# SUR
# LES SUITES OU SÉQUENCES
# DANS
# LA LOTTERIE DE GENES
# *PREMIER MÉMOIRE*[*]

NICOLAUS BEGUELIN

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§. 1.

The doctrine of probabilities belongs to Metaphysics as much, & more perhaps, than to Geometry. If the latter lends its calculus, the other furnishes the principle which serves as basis to them.

But is there not in the main a quite elaborate discussion, as the one which turns on account of the business of the diverse sciences; & which signifies to its progress to demand from the one, some matters which the other has already treated with so much sagacity? One can respond that the metaphysician employs the common language; which he puts as much as it is possible in the range of all, instead the analyst hurries himself from the first lines to speak the most correct language, the most concise, the most forcible to the truth, but which is understood only by those who are initiated in the depths of Algebra, & this is not the business of everyone.

§ 2. There is more: since the exact discussion of a matter requires the help of the calculus, the metaphysician & the geometer will treat it differently; & it is probable that so far as the first will be able to go, the clarity, the evidence, & likewise the brevity, will be preferably on his side. Let one pardon me this paradox in favor of Metaphysics. One exalts so frequently Geometry to its detriment, as it must be permitted also to accept the cases, where this former has some advantage over the other: besides vanity enters here for nothing, since this advantage of the Metaphysician is due only to the same superiority of the Algebraist. These are two voyagers cast upon an unknown coast, which they seek to discover. The one who no obstacle stops, plunges head lowered into the thick forest which is present before him, marches with a sure & firm step to the middle of the obscurity of the woods, diverts to right & to left the thorns which close

[*]Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 29, 2009

the passage to him, clears the brush, jumps the ravines, & across a thousand difficulties which he has surmounted, he arrives in the end by a dark & mysterious route to the end which he has in view without having seen nor known it before.

The other, less vigorous & more timid, as the aspect of an unknown forest frightens, seeks to place himself on some height whence he is able to see as a bird to discover the less arduous routes of this woods, & to see by one same glance the point from where he can depart from it, & the one where he must arrive. It is not doubtful, that this last will succeed to the end, by a shorter path, less rugged & more illuminated, than the first. But one does not always find these points of elevated view; & there are quite some impenetrable forests to the Metaphysician; it is this to which I will agree at my turn very readily.

§ 3. A *sequence* in the Genoise Lottery is a series of numbers pulled out in the same drawing, of which the one would exceed the other only by the unit. Two such numbers are a *binary sequence*; three a *ternary* sequence, &c. In the remainder there is no need that these numbers be pulled out precisely in the order of sequence, it suffices that those of one same drawing be susceptible to such an arrangement. Thus a drawing which will give successively the numbers $7,\ 13,\ 17,\ 8,\ 24$ will contain the binary sequence $7,\ 8$.

§ 4. One can consider the sequences in two ways that the numerals of a lottery generate: the first way, is to adhere only to the natural series of numbers. This is thus what Mr. Euler has made in his excellent Memoir on this subject. The second way, is to imagine by fiction all the numbers arranged in a circle, this which adds one more sequence, namely the one which forms between the greatest numeral & the least, in forming the circle. This is the assumption which Mr. Bernoulli has adopted. Each reckoning has its advantage & its inconvenience, the one of Mr. Euler has more regularity in the abstract calculus, the other has greater relevance to the plan of a lottery, where all the numerals must be indifferent to choice. I myself propose here to unite them; & to consider the sequences under this double aspect.

§ 5. By assuming that one wishes to introduce the chance of the sequences of one or the other kind into the Genoese Lottery, the question which presents itself for resolution is to know what probability there is that in a certain number of numerals extracted in a single drawing there will be a sequence or not; & consequently what will be the proportion between loss & gain to equal profit?

§ 6. Since the question to determine here is the degree of probability of the events, it is necessary to begin by establishing the rules of this probability. They are founded on the principle of the need of a sufficient reason. In fact the events which one considers here being all possible absolutely speaking; the principle of the contradiction could neither exclude any of them, nor decide which have more or less claim to existence. It is to the principle of sufficient reason alone to make this decision. This principle says to us that when a unique operation, a drawing of a lottery, for example, can give rise to many different events, if there is no reason, why one of these events must result from this operation rather than one of the others, the probability is equal for each of them; but that if, of two possible events, one must exist by a greater number of cases than the other, & if all these cases be equally possible, the probability of these two events will be exactly proportional to the numbers of the cases which are able to produce them; whence results the two rules for computing the probability of any event whatever. Here is the first: *Compare the number of cases, where this event is able to exist by* *the operation, to the number of all the cases that this operation is able to produce, & the ratio of these two numbers will express the degree of probability of the event in question*: if, for example, the numerals of a drawing can give in all 50 different combinations: if among these combinations there are able to be 5, which contain some sequences, the probability for these sequences will be expressed by the ratio of 5 to 50, or of 1 to 10, & consequently by the fraction $\frac{1}{10}$, related to unity. This is the rule which the Geometers ordinarily employ in the computation of the probability of a particular case, because they consider this case out of its relationship with the opposing cases.

