SUR
LES SUITES OU SÉQUENCES
DANS
LA LOTTERIE DE GENES
SECOND MÉMOIRE[*]

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§. 1.

I have said in my first Memoir on this subject, that it was indifferent in following my method to commence with the Solution of the general Problem, or with that of the particular cases. This is what I propose myself to prove today by the investigation of the two general formulas for all the species of sequences whatever, taken in the two different senses under which Messrs. Euler & Bernoulli have regarded them. But that which I myself propose principally here, & that which I believe more important than the discovery of these formulas, is to show the utility of the metaphysical principles in the employ of the calculations of the algebra, & to give a sample of the speculative philosophy applied to the analysis.

§ 2. What does one seek when one wishes to attain by algebra a general formula? It is to find a rule which is able to serve to decide all the particular cases which it must contain; just as in Jurisprudence a general law on the matter of succession, for example, must be applied to all the cases of succession which present themselves everyday for determination in a State.

The laws are formed, either by induction from particular cases, what one names in civil law the jurisprudence of the arrests, or else the legislature deduces these laws from the same nature of the general object which they concern; one finds in Roman Law some frequent examples of both species of these laws. All the diverse branches of the sciences offer us equally the two kinds; but nearly in all the first specie is that which one encounters most often. Medicine, Astronomy, Physics, Chemistry, Botany, have become sciences only by dint of comparing some individual observations, & by forming some more or less general rules from them, in proportion to the number more

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or less great of the particular cases to which these laws are applied. Speculative philosophy, & Geometry have in common the double advantage of rising from individual cases to general laws, & to see in the general notions, the laws which agree to each individual contained under these notions.

§ 3. Let it be that one rises from particular to general, or that one descends from general to particular; the passage from one to the other is based on the principle of sufficient reason, & by analogy which is only a repeated application of this principle. Why do we conclude from the particular to the general? If our conclusion is legitimate, it is only because the same reason which decides the state of an individual case, decides equally the state of all the others. It is only as the diverse particular cases are determined by one same principle, that we can arrange them under one same general rule.

Why do we conclude from general to the particular? It is because we see that the reason for the general determination must be common to all the individuals. It is only as far as the individual subjects have, in spite of their diversity, some common quality to all, which they are able to be submit to one same & single law.

§ 4. We apply this theory to the sequences. What have they in common? It is by being sequences of 2 or of many consecutive numbers. In what can they differ? It is by being of a single or of many species at once, for example, either composed of only *quinaries*, or composed at once of *quinaries*, & of *ternaries*, & of such other specie as one will wish. The first specie contains therefore all the *homogeneous sequences*, & the second all the *mixed sequences*. Each of these two species can yet be either *simple*, or *multiple*; that is repeated many times: a *simple sequence*, will be for example a *quinary*, or a *binary quinary*; a multiple sequence, will be for example a *triple quinary*, or a *double quinary* accompanied by a *binary triple*.

§ 5. All the possible sequences are evidently comprised under one of these four classes. But, in order to prescribe a unique rule for them, it is necessary to reduce these four classes to a single one which contains all of them; without that it will require as many different rules, as there will be irreducible classes.

Now we could not arrange the multiple sequences, in the class of simple sequences, because the simple would not contain that which is not in it. But we can very well arrange the simple sequences under the class of the multiples. Because the multiplicity is only a simplicity or unity taken a certain number of times; & since this number is not determined, it can as well be 1, that which is the case of simple sequences, as 2, 3, 4, &c. that which is the case of multiple sequences. Here is therefore our four classes reduced to two; namely to the *multiple homogeneous* sequences, & to the *multiple mixed* sequences. Now the *mixed* could not be contained under the *homogeneous*, because that which is homogeneous would not be *mixed*. But the mixed can very well represent the homogeneous, because the *mixture* can as well be the blending of homogeneous sequences, as that of heterogeneous sequences. If therefore we find the formula of the multiple mixed sequences, we will have the general law of all the species of sequences.

§ 6. Now the number of cases of any sequence whatever depends on the number of combinations of the files by the complete ranks, & on the different transpositions of which each combination is susceptible; & it could depend only on these two elements. Thus, if we name the number of combinations C, & the number of diverse transpositions that each combination is able to have T, the general formula of the cases of these sequences will be $= T \times C$.

