Risk is random: The magic of the d'Alembert

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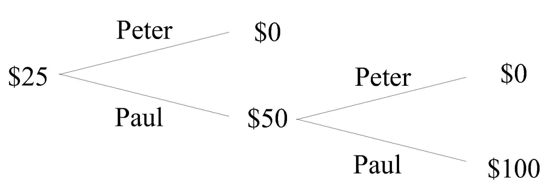

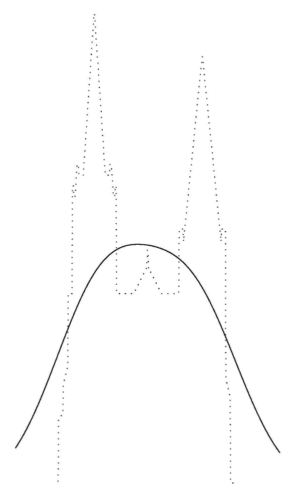

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Abstract

The most common bets in 19th-century casinos were even-money bets on red or black in Roulette or Trente et Quarante. Many casino gamblers allowed themselves to be persuaded that they could make money for sure in these games by following betting systems such as the d'Alembert. What made these systems so seductive? Part of the answer is that some of the systems, including the d'Alembert, can give bettors a very high probability of winning a small or moderate amount. But there is also a more subtle aspect of the seduction. When the systems do win, their return on investment — the gain relative to the amount of money the bettor has to take out of their pocket and put on the table to cover their bets — can be astonishingly high. Systems such as *le tiers et le tout*, which offer a large gain when they do win rather than a high probability of winning, also typically have a high upside return on investment. In order to understand these high returns on investment, we need to recognize that the denominator — the amount invested — is random, as it depends on how successive bets come out.

In this article, we compare some systems on their return on investment and their success in hiding their pitfalls. Systems that provide a moderate gain with a very high probability seem to accomplish this by stopping when they are ahead and more generally by betting less when they are ahead or at least have just won, while betting more when they are behind or have just lost. For historical reasons, we call this *martingaling*. Among martingales, the d'Alembert seems especially good at making an impressive return on investment quickly, encouraging gamblers' hope that they can use it so gingerly as to avoid the possible large losses, and this may explain why its popularity was so durable.

We also discuss the lessons that this aspect of gambling can have for evaluating success in business and finance and for evaluating the results of statistical testing.

1 Introduction

There is a vast literature, dating from the end of the 18th century, on betting systems for casino games: books and pamphlets that teach the systems to gamblers, others that chronicle and deplore the ruin to which their use leads. Mathematicians have occasionally joined the chorus of disapproval, providing general explanations of the futility of such systems and studying some of the systems in detail. Much of this mathematical work can be found in Chapter 8 and in the bibliography of Stewart Ethier's *The Doctrine of Chances* [15].[1]

In this article, we ask a question not so often addressed: what made various betting systems so attractive to novice gamblers? Because the systems were often touted by casinos to encourage more gambling, we can sharpen the question by asking what aspects of the systems helped blind the casinos' customers to the risks they were taking.

The most popular casino betting systems in the 18th and 19th centuries were systems for even-money bets, and we will focus on such systems in this article. The systems were not always strategies in the modern game-theoretic sense, for often they left some decisions to the player, most importantly the decision when to stop betting. In order to study the systems mathematically, we turn them into strategies by imposing a stopping rule; sometimes the strategy will simply play a fixed number of rounds, sometimes it will stop early if a certain gain is attained.

One way of understanding the phenomena that we study in this paper is to think about the expected return on an investment. According to the definition usually used in finance, the *return* on an investment K is the ratio

$$R := \frac{E}{K}, \tag{1}$$

where E is the net gain. If K is a constant but E is a random variable with $\mathbf{E}(E) = 0$, then (1) implies that $\mathbf{E}(R) = 0$. But in the context of casino betting, and also in the context of many financial investments, K itself is random and negatively correlated with E, because the bettor or investor often increases the investment after initial losses. In this case E may be positively correlated with $1/K$, and when $\mathbf{E}(E) = 0$, we have

$$\mathbf{E}(R) = \mathbf{Cov}\left(E, \frac{1}{K}\right) > 0.$$

The same tendency to increase the investment to counterbalance initial losses may make the probability $\mathbf{P}(E > 0)$ very high, at the cost of making losses very large when they do happen.

[1] Other landmarks in the mathematical study of casino games include treatises by Marcel Boll (1936 [6]), Lester Dubins and Leonard Savage (1965 [13]), and Richard Epstein (1965 [14]). There is also a rigorous mathematical literature on "advantage play" — i.e., strategies for playing in situations where the player has an advantage over the house [16, 18, 30], but this is not our topic here.

In the next section, §2, we describe some of the best known betting systems. In §3, we develop a small general theory, which includes Ville's inequality and definitions of important characteristics of betting strategies: the *mean upside return* and the *martingaling ratio*. In §4, we compare a number of systems on their mean upside return, martingaling ratio, and other properties that make them attractive. Finally, in §5, we discuss the relevance of our betting concepts to finance theory and statistical theory.

In an appendix, §6, we discuss the word *martingale*, proofs of Ville's inequality, Doob's conservation of fairness, the games Trente et Quarante and Roulette, and the typical limits casinos put on bets in these games. The betting systems that were most popular in the 19th century were for even-money bets on red or black in Trente et Quarante and Roulette. Although precise details of the rules of these games are not crucial for the themes of this article, the games do constitute the historical context of the systems we are discussing, and so we describe the bets on red and black in these games and the house's advantage in each case.

2 Some classical betting systems for red and black

We now describe some classical betting systems. We begin with the simplest and most classical: the martingale and the paroli. The martingale doubles its bet after every loss, while the paroli doubles its bet after every win. As we explain in §2.1, the two systems have opposite results: the martingale produces a small gain with high probability, while the paroli produces a large gain with low probability.

In §§2.2 and 2.3, we look at two systems that were featured in the earliest surviving books in which more complicated systems were described, G. N. Bertrand's *Trente-un dévoilé* [4] and Alexandre Toussaint de Gaigne's *Mon histoire au Trente-un* [10]. These are the d'Alembert and *le tiers et le tout*. The d'Alembert can be thought of as a disguised or moderate martingale, while *le tiers et le tout* can be thought of as a disguised or moderate paroli.