Here is the second rule, which, although it returns for foundation to the preceding, is however of more use in common life, & principally in games of chance: *Compare the number of cases where an event can exist by one operation, with the number of cases where this operation will not produce it; & the ratio of the two numbers will express the probability for & against this event;* or, this which comes to the same, *this ratio will express the sums that one can wager on all sides in an equal game*. Thus, in the preceding example, the probabilities for & against the sequence will be, according to this last rule, as 5 to 45; & the one who wagers for the sequence can consequently put 1 écu against 9. Because, all 50 cases being equally possible, there is no reason why one would be more probable than the other: they have therefore each the same degree of probability: therefore the one who wagers for the sequence has 5 degrees of expectation, while the one who wagers against him has 45 of them. This latter can consequently risk 9 écus when the other risks *one* of them.

§ 7. In order to resolve the question on the degree of probability of the sequences, there is therefore only to count all the combinations which can result from a drawing, & see next which is more easy, either to count the combinations which contain some sequences, or those which contain none of them. As subtracting one of these sums from the sum of the total combinations, gives always the other, it is clear that there is need to seek only one, & that it is necessary to seek that which is found most easily. I will call *total combinations*, the sum of all the possible cases which one drawing can produce, *pure combinations*, the sum of the cases from this drawing which contain no sequence; & *sequences*, the sum of the cases of the same drawing which contain some *sequences*. One will understand that way, what I understand, by *ambes, ternes, quadernes, totals. Ambes, ternes, pure quadernes. Sequences of ambe, de terne, de quaderne,* &c.[1]

§ 8. Although the Genoise Lottery is sufficiently known, in order that one have familiarity with the method of computing the total combinations, I believe however that one can spread more clarity on this subject by representing it in a more sensible manner; the abstract numbers easily fatigue the imagination, which does not have enough grasp on them. Whoever has seen march some soldiers, knows what are *ranks & files*. The number of files represents very well the number of the numerals, which one extracts in a drawing, as the number of ranks, or some individuals who are following in a file, represents the total number of the numerals which the wheel contains. We imagine therefore after this image that one has arranged many series of medals of ancient Roman emperors, from *Augustus* to *Trajan*[2], & further, if it is necessary: each sequence

[1] *Translators note*: An ambe is two numbers in order, such as 3, 4 or 9, 10. A terne is three in order such as 3, 4, 5. Likewise a quaderne is four in order. Béquelin uses quaderne and quatern interchangably.

[2] *Translators note*: These are Augustus, Tiberius, Caligula, Claudius, Nero, Galba, Otho, Vitellus, Vespasian, Titus, Domitian, Nerva and Trajan.

will form a file, & there will be as many of these arranged files on the same height up and down as one will have of sequences of these Emperors. The first rank will contain the *Augustuses*, the second the *Tiberiuses*, & the last the *Trajans*.

§ 9. If one asks to present in how many ways one can transpose these medals without diminishing neither the files, nor the ranks, it is clear that each medal of the first file can make ambe, with each medal of the second; & each of these ambes to form a terne with each individual of the third, & each of these ternes to give a quaderne with each piece of the fourth file, & thus in sequence as far as one will wish, such that in order to have all the transpositions, there is only to multiply the ranks by themselves, as many times as there are files, or, (what comes to the same,) as many times as a rank contains individuals. Thus five sequences of medals, each of 13 Emperors, forming 13 ranks, on 5 individuals in front can be transposed, without diminishing either the ranks, nor the files, in 13 times 13 times 13 times 13 times 13 ways, that is, as many times as the number 13, raised to the fifth power can contain of units.

But in these transpositions enter all the combinations of simple situation, where diverse ranks contain only the same individuals otherwise placed. Now this diversity is counted for nothing in the Genoise Lottery. *August & Tiberius; Tiberius & Augustus*, would make only one same ambe. *Claudius-Galba-Nero; Galba-Nero-Claudius; Nero-Claudius-Galba*, & the three other transpositions of these three names are counted as for a single terne. Thus, in order to find the combinations without repetition, it is necessary to avoid mixing more than one time the same individuals.

We suppose therefore first two single files: *Augustus* of the first file can be combined with all the individuals of the other file. *Tiberius* of the first, can no longer be combined with *Augustus* of the second file, this would be a repetition; but he will be combined with all the others. *Claudius*, for the same reason, will be combined with all to the exception of *Augustus & Tiberius*; in a word each medal of the first file is combined with that of the second which is found in the same rank as it, & with those of all the ranks which follow that one, to the exclusion of all those which are in the superior ranks. So that if *Trajan* is the last of the first file, he is combined only with *Trajan* of the second. The number of combinations increases therefore from Trajan to Augustus according to the natural progression of the numbers $1, 2, 3, \dots, 13$. Now one knows that the sum of this arithmetic progression is the sum of the two extreme numbers multiplied by the half of the greater, & consequently in our example it will be $14 \times 6\frac{1}{2} = 91$, or in general if the indeterminate number of ranks is called $n$, the sum of the combinations of two files will be $(n+1)\frac{1}{2}n = \frac{n(n+1)}{2}$.