§ 7. But what are the elements which determine the number of combinations? These are the quantities of ranks, & those of the files to combine. Naming therefore the number of ranks R, & that of the bundles to combine F;

$$\frac{(R)(R+1)\cdots(R+F-1)}{1.2\cdots F},$$

will express more precisely, that which C expressed in a manner more vaguely; & substituting the more precise expression into the other, our formula becomes

$$= T \times \frac{(R)(R+1)\cdots(R+F-1)}{1.2\cdots F}.$$

§ 8. Each of these two elements, R, F, has again its integral parts which constitute its total; & it is here where the formula which until the present concerned equally the sequences of Mr. Euler, & those of Mr. Bernoulli, must be separated into two different forms, because the number of ranks differs in these two ways to evaluate the cases of the same sequences. I will determine first the formula in the sense of Mr. Euler.

The number of ranks R, is here equal to the length of the last file, as one has seen in my first Memoir. Now this last file is equal to the length of the first n, less the shortening which results from the arrangement of the following files set in sequence, or out of sequence, that is by the number of levels E, by which each file surpasses its preceding: one can therefore substitute into R, which marked the complete ranks, the expression $n - E$, & the formula becomes

$$= T \frac{(n-E)(n-E+1)\cdots(n-E+F-1)}{1.2\cdots F}.$$

§ 9. Here we came to an element n, which is more susceptible to decomposition, because seeing that it marks the total number of numerals in the lottery, or the total length of each file, one could not determine it more precisely without rendering it an individual number, & without restricting consequently the formula to serve as a rule only for a lottery of a determined number of numerals.

§ 10. The other element E, which designates the number of levels by which each file surpasses its previous, is determined by the number of sequences joined to one of the isolated files; the files in sequences cannot be raised on one another, as of a single level; the isolated files must be raised by two, in order to be out of sequence; & similarly the bundles of diverse sequences are raised by two levels, without which they would form only a single more numerous sequence. Therefore, naming t the number of files, it is clear that if they were all isolated out of sequence, E would be the double of $t-1$. Because the first file has no elevation. But, since there are some sequences, it is necessary to subtract for each sequence as many levels as the sequence contains of files, less the first, which exceeds by two notches the previous bundle. Now the mixed sequences of which we seek here the formula, can contain all the species of sequences, from the most numerous that I will call S, to the least numerous which is the binary 2. Thus the mixed sequence is represented by the descending arithmetic progression

$$S + S - 1 + S - 2 + \cdots + 2.$$

This is not all, our formula must contain the multiples of each of these sequences, because each can be repeated more or less times, we will name therefore M, the number which expresses the repetition of a sequence, by us remembering that this number can be the same for each sequence, & that it can also vary from one sequence to another by more & by less, by means of which the expression of the mixed multiple sequences will be

$$SM + (S-1)M + (S-2)M + \cdots + 2M,$$

& consequently the more determined expression, to substitute into E, will be

$$= 2t - 2 - (S-1)M - (S-2)M \cdots - M,$$

by this substitution the expression of the complete ranks becomes

$$R = n - 2t + 2 + (S-1)M + (S-2)M \cdots + M,$$

And as it is reduced to four elements n, t, S, M, which can no longer receive further decomposition, without substituting the individual numbers which they represent, it has all the precision of which it is susceptible. But, for brevity, putting $n - 2t + 2 = N$, & the arithmetic progression $S - 1 + S - 2 + S - 3 + \cdots 1 = P$, one will have $R = N + PM$, & consequently the sought formula becomes

$$\frac{T(N+PM)(N+PM+1)(\cdots)(N+PM+F-1)}{1.2\cdots(F-1)}$$

.

§ 11. We seek now, in order to complete the article of the combinations, to make for us a more precise idea of the expression F, which designates the number of bundles to combine. Now this number is composed of the one of the multiple mixed sequences, & of the one of the isolated files which do not form a sequence. But we just showed that the number of multiplied sequences is expressed by M, & if we name the one of the isolated files I, we will have for F, the less vague expression $M + I$.

§ 12. The term I is susceptible yet to a greater precision, because there will always be as many isolated files, as there are files in all, less the files employed to form some sequences, the number of these last is expressed by the progression $SM + (S-1)M + \cdots + 2M$, or as we just shortened, to $(P+1)M$, & the total number of files being t, one can substitute into the expression I this latter $t - (P+1)M$, & then one has $F = t - PM$, of which all the elements are as simple & as determined as they can be. Thus our formula which at first was $T \times C$, is presently

$$T\frac{(N+PM)(N+PM+1)(\cdots)(N+t-1)}{1.2\cdots(t-PM)},$$

in which there remains no more, than to estimate more clearly the transpositions T.

§ 13. The rules of transpositions are known & easy to find; *one* thing has only one position, *two* things can be transposed in *two* ways; *three* things in 2 times *three* ways, *four* things in 2 times 3 times *four* ways, & so forth. If among the things to transpose there are two of them similar, their transposition would give no different result; it is necessary therefore to divide the product by 2; if there are 3 of them similar, the product will be divided by 2 times *three* &c.