We conclude, in §2.4, by describing a system introduced by Émile Borel in 1949 [7]. Borel's system is completely impractical for casino play, but it is of interest because of its mathematical transparency.

The betting systems touted by casinos and described by authors about systems usually involved more than varying the size of the bet. In a game of red and black, for example, you can switch from betting on one color to betting on the other. The casino's victim might be encouraged to think that he or she could gain further advantage by betting on the color that was hot or the color that was overdue. In the French literature on red and black, a betting system was sometimes thought of as having two components: the *massage*, which told you how to vary the size of the bet (a unit bet was called a *masse*), and the *marche* or *attaque*, which told you how to vary the color on which you bet and perhaps how to wait to bet; see [24] for details. The *marche* or *attaque* made no difference when the casino's advertised odds were accurate. Even some of the authors who touted systems seem to have realized this, for they put greater emphasis on the *massage* and left the details of the *marche* or *attaque* more to the reader; see for example the 1902 manual by G. d'Albigny [9]. Mathematicians who write about betting systems often simply ignore the *marche* and *attaque*, assuming that a betting system consists only of a *massage*. We will follow this mathematical tradition, and to fix ideas we will suppose that the player always makes a nonnegative bet on black.

2.1 Martingales and parolis

We can distinguish two fantasies that might bring people into a casino. Some casino goers are looking for a sure thing, a way of betting that is certain to win, even if what it wins is modest. If you could win for sure, you could do it over and over and make a living or more. Other casino goers resemble the typical buyer of a lottery ticket. They acknowledge that they are unlikely to win, but they want to try to their luck, and they are looking for a big win.

The simplest recipes that engage these two fantasies in a game of red and black are the martingale and the paroli, respectively.

- The *martingale* is a purportedly sure way of netting 1 monetary unit. You begin by betting 1 unit. (On black; recall that all bets discussed in this article are on black.) If you win, you are done. If you lose, you double your bet, and you keep doubling until you win. In theory, you are sure to win eventually. If you lose k times before winning, then your losses add up to
$$1 + 2 + \ldots + 2^{k-1} = 2^k - 1,$$
but your win of 2^k leaves you with the promised net gain of 1. (See §6.1 and [19] for a discussion of how this use of the word *martingale*, which goes back to the 18th century, is related to the current use of the term in probability theory.)

- The *paroli* doubles not when you lose but when you win. You may begin the same way, by betting 1 unit. You keep betting 1 unit until you win. Then you double your bet, and if you win you double again. If you win k times in a row and then stop betting, you have won $2^k - 1$, from which you must deduct at most a few units that you lost before your first win. You have parlayed a few units of capital into a very large haul. (The word *paroli* came into the European languages from Neapolitan Italian. American gamblers made it into *parlay*.)

For the casino owner touting betting systems to potential customers, the problem with the martingale and the paroli is that their pitfalls are too obvious.

- If the martingaler could keep on doubling indefinitely, he would eventually win. But how long can he afford to double, and how long would the casino permit it? Casinos always limit the size of a single bet, and the limit is usually not much more than 1000 times the minimum bet. As $2^{10} = 1024$, you cannot double more than 10 times. If you do double up to 10 times, you will hit the limit and lose 1023 units about once in 1024 times. Even when you do win, you will often be investing a lot to win a little. If you lose and hence double 6 times before winning, you will have invested 63 units just to walk away with a 1 unit profit.

- The paroli seems less dangerous. Each time you double, you again risk the single unit with which you won the first bet, but otherwise you are only risking what you have won. As many a rogue has explained, you are martingaling with the casino's money. But the futility of the exercise is too obvious. The probability of winning 10 times in a row and netting 1023 units being only 1 in 1024, you can expect to do it only about once in 1024 tries, and you lose 1 unit every time you fail.

There many other systems, however, that behave like the martingale or the paroli while obscuring their dangers. By 1800, it was common to call any system that makes money with high probability a martingale. We will follow this usage here, and we will also call any system that gives some probability for a large gain a paroli. When we examine the systems that have been proposed, we find that the martingales achieve their effect by betting more when they are behind or when they have just lost and perhaps also less when they are ahead or have just won,[2] while the parolis do the opposite. But these stratagems work only within some limits, and these limits have never been fully delineated.

If you have the time and money to play long enough, you can make a martingale out of almost any betting system simply by stopping when you are ahead. This is the simplest way of betting less when you are ahead. Decide what net gain you want, keep playing until you get it, and then stop. For example, if the banker has no advantage and you repeatedly make a unit bet until you are 3 units ahead or have bet 200 times, whichever comes first, you will reach your goal of 3 units about 83% of the time.

2.2 The d'Alembert

How can the casino owner or his confederate hide from the martingaler the risks he or she is taking? A gambler's willingness to start betting without deciding how much they are willing to risk opens up a big opportunity for deception. A mathematician can calculate the amount of capital required to implement a well-defined strategy, i.e., the maximum amount that you would ever need to take out of your pocket to play it through to the end no matter

[2] The notorious Doctor Petiot wrote that the *absolute and principal rule of martingales* is that you should never increase your bet after a gain [21, p. 24]. Pretending to provide a means of escape for Jews from Nazi-occupied France, Petiot murdered them with injections he passed off as preventive medicine, then plundered their cash and valuables. He completed his book, *Le hasard vaincu... Les lois des martingales*, while in detention and published it, handwritten, during his trial. He was convicted and executed by the guillotine in 1946.

how the successive rounds come out. This is also the maximum amount the strategy can lose. If your strategy is simply to make a one-dollar bet on every round for 200 rounds, then this maximum is obviously \$200. But most gamblers would be unable to calculate the maximum required to play a more complicated betting system for a given number of rounds, and they might easily play the system many times without ever experiencing losses near this maximum. If they obtain a good return on their investment each time (a reasonable net gain relative to what they have actually taken out of their pocket), they may easily succumb to the delusion that they have discovered a sure thing or that they have chanced into a durable streak of good luck.