If we add a third file, equal to the preceding, one will find by the same reasons, that *Augustus* of the first file, combined with *Augustus* of the second, forms as many ternes, as the third file has individuals; then combined with *Tiberius* of the second file, it forms some ternes, with *Tiberius* & all the following in the third: next combined with *Claudius* in the second file, it forms some ternes with *Claudius*, & all the following in the third file. One finds a perfect analogy in all the others of the first file, whence results for each of them a number of combinations, equal to the sum of a natural arithmetic progression, $1 + 2 + 3$ &c. of which the greatest term is equal to the number of the rank of this emperor counting the ranks from bottom to top. Thus *Trajan* forms only one combination of terne: *Nerva*, forms three of them, *Domitian*, 6; *Titus*, 10; & so forth. Therefore the sum of the combinations of ternes is expressed, by the sequence of numbers which one names *triangulars*, $1 + 3 + 6 + 10 + 15$ &c. of which each term is the sum of the sequence or natural progression, which corresponds to it. Now there are here as many terms, as there are ranks to combine, & one knows that the sum of the triangular numbers for any number whatever of terms, which I will name $n$, is $= \frac{n}{1}\frac{(n+1)}{2.}\frac{(n+2)}{3}$.

One finds by the same method, that the sum of the combinations of four files, is expressed by the sequence of *first pyramidal numbers*, which result from the addition of the triangulars, as these latter resulted from the addition of the natural numbers. This sequence is therefore $1 + 4 + 10 + 20$ &c. & the sum for a certain number of ranks named $n$ will be: $\frac{n}{1}\frac{(n+1)}{2.}\frac{(n+2)}{3.}\frac{(n+3)}{4}$.

If one adds a fifth file, the combinations which result from it, will consequently be expressed by the sum of the *second pyramidal* numbers, of which each term represents the sum of the first pyramidal numbers to this term here. This sequence is therefore $1 + 5 + 15 + 35$ &c. & the sum for a certain number of ranks, named $n$, is $= \frac{n(n+1)(n+2)(n+3)(n+4)}{1.2.3.4.5}$.

There is no need to push this discussion further, one sees easily that the same analogy subsists to infinity, & I myself would not be stopped even since in the subject I just treated it was permitted to me to suppose all this known, it was only that these combinations are going to serve as foundation to that which I have to say on the sequences; for the same reason I will add yet some observations which do not concern them immediately.

§ 10. The calculus of the combinations of our medals would be precisely the one of the Genoise Lottery, if the same numeral could have exited more than one time in one same drawing. But as this cannot take place, it is necessary to make a small change in the arrangement of our files, let the first to the left remain immobile, & let the second be raised by one level; then *Augustus* ranks with *Tiberius*; & *Nerva* with *Trajan*. The third file will exceed likewise the second by one step, one will have then in the first complete rank, *Augustus, Tiberius, Caligula*; & in the last *Domitian, Nerva, Trajan*. The fourth file advanced similarly, will give in the first complete rank, *Augustus, Tiberius, Caligula, Claudius*, & in the last *Titus, Domitian, Nerva, Trajan*. It will be again likewise for the other files which one will wish to add.

One sees easily that the calculation of the combinations remains always the same; the only difference which is found is that the number of complete ranks has diminished, it is no longer equal to the number of individuals in the first file, as it is when all the files are of equal length. Here the number of complete ranks is always equal to the number of individuals of the shortest file, which is in our arrangement that of the right flank. Thus a sequence of 13 medals gives only 12 ranks with two in front, 11 with three in front, 10 with four in front, & the quines[3] form only 9 complete ranks. If one demands therefore how many of these 5 files of 13 medals will give quines, one has here $n = 9$, consequently this will be $\frac{9.10.11.12.13}{1.2.3.4.5} = 1287$.

§ 11. Since we have avoided the identical combinations by making each of our files rise by one step on the neighbor to the left; it will be easy for us also to avoid the combinations of sequence by making these files take two steps instead of one. One

[3]*Translators note*: A *quine* is the five winning numbers in a lottery.

will have then at the first complete rank, *Augustus Caligula*, if there are only two files; *Augustus, Caligula, Nero*, if there are three of them; the last rank will have in the first case, *Domitian, Trajan*, & in the second, *Vespatian, Domitian, Trajan*; & thus of the other files as one will wish to add.

The calculation of the *pure* combinations will be again here the same as the one of the *totals* § 10. & of the *identicals* § 9. only there will be less ranks to combine; their number will be always equal to the shortest file; & since each file the first excepted is shortened by two numbers, the shortening of the last will be two times the number of files diminished by one unit. Thus our sequence of 13 emperors will give 11 ranks of *pure ambes*, 9 ranks of *pure ternes*, 7 ranks of *pure quadernes*, & 5 ranks of quines without sequences. If one demands therefore how many of these 5 files will give quines, without which any Emperor is arranged in front with his successor or his immediate successors, one has here $n = 5$, & consequently the sum of these *pure quines* is $= \frac{5\times6\times7\times8\times9}{1.2.3.4.5} = 126$.

§ 12. We have found § 10. the *total quines* for the same number of 13 medals$=$ 1287, taking off from those the *pure quines*, the number of cases of sequences remains $= 1161$. Thus the probability of a sequence will be here $= \frac{1161}{1287} = \frac{129}{143}$, & one can wager 1161 against 126, or 129 against 14.

