Bemerkungen zu einer Aufgabe der Wahrscheinlichkeitsrechnung*

By Mr. *Dedekind* Ph. D. at Göttingen†

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In one of the earlier issues of the *Philosophical Magazine* for 1854 G. *Boole* has given a resolution of a probability problem attacked by C. *Cayley*. Although without doubt, that the mistake contained in this attack is already recognized also by others, thus nevertheless the following remarks come perhaps not unwanted to those, who are interested in this beautiful theory.

The problem reads: Given the probability α, that a cause A (which a certain event can produce) comes with the effect, and the probability p, that, if A acts, the event enters; likewise the probability β, that a cause B arrives with the effect, and the probability q, that, if B acts, the event enters: the probability u is sought of the event, under the assumption, that the same can be produced from no other cause than of A and B.

Cayley solves the problem in the following way: Let λ be the probability, that, if A acts, the event is produced also by A; μ the probability, that, if B acts, the event is also produced by B; then is

$$p = \lambda + (1 - \lambda)\mu\beta; \qquad q = \mu + (1 - \mu)\lambda\alpha.$$

From this λ, μ are determined; and the sought probability is

$$u = \lambda\alpha + \mu\beta - \lambda\mu\alpha\beta.$$

Now when Boole found this resolution with more specializations proven as correct, he seeks to prove, that it leads in the case $p = 1$, $q = 0$ to a wrong result. He says: It is plausible, that the probability of the event must be in this case $= \alpha$. Because if the cause A *always* bring forth the events, the cause B *never*, and the entering of the event can be attributed to no other cause, so must the probability "of the event be equal to that of the entering the cause A." Since to this theorem naturally nothing can be objected, and now the resolution, as Cayley represents it, gives in this case either $u = 1$, or $u = \alpha(1 - \beta)$, so *Boole* concludes, that the whole resolution must be incorrect, and gives the final formula of his own resolution, with addition of special restrictions, from which however for this case the desired result $u = \alpha$ can be derived.

*Remarks on a problem in the calculation of a probability.

†Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. August 8, 2010

Meanwhile one does not see at all, where *Cayley* has made an error; and indeed his resolution is also (up to certain restrictions, by which it must be done first *unequivocally*) strictly correct, even in the just now stated case; because one finds easily that $\alpha(1-\beta)$ agrees with α, as α can be nothing other than zero. If the possibility of the entering of the cause A would be openly permitted, i.e. α would not be zero, then also the probability q of the event (under the assumption of the entering the cause B) could not possibly disappear completely, p might be yet so small, only not zero (in this case however $p = 1$ was assumed). The problem posed is therefore *absurd*, when $q = 0$, α and p on the other hand both are accepted differently from zero. This results also as a result of a look at the equations of *Cayley*. If one notes namely, that μ, $(1-\mu)$ λ, α, *according to the nature of their meaning*, cannot be *negative*, then it follows from the *one* equation $q = 0$, as well as $\mu = 0$, as also $\lambda\alpha = 0$, and the other equation changes into $p = \lambda$. Now if p is different from null (it is not necessarily that p be exactly $= 1$), then also must be $\alpha = 0$; and the sought probability u must be always $= 0$; if q or p may be, or both may be $= 0$; as one may not expect it differently.

If now however this reproach also does not meet the above resolution, then it is yet nevertheless at least *incompletely* named, since the conditions are not stated, under which the problem has indeed a real sense, and since furthermore to decide remains extra, one has to choose which of the two values of u, which suffice the above equations. This shall happen here.

One proceeds with the most symmetry, if one eliminates μ from the equations for q and u, and in such a way λ from the equations for p and u. This gives

$$u - \beta q = (1-\beta)\lambda\alpha; \quad u - \alpha p = (1-\alpha)\mu\beta, \tag{1}$$

and if one substitutes these values of $\lambda\alpha$, $\mu\beta$ into the equation for u, then one gets a quadratic equation, through its resolution yields

$$u = \frac{1}{2}(1 - \alpha\beta + \alpha p + \beta q - \rho), \tag{2}$$

where ρ is the *yet ambiguous* square root out of

$$\rho\rho = \begin{cases} (1-\alpha\beta+\alpha p+\beta q)^2 - 4(1-\beta)\alpha p, \\ -4(1-\alpha)\beta q - 4\alpha p.\beta q \end{cases}$$

So that however the problem is solvable, it is *necessary*: first, that ρ be *real*, and further, that u (as a probability) is a *positive proper fraction*. But also this is not yet *sufficient*; and therein actually lies the principal interest of the entire problem. It would still remain without sense, if the auxilliary probabilities λ, μ were not contained likewise between the limits 0 and 1, and it is clear, that with these *last* conditions also the *first* must be fulfilled at the same time. The point is then, to express the conditions, that λ, μ do not lie beyond the limits named. This is easy, since one receives the values of λ, μ from equations (1), if one substituted in them for u the found expression in (2). By this investigation one gets to the following equations:

$$\begin{aligned}
\rho\rho &= (1-2\alpha+\alpha\beta+\alpha p-\beta q)^2+4\alpha(1-\alpha)(1-\beta)(1-p) \\
&= (1-2\beta+\alpha\beta-\alpha p+\beta q)^2+4\beta(1-\beta)(1-\alpha)(1-q) \\
&= (1-\alpha\beta+\alpha p-\beta q)^2-4\alpha(1-\beta)(p-\beta q) \\
&= (1-\alpha\beta-\alpha p+\beta q)^2-4\beta(1-\alpha)(q-\alpha p).
\end{aligned}$$

From the two first forms for $\rho\rho$ it follows, that it requires no special condition for the reality of ρ. But if one sets the forms in connection with the requirements for λ, μ, then it results, that in the expression (2) for u the *positive* square root for ρ must always be taken. If one compares finally the two last forms for $\rho\rho$ with the requirements for λ and μ, then one obtains, as the only *requirement* necessarily, but also completely *sufficient* conditions, that the two differences

$$p-\beta q \quad \text{and} \quad q-\alpha p \tag{3}$$

are *not* allowed to be *negative*.

If one therefore, in order to give an example of the problem, has assumed for α, β, p, q four arbitrary numbers within the limits 0 and 1, then one must examine first, whether the two conditions in (3) satisfy. Observed in passing, one can with such an arbitrary choice just as often meet an absurd example as a suitable one; because the value of the quadruple integral $\iiiint d\alpha\, d\beta\, dp\, dq$ is =1, if one expands the integrations over *all* values of the variable between 0 and 1; on the other hand $= \frac{1}{2}$, if one excludes those values, which the conditions (3) do not satisfy. One can also convince oneself easily of this through *geometrical* examinations.

In the case $q = 0$ examined by Boole the conditions (3) reduce to $\alpha p = 0$; then becomes $\rho = 1 - \alpha\beta$, and therefore $u - 0$; completely in agreement with the above results. Also the case $\alpha = 0$ is of interest. Then it is $\rho = 1 - \beta q$, and therefore $u = \beta q$ the correct result obviously. Here u naturally is independent of q, and nevertheless the condition $p - \beta q \geq 0$ remains in full force, and although for the determination of u out of the value of p no weight falls at all, then it would be nevertheless absurd to assume the probability p of the event *under the assumption*, that the cause A reaches to the effect, smaller than the probability βq, even if this assumption is *actually forbidden* by the requirement $\alpha = 0$. And with this remark, that, as I believe, is suitable for it, to throw a striking light on the peculiarity of this kind of problem, will I break off my contemplations.

Göttingen, 22 July 1854.