One of the oldest casino betting systems is the one that calls for the player to begin with a unit bet, increase the size of the bet by one unit after every loss, and decrease it by one unit, except when it is already only one unit, after every win. Since the late 19th century, this system has been called the *d'Alembert*, on the erroneous theory that it was invented by the mathematician Jean Le Rond d'Alembert. Whatever its origin, the d'Alembert was already one of the most popular systems by the 1790s.

As Bertrand reported in 1798, admirers of the d'Alembert thought it was bound to succeed because the numbers of wins and losses will eventually equalize. Consider what happens when you play the d'Alembert after having bet one unit and lost. Whenever the number of subsequent wins and losses, including this first loss, are equal, your net gain will be equal to the number of wins (or, equivalently, equal to one-half the number of rounds played).[3]

The flaw in this venerable argument for the d'Alembert is that you may run out of money before your wins and losses equalize. But as Jacques-Joseph Boreux explained in 1820 [29], the system often appears to work in practice. If you set a modest goal and play the d'Alembert repeatedly, playing each time until the goal is reached or you are forced to stop (because you run out of money, the séance is over, or your proposed bet exceeds the house limit), you can expect a string of relatively quick successes before you ever fail, and the successes will not usually require taking too much money out of your pocket.[4]

This article undertakes to quantify the magic of the d'Alembert and to compare its ability to seduce to that of some other well known betting systems. Here are some features on which we will make the comparison:

1. The probability with which the system achieves a given modest gain when there is a given limit on how much the player has to lose.

2. How quickly it achieves this goal when it does achieve it. This is important not only because quick success can impress the player but also

[3]This is obvious when the losses and wins alternate, because after each loss, the subsequent win is one unit larger. To complete the proof, it suffices to notice that interchanging the results of two consecutive rounds changes neither the net gain for the two rounds nor the bet size for the round following the two. For a more formal proof, see Ethier [15, pp. 290–291]. Ethier makes the d'Alembert into a strategy by assuming that (1) if the first bet is a win, you stop immediately, and (2) if the first bet is a loss, you stop the first time the number of wins and losses equalize. But this is only one possible stopping rule.

[4]For more on the history of the d'Alembert and other betting systems, see [24].

because the house's advantage, being a percentage of the total money bet, extracts more from the player as more bets are made.

3. The average return on the capital actually risked (the amount taken out of pocket to finance the bets) when it achieves the goal. We call this the *mean upside return*.

4. How well the magnitude of the potential loss is hidden.

We also try to understand the general features of betting systems that allow them to excel on these different dimensions of seduction.

2.3 *Le tiers et le tout* (the third and the whole)

The French name *le tiers et le tout* can be translated as "the third and the whole". To begin, you take 3 units out of your pocket and put them on the table, and you bet 1 of the 3 units. As long as you are winning you always bet a third of what you have on the table. If you win the first bet, for example, you have 4 units on the table, and so you bet $4/3$. But the first time you lose, you bet the whole of what you have left on the table. If you lose this second bet, you take 3 more units out of your pocket to start over.[5]

If you could continue to bet in this way for a long time, you would eventually win many times without losing twice in a row. From a streak in which you win w times and lose l times, with $w > l$ and no losses twice in a role, you will have a net gain of $3 \times (4/3)^{w-l} - 3$ units. When $w - l = 10$, this is approximately 50 units. Before this streak, you may have lost twice in a row one or more times, with a net loss of 3 units every time. So you hope that this happened only a few times. Simulations show that if your capital and the house's limits on bets were great enough that you could play for 200 rounds, you would gain rather than lose money only about 3% of the time, but when you did gain, you would gain on average over 3500 units while taking only about 100 out of your pocket.

The popularity of *le tiers et le tout*, like that of a lottery, derived largely from publicity about individuals who won big with it. Beginning in the early 1860s, many books on betting systems mention a Spaniard named Garcia, said to have won a fantastic amount of money at the Homburg casino, only to return and lose it the following year. Some versions of the story say that he used no system ([1, p. 70], for example); others contend that he played *le tiers et le tout* ([5, p. 147], for example).

2.4 Borel's martingale

Suppose you begin with a unit bet, then multiply the bet size by $1 - \alpha$ every time you win and by $1 + \alpha$ every time you lose, where $0 < \alpha < 1$. This system

[5] This description of *le tiers et le tout* conflicts with our general picture, in which the player never puts on the table more than needed to cover the bets he actually makes. When we calculate how much the system actually requires the player to invest, we will align with the general picture by assuming that the player first puts only one of the three units on the table, adding the second two only when needed.

could not be played in any casino, because it requires precise and sometimes very small bets, but it is of considerable theoretical interest. It was introduced by Émile Borel in a note published in 1949 [7]. Following Ethier [15, p. 114], we call it *Borel's martingale*.

Borel showed that the system's net gain after n rounds is

$$\frac{1}{\alpha}\big(1-(1-\alpha)^X(1+\alpha)^{n-X}\big), \tag{2}$$

where X is the number of wins. Borel's simple proof of (2) used induction on the number of rounds. We can also prove it easily by the following calculation.

Proof. Let x_k, for $k=1,\ldots,n$, be equal to 1 if the kth bet loses, -1 if it wins. Then the net gain after n rounds is

$$\begin{aligned}\sum_{k=1}^{n}(-x_k)\prod_{j=1}^{k-1}(1+\alpha x_j) &= \sum_{k=1}^{n} x_k\left(-\sum_{J\subseteq\{1,\ldots,k-1\}}\prod_{j\in J}\alpha x_j\right)\\ &= \frac{1}{\alpha}\left(-\sum_{\emptyset\neq J\subseteq\{1,\ldots,n\}}\prod_{j\in J}\alpha x_j\right)\\ &= \frac{1}{\alpha}\left(1-\sum_{J\subseteq\{1,\ldots,n\}}\prod_{j\in J}\alpha x_j\right),\end{aligned}$$

and this is equal to (2). □

Borel pointed out that when there is no house advantage and play continues indefinitely, $(1-\alpha)^X(1+\alpha)^{n-X}$ tends almost surely to zero, and hence the net gain (2) tends almost surely to its upper bound $1/\alpha$. The smaller α, the greater this gain but the slower the convergence to $1/\alpha$. Borel suggested that the most interesting values for α might be in the neighborhood of $1/2$ or $1/3$.