This supposed known, what will be the transposition of each rank composed of multiple mixed sequences, and of isolated files? This will be the product of the natural numbers to the one which expresses the sum of the things to transpose contained in this rank divided by the product of these same natural numbers for each thing which will not be susceptible to transposition. Thus T must be expressed by a fraction, of which we will name the denominator D, & the numerator is $1.2.3(\cdots)F$, & substituting for F its precise expression that we just determined, one will have

$$T = \frac{1.2.3\cdots(t-PM)}{1.2\cdots D}.$$

§ 14. The denominator D represents the things which are not susceptible to transpositions; now these things are 1. the multiples of each particular specie of sequences, which among themselves cannot be transposed, although they transpose themselves very well with the sequences of another specie, & with the isolated files. 2. These same isolated files, which among themselves do not transpose, although they transpose themselves very well with the sequences. One can therefore substitute into D, its precise expression M, for the multiples of the sequences, & $t-(P+1)M$: for the isolated files, & then we will have the formula which we sought in all its precision,

$$\frac{1.2.3(\cdots)(t-PM)}{(1.2\cdots(t-(P+1)M)(1.2.(\cdots)M)}\;\frac{(N+PM)(N+PM+1)(\cdots)(n-t+1)}{1.2.(\cdots)(t-PM)}$$

§ 15. If one wishes to confound the transpositions with the combinations, one would erase in the formula the entire numerator of the transpositions, & the entire denominator of the combinations, & the general rule will become

$$=\frac{(N+PM)(N+PM+1)(\cdots)(n-t+1)}{(1.2\cdots(t-(P+1)M)(1.2\cdots M)}$$

§ 16. But, by conserving for the regularity of the development each element in its nature, one can subtract from the transpositions all the factors which mutually destroy themselves, & then the general formula for any sequences whatever in the sense of Euler is

$$\frac{(t-(P+1)M+1)(t-(P+1)M+2)(\cdots)(t-PM)}{(1.2\cdots M)}\times$$
$$\frac{(N+PM)(N+PM+1)(\cdots)(n-t+1)}{1.2\cdots(t-PM)}.$$

§ 17. It will not be uneasy to transform this formula into the one which will give all the cases of any sequence whatever in the sense of Bernoulli.

The combinations differ, as I have indicated in my first Memoir, only in two items. 1. that in an equal number of files there are always one rank less, with the repetition of the greater term, so that, when the formula of the sequences of Mr. Euler is (§ 7.)

$$T\frac{(R)(R+1)(\cdots)(R+F-1)}{1.2\cdots F},$$

that of Mr. Bernoulli will be in this regard,

$$T\frac{(R-1)(R)(R+1)(\cdots)(R+F-2)}{1.2\cdots F}+T\frac{(R-1)(R)(\cdots)(R+F-3)}{1.2\cdots(F-1)}$$

which reduced to a single denominator is,

$$T\frac{(R+2F-2)(R-1)(R)(R+1)(\cdots)(R+F-3)}{1.2\cdots F}$$

& as the transpositions are here the same, there is nothing to change in the found expression for T.

§ 18. 2. The second item in which the two methods differ, is that in the one of Mr. Bernoulli, all the times that the sequences occupy the shortest files, where consequently are found the last numerals, the number of these sequences is increased by *one* for the *binaries*, by 2 for the *ternaries*, & in general by $S-1$, for whatever sequence S, & consequently also of P, for all the mixed sequences whatever, this which gives as many combinations to add, of $R-1$ ranks, by $F-1$ bundles: this addition to the formula will be therefore

$$+T'\times P\frac{(R-1)(R)(R+1)(\cdots)(R+F-3)}{1.2\cdots(F-1)}$$

§ 19. There remains to determine the value of the transposition T'. One sees as well it is not the same as we have already estimated; but it is easy to perceive by what it differs. The sequence that we combine here is fixed in this last place, consequently it is not transposed; there is therefore one thing at least to transpose, that which subtracts the greatest factor from the numerator of T: by the same reason it is necessary to subtract from the denominator, the factor M which is the multiple of this sequence; thus the transposition, by letting the old value of T be substituted, must be $\frac{TM}{F}$, & the addition to make will be

$$\frac{T\times PM}{F}\cdot\frac{(R-1)(R)(R+1)(\cdots)(R+F-3)}{1.2\cdots(F-1)}$$

this which joined really, & setting under a like denominator, gives the general formula,

$$\frac{T\times(R+PM+2F-2)(R-1)(R)(R+1)(\cdots)(R+F-3)}{1.2\cdots F}$$

& substituting into T, R, & F, their more correct expressions which we have determined (§ 10. 12. 16) this formula becomes

$$\frac{(t-(P+1)M+1)(t-(P+1)M+2)(\cdots)(t-PM)}{(1.2\cdots M)}\times$$
$$\frac{(n)(N+PM-1)(N+PM)(\cdots)(n-t-1)}{1.2.3\cdots(t-PM)}$$