§ 13. Generalizing therefore these results, one will have by taking the sequences in the sense of Mr. Euler, for a number $n$ of numerals.

| *Drawing* | *Total combinations* | *Pure combinations* | *Sequences* |
|---|---|---|---|
| of 2 Nos. | $\frac{(n-1)n}{1.2}$ | $\frac{(n-2)(n-1)}{1.2}$ | $\frac{(n-1)n-(n-2)(n-1)}{1.2}$ |
| 3 Nos. | $\frac{(n-2)(n-1)n}{1.2.3}$ | $\frac{(n-4)(n-3)(n-2)}{1.2.3}$ | $\frac{(n-2)(n-1)n-(n-4)(n-3)(n-2)}{1.2.3}$ |
| 4 Nos. | $\frac{(n-3)(n-2)(n-1)n}{1.2.3.4}$ | $\frac{(n-6)(n-5)(n-4)(n-3)}{1.2.3.4}$ | $\frac{(n-3)(n-2)(n-1)n-(n-6)(n-5)(n-4)(n-3)}{1.2.3.4}$ |
| | &c. | &c. | &c. |
| | $\vdots$ | & in general, for a drawing of $t$ numerals. | $\vdots$ |
| $t$ Nos. | $\frac{(n-t+1)(n-t+2)(n-t+3)(\cdots)n}{1.2.3\cdots t}$ | $\frac{(n-2t+2)(n-2t+3)(\cdots)(n-2t+1)}{1.2\cdots t}$ | $\frac{(n-t+1)(\cdots)n-(n-2t+2)(\cdots)(n-2t+1)}{1\cdots t}$ |

§ 14. Under the assumption of Mr. Bernoulli, if the greatest number makes also sequence with the least, there will be again the same calculation, accompanied only by one more way; which I am going to explain. We have avoided the sequences by placing our files into levels of two. But that way we will not avoid the new sequence of *Trajan* with *Augustus*. It is necessary therefore, in order to avoid it, to subtract for the *Augustus* of the first file, the last rank where is found *Trajan*, & to restore this last rank for all the successors of *Augustus* contained in the first file. Thence it follows that *Tiberius*, & all his successors, will be combined with as many ranks as when one counts the sequences in the manner of Mr. Euler, but that *Augustus*, who in that former manner combined with a higher rank, than his successor *Tiberius*, is combined here only with the same number of ranks as that latter one. Whence there results again, that *Augustus & Tiberius* will both have the same number of combinations. Such that the progressions of natural numbers for the ambes, of triangular numbers for the ternes, of first pyramidal numbers for the quadernes, of second pyramidal numbers for the quines, &c. that all these progressions, I say, beginning always with unity at *Trajan*, will go no more than up to the highest rank of *Tiberius* exclusively, & that to their sum it will be necessary to add the repetition of the last term, for the combinations of *Augustus*; this which will give the sum of the pure combinations, & consequently also the sequences in the sense of Mr. Bernoulli.

If for example the number of medals in the first file is always 13, the second file will have only 11 at its height, & will form consequently only 11 ranks of ambes. But *Augustus* cannot be combined with *Trajan*, it is as if he would have for himself only 10 ranks. Now *Tiberius* can also be combined with 10 ranks; *Caligula* with 9; *Claudius* with 8; &c. It is therefore as if the natural progression were only 10 terms, & that the tenth was repeated. But the sum of a natural progression of 10 terms, is $(10+1) \times \frac{10}{2} = 55$; add again one time the last term 10, for the pure combinations of *Augustus*, & you have the sum of all the pure combinations$= 65$. According to Mr. Euler one will have found $= 66$. The total combinations are here $91$; therefore there are according to Mr. Euler 25 cases of sequences, & according to Mr. Bernoulli 26.

It is the same process for the pure ternes: the third file in our example ranges no more than to 9 in depth. One will have therefore 9 complete ranks, & consequently a progression of triangular numbers of 9 terms; but in order to avoid the circular sequence, this progression has no more than 8 terms with redoubling of the last: it is therefore $n = 8$, which gives $\frac{8.9.10}{1.2.3} = 120$.

In order to find the last or highest term of this progression to add, there is only to consider that *Augustus* of the first file, of which one seeks the combinations, forms with these 8 ranks of two files, precisely as many terms as these 8 ranks give ambes; now, if $n$ is $= 8$, one has $\frac{8\times 9}{2} = 36$, for the sum of the ambes, & consequently for the sum of the combinations of *Augustus*, or, what is again the same thing, for the highest term of the triangular progression which one sought, & that one would have been able to find also by its relationship alone that each term of the triangular numbers is the sum of the natural sequence of numbers up to it: $1+2+3+4+5+6+7+8$. Adding therefore 36 to 120, one has the sum of the pure ternes, in our case, according to Mr. Bernoulli $= 156$. According to Mr. Euler, one will have obtained a triangular progression of 9 terms of which the sum was $= \frac{9.10.11}{1.2.3} = 165$ pure ternes, the total ternes are here $= \frac{11.12.13}{1.2.3} = 286$: therefore there are in this case according to Mr. Euler $121$ cases of sequences; & $130$ in the sense of Mr. Bernoulli.

§ 15. It would be superfluous, without doubt, to give after this some examples of quadernes & of quines, in this second manner to calculate the sequences. We have said enough on it, so that by the aid of the principle of sufficient reason, & from the analogy of which this principle is the basis, we can generalize the truths contained in these particular examples; & by subtracting successively the individual determinations, realize in a singular case the universal properties of the species & of the genres under which this case was contained. Here, for example, for a number 13 of medals, we have $13-3$ ranks of ambes; $13-5$ ranks of ternes; $13-7$ ranks of quadernes; &c. for the same reason we will have for a number $n$ of numerals; $n-3$ ambes, $n-5$ ternes; $n-7$ quadernes &c. But these numbers $3$, $5$, $7$ subtracted successively correspond to the numbers of the files $2$, $3$, $4$ which we have successively considered. This is the same sufficient reason which founds the relationship of diminution of 3 for two files; & of 5 for three files, consequently also of $2f-1$ for $f$ files.