3 A small theory of sequential betting

Now we explore some general features of sequential betting. In §3.1 we state and discuss Ville's inequality, which tells how the probabilities for what a betting strategy can achieve is limited by the amount of capital it deploys. In §3.2 we discuss the randomness of the amount a player risks and how this randomness permits the existence of betting strategies that have a high mean upside return. In §3.3 we introduce the *martingaling ratio*, a simple measure of the extent to which a betting strategy resembles a martingale.

The theory of this section applies far beyond the games of red and black considered in the preceding section, where successive outcomes are independent and binary. It applies to any betting game in which a player sees outcomes

$y_1, y_2, \ldots$ in sequence and the house invites him to bet on what y_{n+1} will be after seeing $y_1, \ldots, y_n$.[6] We assume agreement on a probability distribution $\mathbf{P}$ for the outcomes, or rather for a sequence $Y_1, Y_2, \ldots$ of random variables such that y_n is the realized value of Y_n. The Y_n may or may not be mutually independent. We write $\mathbf{E}$ for $\mathbf{P}$'s expectation operator. We write $\mathbf{P}_n$ for the conditional probability distribution for $Y_{n+1}, Y_{n+2}, \ldots$ given $y_1, \ldots, y_n$, and $\mathbf{E}_n$ for its expectation operator.[7]

We emphasize the ideal case where the house has no advantage, i.e., any bet the house offers on the nth round comes down to offering a payoff $G_n = G_n(Y_n)$ such that $\mathbf{E}_{n-1}(G_n) = 0$. We also consider the case where the house has an advantage, i.e., $\mathbf{E}_{n-1}(G_n) < 0$.

3.1 Ville's inequality

Write $\mathcal{K}_0$ for the player's initial capital and $\mathcal{K}_n$ for his capital after the nth round of play. In practice the player will stop betting after some finite number of rounds, after which $\mathcal{K}_n$ will not change. But for the moment we do not assume this.

Suppose the player fixes $\mathcal{K}_0$ and adopts a betting strategy $\mathcal{S}$ that takes account only of previous outcomes. This makes his bet on Y_n a function of $y_1, \ldots, y_{n-1}$, and it thereby makes his capital $\mathcal{K}_n$ a random variable, a function of the random variables $Y_1, \ldots, Y_n$. The sequence $(\mathcal{K}_n)_{n \geq 0}$ is then a stochastic process. On our assumption that the house has no advantage, so that each gain G_n satisfies $\mathbf{E}_{n-1}(G_n) = 0$ and $\mathcal{K}_n = \mathcal{K}_{n-1} + G_n$, we have

$$\mathbf{E}_{n-1}(\mathcal{K}_n) = \mathcal{K}_{n-1}. \tag{3}$$

So $(\mathcal{K}_n)_{n \geq 0}$ is a martingale in the sense in which this word is used modern probability theory. When the prices at which the strategy bets on each round n are not necessarily given by $\mathbf{E}_{n-1}$ but may instead by less favorable, $(\mathcal{K}_n)_{n \geq 0}$ is still a supermartingale. (See §6.1.)

In this context, we can assert that a strategy must be prepared to risk a lot to have a high probability of a modest gain. This assertion is made precise by Ville's inequality, which says that if the initial capital $\mathcal{K}_0$ is positive and $\mathcal{S}$ never risks more than this initial capital $\mathcal{K}_0$ no matter how its bets come out, so that all the random variables $\mathcal{K}_n$ are nonnegative, then

$$\mathbf{P}\left(\mathcal{K}_n \geq c\mathcal{K}_0 \text{ for some } n\right) \leq \frac{1}{c} \tag{4}$$

[6] The pretense that the sequence continues forever is harmless, as our story is over whenever the bettor stops betting.

[7] We assume that $\mathbf{P}_n$ and $\mathbf{E}_n$ exist. They do exist in our games of red and black, where $Y_1, Y_2, \ldots$ are mutually independent and $\mathbf{P}_n$ and $\mathbf{E}_n$ do not even depend on n. But in the general theory the objects $\mathbf{P}_n$ and $\mathbf{E}_n$ depend on the first n outcomes and are therefore themselves random objects.

for any positive constant c. The inequality continues to hold when the house has an advantage. We discuss its proof in §6.2.[8]

Suppose, for example, that you want a strategy that will gain at least one unit with probability at least $1 - p$, where p is small. Adding one unit to $\mathcal{K}_0$ is the same as multiplying it by $c := (\mathcal{K}_0 + 1)/\mathcal{K}_0$, and (4) says that this can happen with probability $1 - p$ only if

$$1 - p \leq \frac{\mathcal{K}_0}{\mathcal{K}_0 + 1},$$

or

$$\mathcal{K}_0 \geq \frac{1 - p}{p}.$$

For example, to have a 99% probability of gaining \$1, a strategy needs to start with at least \$99.

For a given betting strategy $\mathcal{S}$, let us write $\mathcal{K}_\mathcal{S}$ for the least value of the initial capital $\mathcal{K}_0$ such that the martingale $(\mathcal{K}_n)_{n\geq 0}$ is nonnegative. We call $\mathcal{K}_\mathcal{S}$ *the capital required by* $\mathcal{S}$. If $\mathcal{S}$ always stops betting after some finite number of rounds N (possibly depending on the previous outcomes and thus random), its net gain is the random variable

$$\mathtt{Gain}_\mathcal{S} := \mathcal{K}_N - \mathcal{K}_0.$$

We call

$$\frac{\mathtt{Gain}_\mathcal{S}}{\mathcal{K}_\mathcal{S}}$$

$\mathcal{S}$'s *return on the capital it requires*. Ville's inequality tells us that the best such return we can expect with probability $1 - p$, for p small, is approximately p.