Now, if we wish to confound the transpositions with the combinations, it will be in this more concise formula,

$$\frac{n(N+PM-1)(N+PM)(\cdots)(n-t-1)}{(1.2\cdots(t-(P+1)M)(1.2\cdots M)}$$

§ 20. There remains to indicate the application of these formula to some particular cases.

Let the total number of numerals of the lottery be always n, & let in each drawing one extract 7 of them, one asks how many possible cases there will be for each specie of sequences, taken in the sense of Mr. Euler.

The general formula for these sequences (§ 16) becomes, by substituting 7 into t, & $n-12$ into N,

$$\frac{(8-(P+1)M)(9-(P+1)M)(\cdots)(7-PM)}{(1.2\cdots M)}\times$$
$$\frac{(N-12+PM)(n-11+PM)(\cdots)(n-6)}{1.2.3\cdots(7-PM)}$$

I. For the homogeneous *simple* sequences one has $M=1$.

a, for the VII^naries $P=S-1=6,$ this which gives $\frac{1}{1}\times\frac{n-6}{1},$

b, for the VI^naries $P=5$, this which gives $\frac{2}{1}\times\frac{(n-7)}{1}\frac{(n-6)}{2},$

c, for the V^naries $P=4,$ this which gives $\frac{3}{1}\times\frac{(n-8)}{1}\frac{(n-7)}{2}\frac{(n-6)}{3},$

d, for the IV^naries $P=3,$ this which gives $\frac{4}{1}\times\frac{(n-9)}{1}\frac{(n-8)}{2}\frac{(n-7)}{3}\frac{(n-6)}{4},$

e, for the III^naries $P=2$, this which gives $\frac{5}{1}\times\frac{(n-10)}{1}\frac{(n-9)}{2}\frac{(n-8)}{3}\frac{(n-7)}{4}\frac{(n-6)}{5},$

f, for the II^naries $P=1,$ this which gives $\frac{6}{1}\times\frac{(n-11)}{1}\frac{(n-10)}{2}\frac{(n-9)}{3}\frac{(n-8)}{4}\frac{(n-7)}{5}\frac{(n-6)}{6}.$

II. For the *homogeneous multiple sequences*

a, for the *doubles*, one has $M=2,$

The double ternaries; $P=2$, this which gives

$$\frac{2.3}{1.2}\times\frac{(n-8)}{1}\frac{(n-7)}{2}\frac{(n-6)}{3}.$$

The double II^naries; $P=1$, this which gives

$$\frac{4.5}{1.2}\times\frac{(n-10)}{1}\frac{(n-9)}{2}\frac{(n-8)}{3}\frac{(n-7)}{4}\frac{(n-6)}{5}.$$

b. for the *triples*, one has $M=3$.

The triple II^naries, $P=1$, this which gives

$$\frac{2.3.4}{1.2.3}\frac{(n-9)}{1}\frac{(n-8)}{2}\frac{(n-7)}{3}\frac{(n-6)}{4},$$

III. *For the simple mixed sequences*,
one has $M = 1$.

a, for the V^{th} + II^{nd}, one has $\left.\begin{array}{r} P = 4 + 1 = 5 \\ (P+1) = 5 + 2 = 7 \end{array}\right\}$ this which gives,

$\frac{1.2}{1} \times \frac{(n-7)}{1} \frac{(n-6)}{2}$. *b*, for the VI^{th} + III^{rd}, one has $\left.\begin{array}{r} P = 3 + 2 = 5 \\ (P+1) = 4 + 3 = 7 \end{array}\right\}$ this which gives,

$\frac{1.2}{1} \times \frac{(n-7)}{1} \frac{(n-6)}{2}$,*c*, for the IV^{th} + II^{nd}, one has $\left.\begin{array}{r} P = 3 + 1 = 4 \\ (P+1) = 4 + 2 = 6 \end{array}\right\}$ this which gives,

