EXTRACT FROM CHAPTER V
OF THE
THÉORIE ANALYTIQUE DES PROBABILITÉS: APPLICATION DU CALCUL DES PROBABILITÉS A LA RECHERCHE DES PHÉNOMÈNES ET DE LEURS CAUSES
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Finally one would be able to make use of the Calculus of Probabilities in order to rectify curves or to square their surfaces. Without doubt, the geometers will not employ this means; but, as it gives me place to speak of a particular kind of combinations of chance, I will expose it in a few words.

We imagine a plane divided by parallel lines, equidistant by the quantity a; we conceive moreover a very narrow cylinder, of which $2r$ is the length, supposed equal or less than a. One requires the probability that in casting it on it, it will encounter one of the divisions of the plane.

We erect on any point of one of these divisions a perpendicular extended to the following division. We suppose that the center of the cylinder be on this perpendicular and at the height y above the first of these two divisions. In making the cylinder rotate about its center and naming by ϕ the angle that the cylinder makes with the perpendicular, at the moment where it encounters this division, 2ϕ will be the part of the circumference described by each extremity of the cylinder, in which it encounters the division; the sum of all these parts will be therefore $4\int \phi dy$, or $4\phi y - 4\int y d\phi$; now one has $y = r\cos\phi$; this sum is therefore

$$4\phi y - 4r\sin\,\phi + \text{const.}$$

In order to determine this constant, we will observe that the integral must be extended from y nul to $y = r$, and consequently from $\phi = \frac{\pi}{2}$ to $\phi = 0$, this which gives

$$\text{const.} = 4r;$$

thus the sum of which there is concern is $4r$. From $y = a - r$ to $y = a$, the cylinder is able to encounter the following division, and it is clear that the sum of all the parts relative to this encounter is again $4r$; $8r$ is therefore the sum of all the parts relative to the encounter of one or of the other of the divisions by the cylinder, in the movement of its center the length of the perpendicular. But the number of all the arcs that it describes in rotating in entirety with respect to itself, at each point of this perpendicular, is $2a\pi$; this is the number of all the possible combinations; the probability of the encounter of one of the divisions of the plane by the cylinder is therefore $\frac{4r}{a\pi}$. If one casts this cylinder a great number of times, the ratio of the number of times where the cylinder will encounter one of the divisions of the

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plane to the total number of casts will be, by n° 16, very nearly, the value of $\frac{4r}{a\pi}$, that which will make known the value of the circumference 2π. One will have, by the same section, the probability that the error of this value will be contained within some given limits, and it is easy to see that the ratio $\frac{8r}{a\pi}$ which, for a given number of projections, renders the least error to fear, is unity, this which gives the length of the cylinder equal to the interval of the divisions, multiplied by the ratio of the circumference to four diameters.

We imagine now the preceding plane divided again by some lines perpendicular to the preceding, and equidistant by a quantity b equal or greater than the length $2r$ of the cylinder. All these lines will form with the first a sequence of rectangles of which b will be the length and a the height. We will consider one of these rectangles; we suppose that in its interior one draws at the distance r from each side some lines which are parallel to them. They will form first an interior rectangle, of which $b - 2r$ will be the length, and $a - 2r$ the height; next two small rectangles, of which r will be the height, and $b - 2r$ the length; then two other small rectangles of which r will be the length and $a - 2r$ the height; finally, four small squares of which the sides will be equal to r.

As long as the center of the cylinder will be placed in the interior rectangle, the cylinder, in rotating on its center, will never encounter the sides of the large rectangle.

When the center of the cylinder will be placed in the interior of one of the rectangles of which r is the height and $b - 2r$ the length, it is easy to see, by that which precedes, that the product of $8r$ by the length $b - 2r$ will be the number of corresponding combinations, in which the cylinder will encounter one or the other of the sides b of the great rectangle. Thus $8r(b - 2r)$ will be the total number of combinations corresponding to the cases in which, the center of the cylinder being placed in one or the other of these small rectangle, the cylinder encounters the outline of the great rectangle. By the same reason, $8r(a - 2r)$ will be the total number of combinations in which, the center of the cylinder being placed in the interior of the small rectangles of which r and $a - 2r$ are the dimensions, the cylinder encounters the outline of the great rectangle.

There now remains for us to consider the four small squares. Let ABCD be one of them. From the angle A common to this square and to the great rectangle, as center, and from the radius r, we describe a quarter circumference terminating itself at the points B and D. As long as the center of the cylinder will be comprehended within the quarter circle formed by this arc, the cylinder, in turning, will encounter the outline of the rectangle in all its positions; the number of combinations in which this will take place is therefore equal to the product of 2π by the area of the quarter circle, and consequently it is equal to $\frac{\pi^2 r^2}{2}$. If the center of the cylinder is in the part of the square which is outside of the quarter circle, the cylinder, in turning around its center, will be able to encounter one or the other of the two sides AB and AD extended, without ever encountering both at the same time. In order to determine the number of combinations relative to this encounter, I conceive on any point of side AB, distant by x from point A, a perpendicular y of which the extremity is beyond the quarter circle. I place the center of the cylinder on this extremity, from which I let down four straight lines equal to r, and of which two descend onto the side AB extended, if that is necessary, and two others onto the side AD similarly prolonged. I name 2ϕ the angle comprehended between the first two lines, and $2\phi'$ the angle contained between the second two. It is clear that the cylinder, in turning on its center, will encounter the side AB extended as often as one of its halves will be within the angle 2ϕ, and that it will encounter the side AD extended as often as one of its halves will be within the angle $2\phi'$; the total number of all combinations in which the cylinder will encounter one or the other of these sides is therefore $4(\phi + \phi')$; thus this number, comparatively to the part of the square exterior to the quarter circle, is

$$4\int(\phi+\phi')dx\ dy;$$

now one has evidently

$$x = r\cos\ \phi', \qquad y = r\cos\ \phi;$$

the preceding integral becomes thus

$$4r^2\iint(\phi+\phi')d\phi\,d\phi'\sin\phi\sin\phi',$$

and it is easy to see that the integral relative to ϕ' must be taken from $\phi' = 0$ to $\phi' = \frac{\pi}{2} - \phi$, and that the integral relative to ϕ must be taken from $\phi = 0$ to $\phi = \frac{\pi}{2}$, that which gives $\frac{1}{2}r^2(12-\pi^2)$ for this integral. In adding to it $\frac{\pi^2 r^2}{2}$, one will have the number of combinations relative to the square, and in quadrupling this number and joining it to the preceding numbers of combinations relative to the encounter of the outline of the great rectangle by the cylinder, one will have, for the total number of combinations,

$$8(a+b)r - 8r^2.$$

But the total number of possible combinations is evidently equal to 2π multiplied by the area ab of the great rectangle; the probability of the encounter of the divisions of the plane by the cylinder is therefore

$$\frac{4(a+b)r - 4r^2}{ab\pi}.$$