3.2 The randomness of risk

For the sake of clarity, let us insist on the convention that when a player makes a bet he must have on the table the money to cover it, that he leaves any winnings on the table until the end of his betting, and that he adds to what he has on the table on each round only enough to cover his next bet. In general, covering the bet means covering the greatest loss that might result from the bet. When we consider only even-money bets, this greatest loss is simply the amount of the bet — the amount he will win if he wins the bet and lose if he loses the bet. It does not matter where the player gets the money. He may take it out of his pocket or borrow it from a friend or from the casino itself.

Under these assumptions, the total amount of money the player puts on the table in the course of a finite sequence of bets is well defined and depends only

[8] Proofs of the inequality are given by [31, p. 100], [12, p. 314], and [28, p. 132]. See [27, p. 170] for a vast generalization of the inequality, under which the probabilities on each round are not necessarily conditional probabilities from some initial probability distribution, the bets authorized on each round may fall short of defining a complete probability distribution for that round's outcome, and you may follow a strategy using information that comes from outside the game or simply decide how to bet as you go along.

on the bets the player makes and their outcomes. Let us write $\texttt{risk}$ for this total investment, the amount he risks. Let us also write $\texttt{gain}$ for his net gain from a given finite sequence of bets, and let us write

$$\texttt{return} := \frac{\texttt{gain}}{\texttt{risk}}. \tag{5}$$

This quantity is *the player's return on his investment*. It can be positive or negative, but it cannot be less than -1. The player cannot lose more than he risks.

There is nothing novel about (5). It is the standard definition of return on investment in finance. But in the financial literature — textbooks, research articles, and the popular press — it is most often assumed, sometimes accurately and sometimes falsely, that the denominator in (5) is fixed, like the amount of a certificate of deposit in a savings bank. Here, in the casino, this denominator obviously has a random aspect. It depends in part on the random outcomes in the game.

If the player follows a strategy $\mathcal{S}$ that tells him how to move on each round as a function of the outcomes of previous rounds and always tells him to stop after a finite number of rounds, then the numerator and denominator of (5) are both completely determined by the random outcomes in the game. We may then write

$$\texttt{Return}_{\mathcal{S}} := \frac{\texttt{Gain}_{\mathcal{S}}}{\texttt{Risk}_{\mathcal{S}}}, \tag{6}$$

with capital letters indicating that all three quantities are random variables.

As we will see, many strategies more than double the money the player risks with high probability. In this case the *mean return*, $\mathbf{E}(\texttt{Return}_{\mathcal{S}})$, which combines a low probability for values between -1 and 2 with a high probability for values greater than 2, can itself be 2 or more. Such a return, 200%, would be very impressive or even astonishing in many financial contexts. It also contrasts sharply with what Ville's inequality told us about the return on required capital that a strategy can guarantee with high probability. A strategy's required capital is a fixed number, whereas the investment risked by a player playing the strategy is a random number, bounded above by the capital required by the strategy.

When we are studying parlaying strategies, which lose money with high probability but win a lot when they do win, we will also be interested in

$$\mathbf{U}_{\mathcal{S}} := \mathbf{E}\left(\texttt{Return}_{\mathcal{S}} \mid \texttt{Gain}_{\mathcal{S}} > 0\right),$$

which we call $\mathcal{S}$'s *mean upside return*. Because $\texttt{Return}_{\mathcal{S}}$ is bounded below by -1, $\mathbf{U}_{\mathcal{S}}$ differs little from $\mathbf{E}(\texttt{Return}_{\mathcal{S}})$ when $\mathbf{P}(\texttt{Gain}_{\mathcal{S}} > 0)$ and $\mathbf{E}(\texttt{Return}_{\mathcal{S}})$ are both high.

3.3 Martingaling

Consider again a strategy $\mathcal{S}$ that always stops playing after a finite number of rounds, so that $\texttt{Gain}_{\mathcal{S}}$ is well defined. When the house has no advantage,

$\mathbf{E}(\texttt{Gain}_{\mathcal{S}}) = 0$. We can expand this equation to

$$\mathbf{P}(\texttt{Gain}_{\mathcal{S}} > 0)\,\mathbf{E}\left(\texttt{Gain}_{\mathcal{S}} \mid \texttt{Gain}_{\mathcal{S}} > 0\right) \\ + \mathbf{P}(\texttt{Gain}_{\mathcal{S}} < 0)\,\mathbf{E}\left(\texttt{Gain}_{\mathcal{S}} \mid \texttt{Gain}_{\mathcal{S}} < 0\right) = 0. \tag{7}$$

Assuming that $\mathcal{S}$ calls for some betting, so that $\mathbf{P}(\texttt{Gain}_{\mathcal{S}} = 0) < 1$, (7) implies that

$$\frac{\mathbf{P}(\texttt{Gain}_{\mathcal{S}} > 0)}{\mathbf{P}(\texttt{Gain}_{\mathcal{S}} < 0)} = \frac{\mathbf{E}\left(-\texttt{Gain}_{\mathcal{S}} \mid \texttt{Gain}_{\mathcal{S}} < 0\right)}{\mathbf{E}\left(\texttt{Gain}_{\mathcal{S}} \mid \texttt{Gain}_{\mathcal{S}} > 0\right)}. \tag{8}$$

In words: the odds in favor of gaining rather than losing are equal to the ratio of the expected size of the loss when you lose to the expected value of the gain when you gain.

Building on the examples of §2, we say that a gambler who seeks a high probability of a net gain at the cost of risking a much larger net loss is martingaling. We say that the betting strategy $\mathcal{S}$ is *martingaling* if the ratio

$$\mathbf{M}_{\mathcal{S}} := \frac{\mathbf{P}(\texttt{Gain}_{\mathcal{S}} > 0)}{\mathbf{P}(\texttt{Gain}_{\mathcal{S}} < 0)} \tag{9}$$

is greater than 1, and we call this ratio $\mathcal{S}$'s *martingaling ratio*.

