Recherches sur une question de l'analyse des probabilités, relative à une série d'épreuves à chances variables, et qui exige la détermination du terme principal du développement d'une factorielle, formée d'un grand nombre de facteurs (Extract)*

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"D'Alembert has treated many questions of probabilities in the *Dictionnaire mathématique de l'Encyclopédie*: one of the more simple is thus enunciated: "Pierre holds eight cards in his hands, which are one ace, one deuce, one three, on four, one five, one six, one seven and one eight, which he has shuffled; Paul wagers that drawing them one after the other, he will guess them in measure as he will draw them: one demands how much must Pierre wager, against one, that Paul will not succeed in his enterprise?" D'Alembert calculates the expectation of Paul by the fraction

$$\frac{1}{8}\cdot\frac{1}{7}\cdot\frac{1}{6}\cdot\frac{1}{5}\cdot\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{40320};$$

he forms in consequence the exact ratio of the stakes in the game in the open wager. He says next: "If Paul wagered to bring forth or to divine justly in one of the seven coups only, his expectation would be

$$\frac{1}{8}+\frac{1}{7}+\frac{1}{6}+\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2};$$

and consequently the stake of Pierre to the one of Paul, as

$$\frac{1}{8}+\frac{1}{7}+\frac{1}{6}+\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}\quad\text{to}\quad 1-\frac{1}{8}-\frac{1}{7}-\frac{1}{6}-\frac{1}{5}-\frac{1}{4}-\frac{1}{3}-\frac{1}{2}.$$

Two other analogous questions lead, as this one, d'Alembert to some manifestly inexact solutions (article *Cartes* of the Dictionary cited): the sum $\frac{1}{8}+\frac{1}{7}+\frac{1}{6}+\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. December 31, 2009 is a number greater than unity; it cannot be the measure of a probability. This single remark would have cautioned d'Alembert of his oversight and had restored it to the exact solution of his questions; because they suppose only the most common principles of the doctrine of chances; he had found for response to the second question the fraction

$$\frac{1}{8}\left(\frac{1}{7}+\frac{1}{6}+\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1\right)=0.299107\ldots;$$

this is the chance of Paul to divine a single point out of the seven drawings, among the eight cards. The other cases of this problem are resolved also most easily.

"The problem treated by d'Alembert presents some variable aleatory chances of one event to another, and under this ratio of numerous questions of probabilities can be related to the methods which will serve to resolve them. It offers, for example, the same chances as the following question: one has contained in an urn two equal balls, one white, the other black, finally to operate the drawing in the lot of one ball in the urn; one restored after the drawing the two balls accompanied by a second black ball, and one extracts from the urn, at random, a single ball among the three which it contains; for a third drawing to operate in the urn, one will restore the three preceding balls by adjoining to them a new black; one will continue thus to augment the number of blacks for each new drawing to execute in the urn, which will contain constantly only a single white. This posed, one demands the probability that, out of n successive drawings, one will not have brought forth the white ball a single time; one demands also the probability to extract one time only the white; or else to extract it two times, or three times, etc., in the n drawings; or finally to bring forth only the white in all the drawings. In this problem, the variable chances of a coup to the other decreases gradually in the place to increase as in the first problem, but it is evident that the results are the same.

"In the theory of probabilities, as in that of whole numbers, it arrives sometimes that the simplest problems give opening to some questions which offer to the analysts according to difficulties, and which require, in order to be treated, methods totally different than those to which one could attend in approaching them. The respective chances of the two players being supposed constants at each trial or at each coup, Jakob Bernoulli proposed to determine the character of the highest probability which must present itself in a long series of repeated trials: from this question is resulted a fundamental theorem in the doctrine of chances; but it is no less necessary that the competition of the successive efforts of Moivre and of Laplace in order to arrive to the complete expression of the elements, and of the consequences which are naturally outputs of this theorem. Poisson has yet added to these good researches some important results which extend to the cases where the variable chances from one trial to another, under some conditions.

"Having designated by n the number of trials of the second enunciation that we have given above, the diverse chances favorable to one of the two adversaries are furnished by the development, according to the powers of the letter x, of the algebraic function

$$\frac{(1+x)(2+x)(3+x)\cdots(n+x)}{2\cdot 3\cdot 4\cdots(n+1)}=X,$$

so that this development being represented by

$$X = P + P_1 x + P_2 x^2 + P_3 x^3 + \cdots + P_n x^n,$$

the first coefficient P is the probability to bring forth no white ball in the n drawings; P_1 is the probability to bring forth one white ball alone in these same drawings; P_2 is the probability to bring forth two whites, and thus of the other coefficients $P_3,\ P_4$, etc., to P_n which expresses the probability to bring forth n whites in the n drawings. We have already remarked that at each coup or at each trial, the chances of the two adversaries, in this game, are continually variable, instead of being constants, as in the question of Bernoulli; a research analogous to the first proposition of Bernoulli presents itself nevertheless: one demonstrates in effect, rather promptly that there exists a certain probability P_h which surpasses all the others, in order that a number give n trials; but it has been more difficult to assign the rank and the value of this greater probability, when the number of trials become considerable. According to the form of the polynomial X, this question returned evidently to determine in a factorial of order n

$$(1+x)(2+x)(3+x)\cdots(n+x),$$

transformed into a polynomial

$$A + A_1 x + A_2 x^2 + \text{etc.} + A_n x^n,$$

which is the greatest of all whole numbers

$$A,\ A_1,\ A_2, \ldots, A_n\text{:}$$

this greater coefficient is that which I name the principal term. These sorts of factorials being presented in a multitude of researches, have much occupied the analysts. Their consideration has lead Lagrange to the first demonstration of the theorem of Wilson, on the first numbers, and for this object he had to establish the mode of derivation of the coefficients of the ones from the other. But these relations can not lead to the end as I come to indicate. It was necessary to me, in order to arrive there, to abandon the resources of simple algebra, and to make intervene some methods which were not familiar to analysts in the times of d'Alembert, and which have been perfected by some recent labors.

