be sufficient. But, if there would be concern of numbers surpassing this limit, the instrument would give only some approximate results which could serve to control the dominant digits obtained by a direct numerical calculation; the knowledge of the approximate results in this case would certainly not lack being quite useful. We remark further, that by changing the order in the addition of the squares, one will obtain, by means of the summing square, many values of the sought sum; by taking their arithmetic mean one will approach yet more the exact value. Thus, the instrument lends itself easily to some repeated tests, that which constitute one of its advantages. In order to give an idea of the degree of precision that one is able to obtain by means of the repeated tests, I will report here the following example, for which I have redone the operation only 3 times.

The concern is to extract the square root of the sum of the following ten squares:

$$123^2 + 175^2 + 210^2 + 253^2 + 300^2 + 330^2 + 482^2 + 523^2 + 540^2 + 674^2.$$

The operation having been made in this order, has given for the sought root the number 1265.

In the second test I have distributed the numbers in the following manner:

$$540^2 + 210^2 + 330^2 + 523^2 + 123^2 + 253^2 + 300^2 + 674^2 + 482^2 + 175^2,$$

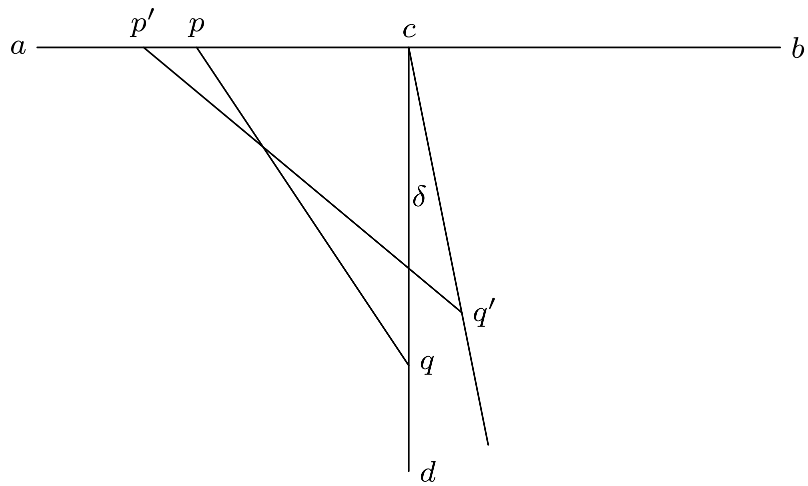

Figure 2: Errors of the instrument

and I have found the number 1266 for the sought root.

Finally, in the third test, the squares were disposed thus as it follows:

$$674^2 + 253^2 + 540^2 + 330^2 + 210^2 + 175^2 + 523^2 + 482^2 + 300^2 + 123^2,$$

and I have found for final result 1264.

The arithmetic mean of these three values, very little different among themselves, is equal to 1265 which deviates from the exact result (1266.6…) only in the *fourth* digit; thus, in this example, taken completely at random, the error has been around only $\frac{1}{800}$ of the exact result. I will add to this, that I have found the three partial results without making use of the micrometrical screws.

We note that, if among the numbers of which one determines the sum of squares, there would be found of them too small, not exceeding, for example, the limit 100, it would be more convenient to operate on those by adopting a multiple scale of that which the instrument bears, for example a scale *double*, *triple*…*tenfold*; in this last case it would be necessary to take 100 parts for 10, 200 for 20, 300 for 30 etc. In this manner the total operation could be composed of two parts: the one for the smallest numbers, and the other for the greatest. One reunites the two results thus obtained by carrying one of the roots on the scale of the rule ab, and the other on the scale cd. Operating next as it has been explicated above, one will arrive to the final result. Moreover, the usage itself of the instrument that one will have to its disposition, will set quite swiftly to the fact of that which will be able to contribute to shorten or to facilitate the operations that one executes.

If the number of squares to sum is too great, in a way that the scale of the rule ab is not sufficient to indicate the square root of their sum, one will partition these numbers into groups, on which one will operate separately. The partial results will be able to be next reunited by means of the summing square by adopting a *reduced scale* or *submultiple*, for example a *subdouble*, *subtriple* scale etc.