If the house has an advantage, so that $\mathbf{E}(\texttt{Gain}_{\mathcal{S}}) < 0$, then we obtain the inequality

$$\frac{\mathbf{P}(\texttt{Gain}_{\mathcal{S}} > 0)}{\mathbf{P}(\texttt{Gain}_{\mathcal{S}} < 0)} < \frac{\mathbf{E}\left(-\texttt{Gain}_{\mathcal{S}} \mid \texttt{Gain}_{\mathcal{S}} < 0\right)}{\mathbf{E}\left(\texttt{Gain}_{\mathcal{S}} \mid \texttt{Gain}_{\mathcal{S}} > 0\right)} \tag{10}$$

instead of the equality (8). Let us call the right-hand side of (10) $\mathcal{S}$'s *gain-loss ratio* and give it its own symbol:

$$\mathbf{GL}_{\mathcal{S}} := \frac{\mathbf{E}\left(-\texttt{Gain}_{\mathcal{S}} \mid \texttt{Gain}_{\mathcal{S}} < 0\right)}{\mathbf{E}\left(\texttt{Gain}_{\mathcal{S}} \mid \texttt{Gain}_{\mathcal{S}} > 0\right)}.$$

As we will see, the house's advantage may not hamper very much a player's ability to obtain a large martingaling ratio. Whatever the player does to obtain a given martingaling ratio when the house has no advantage, doing even more of it may give him a similar martingaling ratio when the house has an advantage. A martingale that doubles up to 5 times will win more than 98% of the time if the house has no advantage. In American Roulette, where the player wins only $18/38$ of the time, this probability drops below 98%, but the player can get it above 98% again by doubling up to 6 times. The real downside is that the gain-loss ratio will be much worse.

We now look at the martingaling coefficients of a few simple strategies. For simplicity, we assume again that the house has no advantage.

The first example, with two rounds of betting, gives us some insight into how betting more after you lose and less after you win can give you a high martingaling ratio.

Example 1. *Consider two rounds of game of red and black with even-money bets, with no advantage for the house. As usual, we always bet on black, and we bet* 1 *on the first round. On the second round, we bet* $1-\alpha$ *if we win on the first,* $1+\beta$ *if we lose, where* $0<\alpha,\beta<1$*. There are four equally likely possibilities:*

$$\begin{array}{ccccc} & gain_1 & & gain_2 & \mathtt{Gain} \\ win & 1 & win & 1-\alpha & 2-\alpha>0 \\ win & 1 & lose & -1+\alpha & \alpha>0 \\ lose & -1 & win & 1+\beta & \beta>0 \\ lose & -1 & lose & -1-\beta & -2+\beta<0 \end{array} \tag{11}$$

The total gain, our random variable $\mathtt{Gain}$*, being positive in three out of four equally likely cases, the martingaling ratio is* 3*.*

Suppose the house does have an advantage, and that each round is a win for the player with probability $p<1/2$*. Under this assumption, the system wins at least one round and hence makes money with probability* $1-(1-p)^2$ *and loses both rounds and loses money with probability* $(1-p)^2$*, for a martingaling ratio of*

$$\frac{1-(1-p)^2}{(1-p)^2}=\frac{1}{(1-p)^2}-1, \tag{12}$$

which remains greater than 1 *provided*

$$p>1-\sqrt{\frac{1}{2}}\approx 0.29.$$

This example should not be over-interpreted, because the martingaling effect may disappear as additional rounds are played. Martingalers would like to believe that they can martingale repeatedly, but repeated martingaling can quickly become no martingaling at all. This is illustrated by the following example.

Example 2. *Suppose* X *is a random variable that is equal to* 1 *with probability* $2/3$ *and equal to* -2 *with probability* $1/3$*. Then* $\mathbf{E}(X)=0$*, and we can think of buying* X *for zero as a betting strategy. The martingaling ratio for this strategy is* $\mathbf{M}_X=2$*. Suppose* Y *is another random variable, independent of* X *and with the same probability distribution. Then* $\mathbf{M}_{X+Y}=4/5$*.*

Example 3. *Let* $\mathcal{S}$ *be the strategy that follows Borel's martingale for a fixed number* N *of rounds. As in §2.4, write* X *for the number of wins. We see from (2) that* $\mathcal{S}$*'s martingaling ratio is*

$$\mathbf{M}_{\mathcal{S}}=\frac{\mathbf{P}\big((1-\alpha)^X(1+\alpha)^{N-X}<1\big)}{\mathbf{P}\big((1-\alpha)^X(1+\alpha)^{N-X}>1\big)} \tag{13}$$

$$=\frac{\mathbf{P}\left(\frac{X}{N-X}>\frac{\ln(1+\alpha)}{-\ln(1-\alpha)}\right)}{\mathbf{P}\left(\frac{X}{N-X}<\frac{\ln(1+\alpha)}{-\ln(1-\alpha)}\right)}. \tag{14}$$

Because the logarithm is concave,

$$\frac{\ln(1+\alpha)}{-\ln(1-\alpha)} < 1. \tag{15}$$

When there is no house advantage, the median value of the random variable $X/(N-X)$ *is* 1*. So* $\mathbf{M}_{\mathcal{S}} > 1$*.*

When $\alpha = 1/2$*, the quantity* (15) *is approximately* 0.58*, and it follows that the martingaling ratio* (13) *will remain greater than* 1 *for even the most extreme house advantage. The convergence noted by Borel will also continue to hold. If the player's chance of winning on each round is* p*, then for large* n *his net gain* (2) *becomes*

$$2\left(1-\left(\frac{3^{\frac{n-X}{n}}}{2}\right)^n\right) \approx 2\left(1-\left(\frac{3^{1-p}}{2}\right)^n\right),$$

and this tends to 2 *provided only that* $3^{1-p} < 2$ *or, approximately,* $p \approx 0.37$*. It is evident, by continuity, that the remarkable properties of Borel's martingale will still hold if the size of the bet is multiplied by* $1-\alpha$ *after a win and by* $1+\beta$ *after a loss, provided that* α *and* β *are not too different.*

4 Simulations

The empirical comparison of betting systems is delicate, because, as we have emphasized, most systems do not tell the player when to stop and hence are not well-defined strategies. Even systems that do have a stopping rule are typically played repeatedly and hence require, for practical evaluation, an additional stopping rule. It is also important to remember that most players of systems do not play a strategy. They start betting without knowing when they will stop, and on any round they may deviate from the system or switch to another system on a whim. Nonetheless, we can gain insight by imposing various stopping rules on systems and examining how the resulting strategies perform.