Thus, in the sense of the circular sequences, if the number of numerals of the lottery is named $n$, one will have

| *Drawing* | *Pure combinations* |
| --- | --- |
| of 2 Nos. | $\frac{(n-3)(n-2)}{1.2}+n-3=\frac{(n-3)n}{1.2}$ |
| 3 Nos. | $\frac{(n-5)(n-4)(n-3)}{1.2.3}+\frac{(n-5)(n-4)}{1.2}=\frac{(n-5)(n-4)n}{1.2.3}$ |
| 4 Nos. | $\frac{(n-7)(n-6)(n-5)(n-4)}{1.2.3.4}+\frac{(n-7)(n-6)(n-5)}{1.2.3}=\frac{(n-7)(n-6)(n-5)n}{1.2.3.4}$ |
| 5 Nos. | $\frac{(n-9)(n-8)(n-7)(n-6)(n-5)}{1.2.3.4.5}+\frac{(n-9)(n-8)(n-7)(n-6)}{1.2.3.4}=\frac{(n-9)(n-8)(n-7)(n-6)n}{1.2.3.4.5}$ |

| *Drawing* | *Sequences* |
| --- | --- |
| of 2 Nos. | $\frac{(n-1)n-(n-3)n}{1.2}$ |
| 3 Nos. | $\frac{(n-2)(n-1)n-(n-5)(n-4)n}{1.2.3}$ |
| 4 Nos. | $\frac{(n-3)(n-2)(n-1)n-(n-7)(n-6)(n-5)n}{1.2.3.4}$ |
| 5 Nos. | $\frac{(n-4)(n-3)(n-2)(n-1)n-(n-9)(\cdots)(n-6)n}{1.2.3.4.5}$ |

and more generally again for a drawing of $t$ numerals, the sum of the pure combinations will be

$$=\frac{(n-2t+1)(n-2t+2)(\cdots)(n-t+1)n}{1.2.3\cdots t}$$

& that of the sequences will be

$$=\frac{(n-t+1)(n-t+2)(\cdots)(n-t+1)-(n-2t+1)(n-2t+2)(\cdots)(n-t+1)n}{1.2.3\cdots t}$$

§ 16. The general formula of the total combinations, & of the pure combinations, for any number whatever of numerals $n$, of which one assumes that one extracts at each drawing any number whatever of numerals $t$, being found § 13. & 15. one has also the general formula for the probability that there will be no sequences, by dividing the *pure* combinations by the *totals*; this probability will be according to Mr. Euler:

$$=\frac{(n-2t+2)(n-2t+3)(\cdots)(n-t+1)}{(n-t+1)(n-t+2)(\cdots)n}$$

& according to Mr. Bernoulli

$$=\frac{(n-2t+1)(n-2t+2)(\cdots)(n-t-1)}{(n-t+1)(n-t+2)(\cdots)(n-1)}$$

Now, as soon as I know how much probability there is for the non-sequence, I know at the same time the probability that there is for the sequence; since I have only to subtract from unity the fraction which expresses one of these probabilities, in order to have the other.

Thus the general formula for the probability of sequences in the sense of Mr. Euler is

$$=1-\frac{(n-2t+2)(n-2t+3)(\cdots)(n-t+1)}{(n-t+1)(n-t+2)(\cdots)n}$$

& in the sense of Mr. Bernoulli, this probability is

$$=1-\frac{(n-2t+1)(n-2t+2)(\cdots)(n-t-1)}{(n-t+1)(n-t+2)(\cdots)(n-1)}$$

§ 17. Naming, for brevity, the product of the factors $(n-2t+2)(n-2t+3)(\cdots)(n-t-1)=a$, & that of the factors $(n-t+1)(n-t+2)(\cdots)(n-1)=b$, the probability of the sequences will be according to Mr. Euler

$$=\frac{bn-a(n-t)(n-t+1)}{bn}$$

& according to Mr. Bernoulli

$$=\frac{b-a(n-2t+1)}{b}.$$

Thus the first is to the second :: $bn-a(c+tt-t).bn-nc$, by setting for brevity $nn-2nt+n=c$.

Here is all that which there is to examine in regard to the sequences in general; I suppress a number of consequences & observations which one can deduce from this theory, in order to pass to the examination of a second question.

§ 18. If instead of simply introducing the game of the sequences into the Genoise Lottery, one wished to set some different prizes into the diverse species of sequences of 2, of 3, or of many numbers, the question would be to find the number of each specie of particular sequence binary, ternary, quaternary, &c. which comprise the sum of the *total* sequences, which we just found. Because, although Mr. Bernoulli names *binary sequences*, the sequences which I just determined, they can not be called thus as all the sequences of more than two numbers are necessarily sequences of two numbers at the same time; & it is for this consideration that I have called them rather *totals*, because they contain all the particular species at once: for the same reason I will call *binaries*, those which have only two numbers which follow themselves; *ternaries*, those which have only three such &c.

If one wishes therefore to determine the number of cases of each of these particular species, one could equally do it, by starting always on the principle that I have followed to now; nothing prevents even starting in this research with the general Problem which has no difficulties. However, in order to spread more clarity on the method that I employ, I will begin with the particular Problems.