$\frac{2.3}{1} \times \frac{(n-8)}{1} \frac{(n-7)}{2} \frac{(n-6)}{3}$,*d*, for the sequence of III + II, one has $\left.\begin{array}{r} P = 2 + 1 = 3 \\ (P+1) = 3 + 2 = 5 \end{array}\right\}$ this which gives,
$\frac{3.4}{1} \times \frac{(n-9)}{1} \frac{(n-8)}{2} \frac{(n-7)}{3} \frac{(n-6)}{4}$,

IV. For the *multiple mixed sequences*.

a, the doubles, one will have $M = 2$; $M = 1$.
for the sequences III + II + II, one has $P = 2,\ P = 1$,

$$\begin{array}{r} PM = 2 \times 1 + 1 \times 2 = 4 \\ (P+1)M = 3 \times 1 + 2 \times 2 = 7, \end{array} \text{this which gives } \frac{1.2.3}{1.1.2} \times \frac{(n-8)(n-7)(n-6)}{1.2.3},$$

§ 21. Since the development of the formula for the circular sequences will be perfectly similar, it would be quite superfluous to show the application of it here. But that which merits to be observed, & furnishes a singular enough paradox is that the development of these formulas indicates that one can pass entirely by them & that by observing a certain order one can write immediately the numbers of transpositions, & of the combinations of each specie of sequence, thanks to the exact harmony which rules among them.

It is this which I am going to indicate by seeking the sequences in the sense of Mr. Bernoulli for a drawing of 7 numerals.

I. *The simple sequences*.

a, sequences of VII	$\frac{1}{1} \times \frac{n}{1}$
b, sequences of VI	$\frac{2}{1} \times \frac{n(n-8)}{1.2}$
c, a, sequences of V	$\frac{3}{1} \times \frac{n(n-9)(n-8)}{1.2.3}$
d, a, sequences of IV	$\frac{4}{1} \times \frac{n(n-10)(n-9)(n-8)}{1.2.3.4}$
e, a, sequences of III	$\frac{5}{1} \times \frac{n(n-11)(n-10)(n-9)(n-8)}{1.2.3.4.5}$
f, a, sequences of II	$\frac{6}{1} \times \frac{n(n-12)(n-11)(n-10)(n-9)(n-8)}{1.2.3.4.5.6}$

Now all the combinations of mixed sequences are contained in the limits of combinations of simple sequences, & they will go neither to the greatest nor to the binary; they will be precisely equal to the combination of the simple sequence which has the same number of bundles to combine.

Thus the simple sequence of 5 numbers, & the mixed sequences, or multiples of $4{+}2$, of $3{+}3$, of $3{+}2{+}2$, will have precisely the same combination. The combinations of sequences $5+2$; $4+3$, are the same as those of the simple sequence 6, etc.

The transpositions of the multiple mixed sequences are that of the corresponding simple sequence multiplied by those of the superiors as much as there are mixed sequences, or multiples, & divided by the product of the numbers of this multiplicity.

Therefore

II One has the *multiple homogeneous sequences*.

a, sequences of III + III $\quad \frac{2.3}{1.2} \times \frac{(n)(n-9)(n-8)}{1.2.3}$

b, sequences of II + II $\quad \frac{4.5}{1.2} \times \frac{n(n-11)(n-10)(n-9)(n-8)}{1.2.3.4.5}$

c, a, sequences of II + II + II $\quad \frac{2.3.4}{1.2.3} \times \frac{n(n-10)(n-9)(n-8)}{1.2.3}$

Therefore again one has

III One has the *simple mixed sequences*.

a, sequences of V + II $\quad \frac{1.2}{1} \times \frac{(n)(n-8)}{1.2}$

b, sequences of IV + III $\quad \frac{1.2}{1} \times \frac{n(n-8)}{1.2}$

c, a, sequences of IV + II $\quad \frac{2.3}{1} \times \frac{n(n-9)(n-8)}{1.2.3}$

f, a, sequences of III + II $\quad \frac{3.4}{1} \times \frac{n(n-10)(n-9)(n-8)}{1.2.3.4}$

Therefore again one has

IV. the *multiple mixed sequences*.

of III + II + II $\quad \frac{1.2.3}{1.2} \times \frac{n(n-9)(n-8)}{1.2.3}$

§ 22. This affinity of the sequences among themselves, & the manner of deducing them from one another will be yet more sensible, if we had conserved the fraction which expresses the transpositions in its original formula (§ 14.), where the numerator expresses all the factors from 1 to the total number of bundles or of the things to combine, & where the denominator contains all the factors from 1 to the number of isolated files, & to the one of each multiple of the sequences.

We seek, for example, in this manner all the species of sequences in the sense of Mr. Bernoulli, for a drawing of 5 numerals.