It is impossible to survey thoroughly the vast range of betting systems that have been created. There was already a huge literature in French on betting systems during the 19th century, and by the end of the century the creation of new systems was a pastime as well as a business. The Belgian poet Maurice Maeterlinck, writing in 1919, noted that the *La Revue de Monte-Carlo* had published a new system in every issue since its founding in 1905. Books full of systems also appeared in English in the early 20th century. One "A. T. Player" published a book with 35 systems in 1911, followed by a second edition with 140 systems in 1925 [22]. Here we will consider only systems we have already discussed along with two additional systems, the Labouchere and the Oscar.

4.1 A small catalog of strategies

All the strategies we consider are assumed to stop betting after 200 rounds unless instructed to stop sooner. Sometimes a strategy is assigned a parameter

G, indicating that it must stop as soon as it attains a gain of G. Some also have other parameters telling them when to stop. The maximum of 200 rounds is chosen to reflect the number of rounds that a gambler would be able to play in a day or two; see §6.5.

We use the following acronyms for strategies based on systems already discussed.

- CB (constant bet). Always make a unit bet until a stopping criterion is satisfied.
- PM (pure martingale), parameter D. Make a unit bet on the first round. If you win, stop. If you lose, double your bet up to D times until you win. Then stop. See §2.1.
- SM (small martingales), parameter D. Play the pure martingale with parameter D repeatedly, for some small value of D such as 2, 3, or 4. If the goal isn't reached after doubling D times, start the strategy over with a 1 unit bet.
- PP (pure paroli), parameter D. Make unit bets until you win. Then double your bet up to D times so long as you are winning. Then stop. See §2.1.
- SM (small parolis), parameter D. Play the pure paroli with parameter D repeatedly, for some small value of D such as 2, 3, or 4.
- DA (d'Alembert). See §2.2.
- TT (*le tiers et le tout*). See §2.3.
- BM (Borel's martingale), parameter α. See §2.4.

We will also consider strategies based on two other systems, the Labouchere and the Oscar.

- LA (Labouchere). Begin with the list of numbers $3, 4, 5, 6, 7$, which add to 25. You will add and remove numbers from the list as you play. On each round you bet the sum of the number at the beginning and the number at the end of the current list (So initially you bet $3 + 7 = 10$. If there is only a single number remaining on the list, you bet that amount.) When you win, erase the two numbers (or single number) you bet from the list. When you lose, add the amount lost as an additional number at the end of the list. Usually the list will eventually all be erased, and at that point you have a net gain of 25 and you stop. Henry Labouchere, a British politician with great inherited wealth made this system well known when he boasted, in 1877, that it had "invariably" paid the expenses for his visits to the casino at Homburg [15, p. 313], [24]. A key feature of this system is that you will always have a net gain equal to the sum of the list when the list is all erased. A different list may be used as a starting point, but we only consider versions of the original proposed list $3, 4, 5, 6, 7$ here, scaled in accordance with the desired gain of a strategy.

$\mathcal{S}$	$\mathbf{P}(\texttt{Gain}_\mathcal{S} > 0)$	$\mathbf{M}_\mathcal{S}$	$\mathbf{GL}_\mathcal{S}$	$\mathcal{K}_\mathcal{S}$	$\mathbf{U}_\mathcal{S}$
No house advantage					
constant bet	94.4%	17.2	16.9	200	0.68
martingale, $D = 10$	99.9%	1457	1957	1,024	0.60
paroli, $D = 10$	97.0%	31.9	32.2	200	0.80
d'Alembert	99.6%	264	280	20,100	0.62
le tiers et le tout	98.7%	77.0	70.1	300	0.31
American Roulette (win $18/38$ **of the time)**					
constant bet	94.2%	16.3	16.6	200	0.68
martingale, $D = 10$	99.9%	1665	2028	1,024	0.60
paroli, $D = 10$	96.7%	28.9	30.4	200	0.80
d'Alembert	99.6%	226	243	20,100	0.63
le tiers et le tout	98.5%	67.2	67.8	300	0.31

Table 1: Comparing five strategies for a séance of 200 rounds. Here no limit on the player's capital or the size of a bet is imposed, but the worst case loss $\mathcal{K}_\mathcal{S}$ is shown. All five strategies begin with a 1 unit bet and play until either achieving a profit of at least 1 unit or the sequence of 200 rounds is complete. Note that the observed difference between $\mathbf{M}_\mathcal{S}$ and $\mathbf{GL}_\mathcal{S}$ in the case of no house advantage reflects random sampling error.

- OS (Oscar). Start with a unit bet, increase it by one unit every time you win. This tends to be a slight paroli, but it becomes a martingale when it is played with a stopping goal G. Ethier portrays the system as a 20th-century invention in the United States [15, p. 314].

4.2 Numerical results

The most seductive betting systems give the impression of "easy money" by (1) achieving a high probability of reaching a goal of G units, (2) multiplying on average the capital actually risked (taken out of pocket) by an impressive factor in those cases where the goal is reached, and (3) reaching the above gain in as short a time as possible. Criteria (1) and (2) are measured by the martingaling ratio and mean upside return, respectively.

Table 1 shows the martingaling ratios and mean upside returns for five different strategies based on some of the systems we have been discussing. A quick glance at this table demonstrates the initial allure of these gambling systems: the statistics do not vary much between the case with no house advantage and a small house advantage of 5-6%. In performing further comparisons of betting systems below, we may therefore restrict attention to the case in which there is no house advantage.

The gambler who plays a martingale that doubles up to 10 times has a 99.9% probability to realize a profit of 1 unit. If he plays the d'Alembert until reaching a gain of 1 unit, he will come out ahead 99.6% of the time. By comparison, he comes out ahead just 94.4% of the time by making constant bets. The results are similar in the case where the house has an advantage. We see that for a given strategy the ratio $\mathbf{GL}_{\mathcal{S}}$ exceeds $\mathbf{M}_{\mathcal{S}}$ when the house has an advantage, but in general by a relatively small margin. A strategy's gain-loss ratio in the case when the house a small advantage can be approximated by the martingaling ratio.