1. First, if one extracts only 2 numerals, that is, if one has to combine only two files, it is evident that one will have only some *binary* sequences; & thus the case is already resolved, since here the *total* sequences which we have found for a drawing of 2 numerals, consists only of *binaries*. But, in order to indicate the use of our method, we seek again one time the number of these binaries.

In order to find it, there is only to bring down again the second file by a notch so that each Emperor is found at the side of his immediate successor. Then one will have at the first complete rank *Augustus-Tiberius*; & at the last *Nerva-Trajan*. It is evident that this position represents all the cases of binary sequences; & that the number of these cases is precisely equal to the one of the ranks. Now the first file contains $n$ medals, the second setting in position of sequence contains one less of them; there are therefore $n-1$ ranks, & consequently $n-1$ binary sequences according to Mr. Euler. We join the circular sequence *Trajan-Augustus*, & we will have the binaries, according to Mr. Bernoulli $=n$.

§ 19. II. If one extracts 3 numerals, the case is also actually resolved. Because we know the number of total sequences. Now there can be here only some binaries, & some ternaries. But the ternaries are known, since it is the greatest sequence that the drawing of 3 numerals can supply. This sequence is constant for all the drawings in the sense of Mr. Bernoulli, & always equal to the total number of numerals $= n$. Because, if I imagine 20 trees planted in a circle, & if I choose one of them at will for the beginning of the first sequence, either binary, or ternary &c. the second will be the beginning of the second sequence; the third of the third; & consequently the twentieth & last, of the twentieth & last. In the sense of Mr. Euler, this sequence is always equal to the total number of numerals $n$, less the number of files, or of the numerals which one extracts, diminished by unity. Because the greatest sequence assumes all the files arranged in sequences, or in levels of one number; there will be precisely as many complete ranks, as the first file contains individuals, less the number of following files.

We see however the usage of our method also for this case. The three files laying in sequences give in the first complete rank *Augustus Tiberius Caligula*, in the last *Domitian Nerva Trajan*. This position represents at a glance all the possible ternaries; because as soon as one would derange one of the three files, there would no longer be ternary sequences: the number of these sequences is therefore equal to that of the complete ranks, & this latter to the length of the shortest file; it is in this case $= n - 2$. Therefore, according to the calculation of Mr. Euler, there will be $n - 2$ ternaries.

But, according to Mr. Bernoulli, the two incomplete ranks must be completed by the aid of the circular revolution: one has therefore again the two ranks *Nerva Trajan Augustus*, & *Trajan Augustus Tiberius*, to add to the sum $n-2$, which that way becomes $= n$.

In order to find now the binaries, there is only to raise the third file again by one level. Then each binary sequence which forms the first two files are combined with all the individuals in the third which is no more than $n - 3$ numbers. The first rank will be *Augustus Tiberius Claudius*, the last *Titus Domitian Trajan*. As the first two files must remain immobile in order to form a sequence, it makes only a single bundle, & one must consider them here as one single file which is combined with the third. This reverts therefore to the calculation of the ambes, which we have explained above, & the sum of these binaries, for $n - 3$ ranks, will be

$$= \frac{(n-3)(n-2)}{1.2}$$

Under the assumption of Mr. Bernoulli, the first sequence *Augustus Tiberius*, could not combine with *Trajan*, last of the third file, because this would give a ternary; thus, by the principles explained above, the sum of the binary combinations will be

$$\frac{(n-4)(n-3)}{1.2} + (n-4) = \frac{(n-1)(n-4)}{2}.$$

We have assumed the first two files immobile in their position of sequence, by making the third slide. But nothing prevents that one should not similarly move the first, by setting the last two immobile in state of sequence: it suffices for this effect to raise by one level the second file: the first rank will be *Augustus Caligula Claudius*,

& the last, *Titus Nerva Trajan*, & as besides the same number of ranks & of bundles subsist, there is no reason why the result must differ in the rectilinear sequences. This combination, made in contrary sense, will give therefore again $\frac{(n-3)(n-2)}{1.2}$ binaries.

But in the circular sequences there is for this position one more sequence to add, namely that of *Trajan Augustus*, which is combined also with $n-4$ numbers, that is from its corresponding *Domitian*, by going up to *Caligula* inclusively, this which joined to the number of the binary cases found in the first combination gives $\frac{(n-1)(n-4)}{2} + (n-4) = \frac{n(n-4)}{2}$.

Thus, if of $n$ numerals one extracts 3 of them at random, there will be in the sense of Mr. Euler

$n-2$ cases of ternaries

& $(n-3)(n-2)$ cases of binaries.

In the sense of Mr. Bernoulli, one will have

$n$ cases of ternary sequences

& $n(n-4)$ cases of binaries.

§ 20. III. Let one extract now 4 numerals, this is the case of 4 files, which placed first all by level in sequence, give $n-3$ complete ranks. This is therefore the number of quadernary sequences following the method of Mr. Euler.

Completing the imperfect ranks in favor of the circular sequences, one will have under this assumption $n$ quaternaries.

In order to find the ternaries, it is necessary to imagine the first three files fixed in their state of sequence, & raise the fourth by one number, the first rank will be *Augustus Tiberius Caligula, Nero*.

One will have in all $n-4$ complete ranks, & two bundles to combine: the sum of the natural progression which expresses this combination is consequently $\frac{(n-4)(n-3)}{2}$.