I. The simple sequences.

of V. will be $\frac{1}{1} \times \frac{n}{1}$

of IV $\frac{1.2}{1.1} \times \frac{n(n-6)}{1.2}$

of III $\frac{1.2.3}{1.2.1} \times \frac{n(n-7)(n-6)}{1.2.3}$

of II $\frac{1.2.3.4}{1.2.3.1} \times \frac{n(n-8)(n-7)(n-6)}{1.2.3.4}$

II. The multiple homogeneous sequences.

of II + II $\frac{1.2.3}{1.1.2} \times \frac{n(n-7)(n-6)}{1.2.3}$

III. The mixed sequences

of III + II $\frac{1.2}{1.1} \times \frac{n(n-6)}{1.2}$,

§ 23. Finally this same affinity subsists again in the developed sequences for the abridged formulas, (§ 15. 19.) which conserve only the numerator of the combinations, & the denominator of the transpositions. It is this which I just indicated by deducing the cases of all the sequences for the Genoese Lottery; in the sense of Mr. Bernoulli.

I. *a*, The highest *simple sequences* are the quinaries therefore there are $\frac{90}{1}$,

b, Of the quaternaries $\frac{90.84}{1.1}$,

c, the ternaries $\frac{90.83.84}{1.2.1}$,

d, the binaries $\frac{90.82.83.84}{1.2.3.1}$,

II. the binary doubles $\frac{90.83.84}{1.2.1}$,

III. The binary ternes $\frac{90.84}{1.1}$.

As the total number of the combinations of a drawing of 5 numerals is

$$\frac{90.89.88.87.86}{1.2.3.4.5}$$

The probability of the quinary sequences is $=\frac{2.3.4.5}{89.88.87.86}$,

of the quaternaries $=\frac{2.3.4.5.84}{89.88.87.86}$,

of the simple ternaries $=\frac{3.4.5.83.84}{89.88.87.86}$,

of the simple binaries $=\frac{4.5.82.83.84}{89.88.87.86}$,

of the binary doubles $=\frac{3.4.5.83.84}{89.88.87.86}$,

of the binary ternes $=\frac{2.3.4.5.84}{89.88.87.86}$, Now it is the sum of the probabilities of these 6 different species of sequences, that Mr. Bernoulli has calculated under the name of binary sequences. But, if one wishes to take account & to admit some wagers on all the inferior sequences which result from the superiors, as each superior sequence contains so many of the inferiors, as there is of difference in their number, & one beyond, as, for example, the quinary contains 2 quaternaries, 3 ternaries, 4 binaries, one will have

$$\begin{aligned} \text{Probability of the quinary sequences } & \frac{1.120}{89.88.87.86}, \\ \text{of the quaternary sequence } & \frac{2.120 + 1.120.84}{89.88.87.86}, \\ \text{of the ternary sequence } & \frac{3.120 + 3.120.84 + 1.60.83.84}{89.88.87.86}, \\ \text{of the binary sequence } & \frac{4.120 + 6.120.84 + 4.60.83.84 + 1.20.82.83.84}{89.88.87.86}. \end{aligned}$$

Whence one sees that the sum of the single binary sequences is much greater than the sum of the six species of separate sequences.

§ 24. In order to indicate again by a single example the remarkable order which rules in this case of the diverse sequences, & the facility of deducing all of them from one same principle, I am going to develop the case of a drawing of 6 numerals in the sense of Mr. Euler.

I. The *simple sequence*

of 6 has only one choice to combine 1 sequence	$\frac{n-5}{1}$,
of 5, has 2 bundles 1 sequence, 1 isolated file	$\frac{(n-6)(n-5)}{1.1}$,
of 4, has 3 bundles one sequence, 2 isolated files	$\frac{(n-7)(n-6)(n-5)}{1.1.2}$
of 3, has 4 bundles 1 sequence, 3 isolated files	$\frac{(n-8)(n-7)(n-6)(n-5)}{1.1.2.3}$
of 2, has 5 bundles 1 sequence, 4 isolated files	$\frac{(n-9)(n-8)(n-7)(n-6)(n-5)}{1.1.2.3.4}$

II. The *homogeneous multiple sequences*

of 2 + 2, has 4 bundles $\frac{(n-8)(n-7)(n-6)(n-5)}{1.2.1.2}$

2 sequences, 2 isolated files.

of 2 + 2 + 2, has 3 bundles $\frac{(n-7)(n-6)(n-5)}{1.2.3}$,

3 homogeneous sequences.

of 3 + 3, has 2 bundles $\frac{(n-6)(n-5)}{1.2}$

2 homogeneous sequences.