We will terminate this note by the approximate calculation of the limit of the error that is able to result from the imperfection of the instrument. We suppose that, in the case of an absolute precision of the one here, one seeks the value of the root $\sqrt{a_1^2 + a_2^2}$. One will bear $a_1 = pc$ (Fig. 2) on the scale of the rule ab, and $a_2 = cq$ on the scale of the rule cd; we admit that the angle acd is rigorously right, and that pc and cq represent exactly the numbers a_1 and a_2. Under this hypothesis, the indication of the instrument which will transport by means of a system of two rules ef and fg the length pq on the scale of ab, is found completely exact, and will represent the square root $\sqrt{a_1^2 + a_2^2}$. Now, the imperfection of the instrument will give necessarily place to some errors that we will be able to reduce to three. And, first, we note that instead of the true right triangle cpq, we will obtain another oblique angled; let $cp'q'$ be this erroneous triangle. The three errors will be: 1° on the angle at c which, instead of being rigorously right, will differ from it by a certain quantity $\delta = qcq'$, and will be consequently equal to $90^\circ + \delta$; 2° instead of the true length $pc = a_1$, one will have another $cp' = a_1 + \epsilon_1$; 3° instead of the true length $cq = a_2$, one will have $cq' = a_2 + \epsilon_2$. The error δ on the angle will result first from this that the rule ab will not glide completely perpendicular to cd, and from this that the points p' and q' will not correspond rigorously, the first, to *zero* of the scale ab, and the second, to the *zero* of the sliding scale of the rule cd. This last cause, joined to the small inevitable inequalities of the divisions of the scales and to the defects of observation itself, will produce also the errors ϵ_1 and ϵ_2.

This put, by representing by ω the total error committed in the determination of the length

$$pq = \sqrt{a_1^2 + a_2^2},$$

this error ω will be clearly equal to the difference $\pm(pq - p'q')$; now, the triangle $cp'q'$ gives

$$p'q' = \sqrt{(a_1 + \epsilon_1)^2 + (a_2 + \epsilon_2)^2 + 2(a_1 + \epsilon_1)(a_2 + \epsilon_2)\sin\delta};$$

one will have therefore, by neglecting the powers of the errors superior to the first,

$$\omega = \pm\sqrt{a_1^2 + a_2^2}\left(1 - \sqrt{1 + \frac{2(a_1\epsilon_1 + a_2\epsilon_2 + a_1 a_2\delta}{a_1^2 + a_2^2}}\right).$$

Developing the radical, and conserving, as just now, only the first powers of ϵ_1, ϵ_2, δ, one will have

$$\omega = \frac{a_1\epsilon_1 + a_2\epsilon_2 + a_1 a_2\delta}{a_1^2 + a_2^2}.$$

Such is the very simple expression of the sought error by setting aside its sign; we see now what will be able to be very nearly its limit. For that we observe that the rule ab glides the length of a prismatic filament subject to the rule cd, and that moreover the axes at e and g (Fig. 1), are previously disposed in a manner to correspond respectively, with as much precision as possible, the first, to *zero* of the scale ab, and the second, to the *zero* of the sliding scale l. In this manner it is clear that the error δ of the angle will be able to be only insensible. We suppose that it goes even to *one fourth of degree*; one will have very nearly

$$\delta = \frac{3.141}{2 \times 360} = 0.0043\dots$$

As for the errors ϵ_1 and ϵ_2, one will exaggerate certainly the imperfection of the instrument by supposing that this error is able to go to the *fifth* part of a direct division of the scales of the two rules; now, this fifth part is equivalent to *two* wholes, that is to two parts indicated by the vernier. One will take therefore $\epsilon_1 = \epsilon_2 = 2$, and one will have

$$\omega = \frac{2(a_1 + a_2) + a_1 a_2 \times 0.0043}{\sqrt{a_1^2 + a_2^2}},$$

by admitting the most unfavorable case, namely the one where all the errors are in the same sense.

In the result that we just found we have set aside the nearly insensible error which could result from the transport of the length $p'q'$ on the rule ab.

We apply our formula to the case where one would have, for example,

$$a_1 = 300, \qquad a_2 = 400;$$

the true value of the root $\sqrt{a_1^2 + a_2^2}$ is 500. We see what error ω there would have place to fear by operating by aid of the instrument. One would have

$$\omega = \frac{2.700 + 120000 \times 0.0043}{500} = \frac{1916}{500} < 4.$$

Therefore, under this hypothesis the error would be only on the simple units. Now, one is able to be assured directly by operating with the instrument, that, in the example cited, the error will be completely insensible, and will be equivalent only to a fraction nearly inappreciable from the admitted unit.

By considering the results of the tests to which I have submitted my *summing square*,—a first exemplar which, by that itself, is not able to claim to perfection,—I do not doubt that an able engineer, by increasing a little the dimensions of this instrument, not arrive to give to it a high degree of precision. Then it will be able to serve not only by controlling some direct calculations already done, but yet it will be able to be employed to execute the most painful part of these calculations, at least when the coefficients of the elements in the equations of condition will not pass beyond a certain limit.