"When the question of a factorial proceeding from some factors only, the calculus immediately gives its coefficients, and one has

$$\begin{aligned}
(1+x)(2+x) &= 2 + 3x + x^2,\\
(1+x)(2+x)(3+x) &= 6 + 11x + 6x^2 + x^3,\\
(1+x)(2+x)\cdots(4+x) &= 24 + 50x + 35x^2 + 10x^3 + x^4,\\
(1+x)(2+x)\cdots(5+x) &= 120 + 274x + 225x^2 + 85x^3 + 15x^4 + x^5,\\
&\\
(1+x)(2+x)\cdots(9+x) &= 362880 + 1026576x + 1172700x^2 +\\
&\quad + 723680x^3 + 269325x^4 + 63273x^5 + 9450x^6 + 870x^7 + 45x^8 + x^9.
\end{aligned}$$

These examples show that the principal term is A, in the first factorials, and that it becomes A_2 for the ninth and likewise from the seventh, thus one can see it in the treatises of Stirling and of Kramp. But this calculation is impractical for a great number of factors, and makes nothing known on the sought law. Here is what the solution consists to which I am arrived, by supposing that n is a number a little great or very considerable. Let there be

$$L = \log(n+1) - \alpha - \frac{1}{2(n+1)} - \frac{1}{4(n+1)(n+2)} - \frac{1}{18(n+1)(n+2)(n+3)} - \text{etc.};$$

α is a constant equal to $0.4227842\ldots$, and the logarithm of $(n+1)$ is hyperbolic: one will take, for the index h, of A_h, the whole number immediately superior to L; now, one can prove in the Memoir, 1° that the series of whole numbers $A < A_1 < A_2 < \cdots < A_{h-2}$ will always be ascending; 2° that it will be able, in certain cases, to remain increasing to A_{h-1}, or even to A_h, but it will be necessarily decreasing from $A_h > A_{h+1} > A_{h+2} > \cdots$ to A_n: the origin of the coefficient α, thus as of the other coefficients of the sequence, will be explicated in the Memoir.

"If one applies this formula to the case of $n = 9$, one will have

$$L = \log(10) - \left[0.4228 + \frac{1}{20} + \frac{1}{440} + \text{etc.}\right]:$$

now $\log(10) = 2.30258$; there results from it $L = 1.807\ldots$ The whole number h consecutive to L being 2, the term A_h is, in this example, A_2; this is, in effect, the one to depart from which $A_3,\ A_4, \ldots, A_9$ cease not to decrease in the factorial $\frac{\Gamma(x+10)}{\Gamma(x)}$ which we have reported.

"Thus, when one has formed the product of the natural numbers $1,\ 2,\ 3, \ldots, n$, and that this product is $A = 1.2.3\ldots n$; that one has formed the sum A_1 of the products $n-1$ to $n-1$ of these numbers; the sum of the products A_2 of these same wholes, taken $n-2$ by $n-2$, and thus of the other sums $A_3,\ A_4$, etc.; the greatest of all these sums, or the principal term is found corresponding to an index h, or $h-1$, or $h-2$, h being nearly the whole number superior to $\log(n)$ when n is a very great number.

"In the question of probabilities proposed above, the greatest of the chances on a number n of trials corresponds thus to one of the three indices $h-2$, $h-1$, or h, the number h being determined by the formula that we come to report; and when this number of trials is very great, h is nearly equal to $\log(n)$, that is to say that the mean chance to choose will be that to extract the white ball in the n drawings, a number of times marked by the integer superior to $\log(n) - \alpha$, or by the one of the two inferior integers: then these three chances differ little the one from the other.

"One sees now at what point the result to which we arrive seems to extend from that which the first proposition of Bernoulli teaches: this proposition proves that the index of the highest probability, in the very numerous repetitions of the trials by constant chances, is proportional to the great number n of repetitions of the trial; in our problem, where the chances vary without ceasing from one trial to the other, the index of the high probability is fournished by the hyperbolic logarithm of the number of repetitions, instead of being proportional to that number shared according to the constant ratio of the probabilities of the simple trial.

“Poisson has treated in a very general manner the probabilities which result from the indefinite repetition of the trials by variable chances, finally to extend the theorems of Bernoulli, or rather to find the rule which must be substituted for them, when the chances opposed of two events are no longer constants at each trial: under this relation, the question of which I just occupied myself will seem to return into the category of those that Poisson had in view; but one will remark easily, by following the quite delicate analysis of the illustrious geometer, that it supposes that the variable probability of one of the events in the successive trials does not decrease without ceasing, in a manner to nullify itself if the number of trials is infinite; the author excepts formally this case which is precisely the one of the problem that I have wished to resolve. One will remark, further, that all the analysis of Poisson rests expressedly on the assumption of the immense number of repetitions, that he names μ, and that he intends to treat as infinite: in my speculations, the number n is supposed of a certain size, of 8 or 10, units for example, and can be next indefinitely increased. These circumstances of the problem have required an analysis which differs essentially from that of Poisson; but I myself am assured that in the case of the infinite number of repetitions, or at least in the case of n immense, the proportion enunciated in article 95 of the work *Sur la probabilité des jugements en matière criminelle*, can furnish the same result as my formula, so that this proposition arrives to a case which seems beyond the conditions of his demonstration.”