III. *The mixed sequences*

that of 3 + 2, has 3 bundles $\frac{(n-7)(n-6)(n-5)}{1.1.1}$

1 sequence of 3, 1 sequence of 2, 1 isolated file.

that of 4 + 2, has 2 bundles $\frac{(n-6)(n-5)}{1.1}$,

1 sequences of 4, 1 sequence of 2.

§ 25. It would be tempting on this subject, to indicate all the relationships that one is able to observe. I have had much less in view the solution of the problem, than to indicate a bright method of proceeding there which conserves the order in the results, & which indicates the application of the principles of the speculative Philosophy to the questions of calculation.

It is by this consideration that after having shown how in the general notions one must disentangle the rules which agree with the less general notions, & with the individuals. I am going to indicate by the opposing route by which manner it is necessary to raise from the knowledge of the individual determinations in one singular case, in the knowledge of the general laws of all the diverse species, & of all the genres under which this singular case is contained.

We choose for this effect the unique arrangement *Augustus Tiberius Caligula, Nero Galba, Vitellius Vespatian; Domitian*. It contains a ternary, a binary double, & one isolated file: this example must contain the laws of all the simple sequences, multiples & mixtures; one more complicated would only render the research more difficult without necessity, since it would add some superfluous determinations; an example less loaded would limit the research in some less general cases, which would contain only the mixed sequences, without their multiples, or the homogeneous without the mixtures, or the simples without multiples, nor mixed.

We see therefore how many of the ternary sequences & of binary doubles this unique case can furnish in the sense of Mr. Euler, & in order to avoid the prolix we substitute numbers for the medals, all the possible cases will be found successively to be the following.

$$
\begin{array}{cccc}
(1,2,3), & (5,6), & (8,9), & 11. \\
(1,2), & (4,5,6), & (8,9), & 11. \\
(1,2), & (4,5), & (7,8,9), & 11. \\
1, & (3,4,5), & (7,8), & (10,11). \\
1, & (3,4), & (6,7,8), & (10,11). \\
1, & (3,4), & (6,7), & (9,10,11). \\
(1,2,3), & 5, & (7,8), & (10,11). \\
(1,2,3), & (5,6), & 8, & (10,11). \\
(1,2), & 4, & (6,7,8), & (10,11). \\
(1,2), & 4, & (6,7), & (9,10,11). \\
(1,2), & (4,5,6), & 8, & (10,11). \\
(1,2), & (4,5), & 7, & (9,10,11).
\end{array}
$$

The total number of numerals is here 11, the number of the files is 8, the last file is shortened to 10, its length is reduced by 1, & it furnishes also only a single complete rank: if it were longer, there would be evidently more ranks in the same proportion to this length. Thus the last file being l, instead of 1, the number of ranks will be l, instead of being 1, its diminution results from the elevation of the files one on the other; & the latter of the bundles & of the sequences; there are 4 bundles, each is raised on the other by 2 numerals on its predecessor, the first bundle has no predecessor, thus it is necessary to count only 3 of them, this which makes a shortening of 6: if there were 4 bundles, the shortening would be consequently of 8, & in general, if there were F bundles, they would produce a shortening of $2(F-1)$. Here the two binaries, & the ternary make a shortening of 3; the reason is that the files in the sequences are raised only by one numeral, each on its preceding, the first file of a sequence has no predecessor, thus there is for each sequence as many numbers to shorten as the sequence contains of files less one. If therefore there were m *binaries*, this would be a shortening of m numbers; if there were m ternaries, they would produce a shortening of $2m$ numbers; finally if the sequence were S, the shortening that it would produce would be $S-1$; & if there were m parallel sequences they would shorten consequently the last file by m times $S-1$ numbers: now one can imagine some sequences from the greatest S, to the binary 2, & each can be a multiple; thus the shortening of the last file can be

$$= 2(F-1) + m(S-1) + m(S-2) + \cdots m,$$

& naming for brevity the descending progression

$$(S-1) + (S-2) + \cdots + 1 = P,$$

& the total number of numerals n, instead of 11, one will have $l = n - 2(F-1) - mP$.