According to Mr. Bernoulli, for the reasons which would be tedious to repeat, it must be $= \frac{(n-5)(n-4)}{2} + n - 5 = \frac{(n-2)(n-5)}{2}$. Reversing next the process, by setting the last three files in sequence, in order to combine them collectively with the first; the givens being besides the same, the result will be also: therefore the sum of the ternaries will be according to Mr. Euler$= (n-4)(n-3)$, according to Mr. Bernoulli, it will be consequently $= (n-2)(n-5)$. But, because of the circular sequences, after the last rank which is here *Vespasian Domitian Nerva Trajan*, one must again add the ranks of the two ternaries *Nerva Trajan Augustus*, & *Trajan Augustus Tiberius*, which are combined each $n-5$ times; therefore the sum of the ternary cases becomes $= n(n-5)$. There remains to find the binaries, under the two assumptions. One can in truth have them all in one stroke, by subtracting the found sequences, from the sum of the total sequences. But this would not clarify the method to proceed. We imagine therefore first the 4 files set two by two into separate sequences: the first rank will therefore be *Augustus Tiberius, Claudius Nero*, & the last *Vespasian Titus, Nerva Trajan*, the shortest file will be raised consequently by 4 levels above the first, this which gives $n-4$ complete ranks. Since each pair of files is considered immobile, & to compose only a single total, these are two bundles to combine. The sum of the combinations will be therefore $= \frac{(n-4)(n-3)}{2}$, & as here the sufficient reason of reversing this, since after the reversing we will have only the same individuals in each rank, who are actually; here is already all the possible binary double sequences in the estimation of Mr. Euler.

In that of Mr. Bernoulli, one will have for $n-5$ ranks, with the repetition $\frac{(n-5)(n-4)}{2}$: next we add to the last files the sequence *Trajan Augustus*, which is combined also $n-5$ times, the total will be $= \frac{n(n-5)}{2}$.

Leaving now the first two files in sequence, the third must remain in its actual position, because in descending from it by one level it would form the ternaries which we already have. It is therefore the fourth file which must be raised by a notch, in order to break the sequence which it formed with the third. We have therefore no more than $n-5$ complete ranks, & 3 bundles or files to combine. This is the case of the ternes. The progression of the triangular numbers which takes place here, give the sum of these combinations $= \frac{(n-5)(n-4)(n-3)}{1.2.3}$. But this operation can be repeated 3 times, because the bundle of the files in sequence, can occupy the right, the left & the center of the two isolated files. One will have, either *Augustus-Tiberius, Claudius Galba*, or *Augustus Caligula-Claudius Galba*, or *Augustus Caligula Nero-Galba*; & there is no reason to exclude the one or the other of these three cases: it is necessary therefore to triple the result which gives the sum of the simple binaries according to Mr. Euler$= \frac{(n-5)(n-4)(n-3)}{1.2.3}$.

In the sense of Mr. Bernoulli one will have therefore also 3 times

$$\frac{(n-6)(n-5)(n-4)}{1.2.3} + \frac{(n-6)(n-5)}{1.2}.$$

But this is not all; in the third operation where the bundle of sequences is to the right, composed of the two shortest files, it is necessary again to add the circular sequence *Trajan-Augustus*; which is combined also with $n-6$ ranks, this which gives $\frac{(n-6)(n-5)}{1.2}$, to add to that which above; the total sum of the simple binaries will be therefore in this sense $= \frac{(n-6)(n-5)(n-4)+4(n-6)(n-5)}{1.2} = \frac{n(n-6)(n-5)}{1.2}$.

We summarize these results.

If of $n$ numerals one extracts from them 4 at random, the number of possible cases of sequences in the sense in which Mr. Euler takes them will be

| | |
|---|---|
| $n-3$ | quaternary sequences, |
| $\frac{(n-3)(n-4)}{1}$ | ternary sequences, |
| $\frac{(n-3)(n-4)(n-4)}{1.2}$ | binary sequences. |

In the sense of the circular sequences one has

| | |
|---|---|
| $n$ | quaternary sequences, |
| $n(n-5)$ | ternary sequences, |
| $n(n-5)(n-5)$ | binary sequences. |

§ 21. We expedite again the case of a drawing of 5 numerals; & in order to avoid confusion, we set aside the circular sequences, for the method inverts that which I have followed until here.

I. All the files in sequence give $n-4$ complete ranks, therefore $n-4$ cases of quinary sequences.

II. The first 4 files in sequence of quaderne, & the last raised a notch higher than it was not, gives $n-5$ complete ranks, there are here only two bundles; thus one has $\frac{(n-5)(n-4)}{2}$ combinations, this operation reversed gives again as many of them: therefore in all $(n-5)(n-4)$ cases of quaternary sequences.

III. A. The first three files in sequence, & the fourth raised one level in order to make with the fifth a separate binary sequence, there will be again $(n-5)$ complete ranks: there are two bundles to combine, & the operation can be reversed by setting the bundle of the binaries to the left, & the ternaries to the right; one has therefore two times $\frac{(n-5)(n-4)}{2}$ combinations: therefore $(n-5)(n-4)$ cases of binary ternaries.