There are in our case 8 files, & however they form only 4 things, or bundles to combine. It is that each sequence is only a single tour, & that it absorbs two or more files. Each binary takes 2 of them, & makes only one bundle; the ternary takes 3 of them, & is counted only for *one*: for the same reason any sequence whatever of S numbers, will occupy S files, & will give only one bundle; & mS sequences will absorb mS files, in order to furnish m bundles. Therefore, since it can have all the descending progression of sequences, from the most numerous S, to the binary 2, & since each can be repeated m times, the number of bundles will be equal to the total number of files, less m times this progression, plus m: thus if the number of files is t, that of the bundles F, it will be

$$= t - (P+1)m + m = t - Pm,$$

therefore the number of ranks to combine l will be

$$= n - 2t + 2 + Pm.$$

The four bundles of our case give 12 transpositions, the isolated file occupies 3 times the left, 3 times the right, 3 times it separates the first sequence from the last two, & 3 times it separates the last from the first two. Four things are transposed regularly in 24 ways, the isolated file must occupy 6 times each position; why does it not occupy it only 3 times? It is that it has two similar things, two binaries of which the transpositions with two other dissimilars give always consequently two times the same result; if there were 3 similar things, one could transpose them among themselves in 1.2.3 ways, & consequently they would give 6 times the same result. Therefore, since the number of transpositions is here, $\frac{1.2.3.4}{1\times 2}$, if there were F bundles, all different among themselves with the exception of two, the number of transpositions would be $\frac{1.2.3\cdots F}{1\times 2}$, & if there are m similar bundles instead of 2, the transpositions will be $\frac{1.2.3\cdots F}{1.2\cdots m}$, or putting in place of F, its found value, the transpositions will be $\frac{1.2.3\cdots(t-Pm)}{1.2\cdots m}$.

I can stop myself here. Arriving at the general expression of the transpositions, of the ranks & of the bundles to combine, there remains nothing more to seek: we have come to the singular case in the knowledge of the general law of all the possible sequences; which will be

$$\frac{1.2.3\cdots(t-Pm)}{1.2\cdots m}\frac{(n-2t+2+Pm)(n-2t+2+Pm+1)(\cdots)(n-t+1)}{1.2\cdots(t-Pm)},$$

if the sequences conclude at the greatest number, &

$$\frac{1.2.3\cdots(t-Pm)}{1.2\cdots m}\frac{(n)(n-2t+1+Pm)(n-2t+1+Pm+1)(\cdots)(n-t+1)}{1.2.3\cdots(t-Pm)},$$

if the greatest number is counted to make a sequence with the least.

However one can again go further. The individual case that I have taken for example, having four bundles to combine & a single rank, this combination would be expressed as one knows by the fraction $\frac{1\times2\times3\times4}{1\times2\times3\times4}$, the transpositions are as we indicate $\frac{1.2.3.4}{1.2.1}$, thus the number of sequences of III + II + II, will be

$$\frac{1.2.3.4}{1.2.1}\times\frac{1\times2\times3\times4}{1\times2\times3\times4},$$

where I remark that the numerator of the transpositions is precisely equal to the denominator of the combinations, & that they destroy themselves. But, what is the sufficient reason of this equality? It is necessary to seek in the same origin of this numerator & of this denominator. When in the combination one regards not only the number of the combinable things; but also their situation, the combination will be always a whole number; it becomes a fraction only in the case where one does not wish to have the diverse situations of the things to combine. Now the transpositions express precisely these diverse situations of combinable things: thus in order to exclude one divides the total number of combinations by the total number of transpositions; a number which becomes consequently the denominator of the fraction. It is not therefore surprising that the numerator of the transpositions is equal here to the denominator of the combinations, & since the same reason will hold all the time that one will choose the total number of cases of a specie of any sequence whatever, it is evident that this number can always be expressed by that of the total combinations, divided simply if it is necessary by the number of the identical cases, that is by those which result from the combination of the bundles or of the similar things.

Thus the number of things to combine being $t - Pm$, & that of the ranks, $n - 2t + 2 + Pm$, one will have the cases of each sequence whatever in the sense of Mr. Euler

$$\frac{(n-2t+Pm+2)(n-2t+Pm+3)(\cdots)(n-t+1)}{(1.2\cdots m)(\cdots)(1.2\cdots m)},$$

& in the sense of Mr. Bernoulli, or this same sequence whatever is always repeated as many times as there are numerals in all, the number of cases will be

$$\frac{(n)(n-2t+Pm+1)(n-2t+Pm+2)(\cdots)(n-t-1)}{(1.2\cdots m)(\cdots)(1.2\cdots m)},$$

So that in the each formula the number of the factors of the numerator is equal to the number of things to combine, & the number of factors of the denominator, to that of the multiples of the similar bundles.

That which there is singularly in our example, is that in the sense of Mr. Bernoulli, it contains not any case of the mixed sequence that we have chosen; since we have supposed only a single rank; & that thus the number of cases would be

$$\frac{11\times(11-11)(11-10)(11-9)}{1.2.1.2} = 0.$$

This which does not prevent that this example not furnish thus the general law for the circular sequences.