B. The first three files remaining in sequence, & the two others out of sequence by raising the last by one higher level, give $n-6$ ranks to combine, one has here 3 bundles: it is the case of the combinations of the ternes; the operation can vary in three ways, by making the sequence ternary, to the left, to the right, & in the middle of the two isolated files: one has therefore in total $\frac{(n-6)(n-5)(n-4)}{1.2}$ cases of simple ternaries.

IV. A. The binary ternaries are already contained under the ternaries.

B. The binary doubles can exist in 3 ways. Because the isolated file can be to the left, to the right, or in the middle of the two sequences: in the three cases there are $n-6$ ranks, & 3 bundles to combine; this gives for the triple combination $\frac{(n-6)(n-5)(n-4)}{1.2}$ cases of binary doubles.

C. Finally the simple binaries assume 3 files raised out of sequence, & consequently $n-7$ ranks; of which the first will be *Augustus Tiberius, Claudius, Galba, Vitellius*. But the sequence which is here to the left, can also occupy the right, or the two intermediate places, this which makes 4 different arrangements; & as one has 4 bundles to combine, this gives in all $\frac{(n-7)(n-6)(n-5)(n-4)}{1.2.3}$ for all the cases of simple binaries.

§ 22. We seek again these same results for the circular sequences, always in the case of 5 files; & beginning with the binary sequences.

I. A. The simple binaries suppose $n-7$ ranks, 4 bundles to combine, & 4 different arrangements; one has therefore in this calculation, 4 times $\frac{(n-8)(n-7)(n-6)(n-5)}{1.2.3.4}+\frac{(n-8)(n-7)(n-6)}{1.2.3}=\frac{(n-1)(n-8)(n-7)(n-6)}{1.2.3}$, and when the sequence is formed by the two shortest files, it joins again that of *Trajan Augustus*, which combined with $n-8$ ranks of 3 files gives $\frac{(n-8)(n-7)(n-6)}{1.2.3}$: therefore the sum of the simple binary cases is $=\frac{n(n-8)(n-7)(n-6)}{1.2.3}$.

B. The double binaries suppose $n-6$ complete ranks, & 3 transpositions of three bundles, this which gives three times

$$\frac{(n-7)(n-6)(n-5)}{1.2}+\frac{(n-7)(n-6)}{1.2}=\frac{(n-1)(n-7)(n-6)}{1.2}.$$

But there are two arrangements where the last two files are in sequence, namely when the isolated file is to the left, & when it is found in the middle of the two bundles; in each of these cases one has the sequence *Trajan Augustus* to add, & to combine with $n-7$ ranks of two files, this which gives $\frac{2(n-7)(n-6)}{1.2}$: therefore the total of these cases of binary doubles is $\frac{n(n-7)(n-6)}{1.2}$.

II. A. the simple ternaries are represented by the rank, *Augustus-Tiberius-Caligula, Nero, Otho*, three bundles, three transpositions, & $n-6$ ranks; to which is added, when the ternary is in the last files, the two ternaries, *Nerva-Trajan-Augustus, Trajan-Augustus-Tiberius*, to combine with two files of $n-7$ ranks: the sum of the simple ternaries must therefore be here $\frac{(n-7)(n-6)(n-5)}{1.2} + \frac{3(n-7)(n-6)}{1.2} + \frac{2(n-7)(n-6)}{1.2} = \frac{n(n-7)(n-6)}{1.2}$.

B. The binary ternaries have only two positions, *Augustus-Tiberius-Caligula, Nero Galba*, & *Augustus-Tiberius, Claudius Nero Galba*. Each has $n-5$ ranks, & consequently here $n-6$ with the repetition, there are 2 bundles to combine; in the first position one must add the binary sequence *Trajan Augustus*, & in the second position the two ternaries, *Nerva Trajan Augustus*, & *Trajan Augustus Tiberius*: one has therefore in all,

$$(n-6)(n-5) + 2(n-6) + (n-6) + 2(n-6) = n(n-6).$$

III. The remaining quaternaries to find: There are two bundles to combine with $n-5$ complete ranks, there are no more than two different positions: when the quaternary is to the right of the isolated file, it is stretched out three quaternaries, *Domitian-Nerva-Trajan-Augustus*, *Nerva-Trajan-Augustus-Tiberius*, & *Trajan-Augustus-Tiberius-Caligula*, which are combined with $n-6$ ranks: the total of these quaternaries is therefore

$$(n-6)(n-5) + 2(n-6) + 3(n-6) = n(n-6).$$

V. The number of quinaries is $= n$, this is that which one has already.

Summarizing again these results; it is certain that if of $n$ numerals, one extracts 5 of them at random, the number of possible cases of sequences will be under the assumption of Mr. Euler

| | |
|---|---|
| $n-4$ | quinary sequences |
| $\frac{(n-4)(n-5)}{1}$ | quaternary sequences |
| $\frac{(n-4)(n-4)(n-5)}{1.2}$ | ternary sequences |
| $\frac{(n-4)(n-4)(n-5)(n-6)}{1.2.3}$ | binary sequences |

And in the sense of Mr. Bernoulli

| | |
|---|---|
| $n$ | quinary sequences |
| $\frac{n(n-6)}{1}$ | quaternary sequences |
| $\frac{n(n-5)(n-6)}{1.2}$ | ternary sequences |
| $\frac{n(n-5)(n-6)(n-7)}{1.2.3}$ | binary sequences |

This Memoir being already too extended, I reserve for the following the formulas of the general problem, & the curious observations which result from it.