Fittingly, the martingale has the largest martingaling ratio among the strategies we consider: for a gambler with unlimited capital, the martingale offers a 99.9% probability of gain, trading off a small chance of a large loss for a high chance of a small gain. The martingale thus satisfies one key criterion of a seductive betting system: it is almost guaranteed to yield a profit on any given implementation. For the purpose of yielding a profit of 1 unit, the martingale also performs well in terms of mean upside return. From the table, we see that the paroli and *le tiers et le tout* have lower martingaling ratios than both the martingale and d'Alembert. *Le tiers et le tout* also has a lower mean upside return.

For the purpose of netting of a profit of 1 unit, the martingale seems as good a strategy as any other, but the d'Alembert is competitive. How does the d'Alembert achieve both impressive returns and comparable probability of winning to the martingale? Notice that the initial capital requirement $\mathcal{K}_0$ of the d'Alembert is 20,100, which far exceeds that of any other strategy in the list. But this capital requirement is concealed in the description of the d'Alembert, which asks the bettor to increment his bets by only +1 or -1 after each round. The pure martingale, by contrast, has a much lower capital requirement but fails to disguise that requirement in its protocol to double the bet after each loss. How does the comparison between d'Alembert and other strategies change if the goal increases to 2, 3 or many more units?

Some further numerical results for strategies based on other betting systems, namely the Labouchere and Oscar, and for different parameters of the goal G and stopping rule are shown in Table 2. As in Table 1, we write $\mathbf{M}_{\mathcal{S}}$ and $\mathbf{U}_{\mathcal{S}}$ for the martingaling ratio and mean upside return of a strategy $\mathcal{S}$, respectively. We now write $\mathbf{S}_{\mathcal{S}}$ for the average amount spent (taken out of pocket) in cases where the goal is reached and $\mathbf{N}_{\mathcal{S}}$ for the average number of rounds played in cases where the goal is reached. In addition to these statistics, we include columns to measure how well each strategy does at achieving the stated goal:

- probability of reaching within $\epsilon > 0$ of the goal given a positive gain,

$$\mathbf{G}_{\mathcal{S}}(\epsilon) := \mathbf{P}(\mathtt{Gain}_{\mathcal{S}} \geq G \mid \mathtt{Gain}_{\mathcal{S}} > 0) = \frac{\mathbf{P}(\mathtt{Gain}_{\mathcal{S}} \geq G)}{\mathbf{P}(\mathtt{Gain}_{\mathcal{S}} > 0)},$$

  and

- the average ratio $\mathtt{Gain}_{\mathcal{S}}/G$ of the gain relative to the goal over the trials in which $\mathtt{Gain}_{\mathcal{S}} > 0$,

$$\mathbf{R}_{\mathcal{S}} := \mathbf{E}\left(\frac{\mathtt{Gain}_{\mathcal{S}}}{G} \mid \mathtt{Gain}_{\mathcal{S}} > 0\right).$$

$\mathcal{S}$	$\mathbf{P}(\texttt{Gain}_{\mathcal{S}} > 0)$	$\mathbf{M}_{\mathcal{S}}$	$\mathbf{U}_{\mathcal{S}}$	$\mathbf{S}_{\mathcal{S}}$	$\mathbf{N}_{\mathcal{S}}$	$\mathbf{G}_{\mathcal{S}}(\epsilon)$	$\mathbf{R}_{\mathcal{S}}$
G = 2:							
CB	88.8%	7.9	1.13	3.5	21	100.0%	1
SM ($D = 2$)	94.4%	17.0	0.89	5.5	12	99.9%	0.999
SP ($D = 2$)	90.9%	10.0	1.31	5	18	99.9%	1.4
DA	99.2%	128	0.87	10.5	8	100.0%	1.19
LA	99.9%	>10000	1.09	28.5	11	100.0%	1
OS	99.5%	232	1.00	10	18	99.9%	0.999
BM	99.9%	> 10000	0.93	23.5	197	73.4%	0.999
TT	96.8%	30	0.62	9.5	14	100.0%	1.8
G = 3:							
CB	83.5%	5.1	1.48	4	31	99.8%	0.999
SM ($D = 2$)	91.7%	11.0	1.12	6.5	17	99.9%	0.999
SP ($D = 2$)	87.7%	7.1	1.57	5.5	22	99.9%	1.26
DA	98.9%	89.7	1.06	14	11	100.0%	1.17
LA	99.9%	>10000	1.09	36.5	5	100.0%	1
OS	99.1%	108	1.25	13	28	99.9%	0.999
BM	99.1%	110	1.16	19.5	200	0.6%	0.994
TT	95.6%	21.5	0.85	11.5	18	100.0%	1.7
G = 4:							
CB	78.2%	3.6	1.79	4.5	42	99.6%	0.998
SM ($D = 2$)	88.9%	8.0	1.29	7.5	22	99.8%	0.999
SP ($D = 2$)	83.1%	5.0	1.97	7	31	99.8%	1.31
DA	98.5%	68	1.20	17	15	99.9%	1.15
LA	99.9%	>10000	1.09	63.5	11	100.0%	1
OS	98.1%	53	1.45	16	41	99.7%	0.998
BM	95.9%	23	1.40	15	200	0.0%	0.963
TT	94.5%	17	0.95	13	22	100.0%	1.58
G = 20:							
CB	47.4%	0.90	4.38	6	176	33.7%	0.60
SM ($D = 2$)	57.8%	1.4	3.3	11	109	81.6%	0.883
SP ($D = 2$)	52.2%	1.1	4.74	9.5	114	78.0%	0.884
DA	92.6%	13	2.20	44.5	57	99.8%	1.06
LA	99.9%	>10000	2.08	78	10	100.0%	1
OS	77.6%	3.5	2.80	23.5	182	28.4%	0.542
BM	63.6%	1.7	2.98	7.5	200	0.00%	0.431
TT	79.4%	3.8	2.53	27	53	99.9%	1.36

Table 2: Simulation results for constant bet (CB), small martingale (SM) with $D = 2$, small paroli (SP) with $D = 2$, d'Alembert (DA), Labouchere (LA) with sequence $L = \frac{G}{25}(3, 4, 5, 6, 7)$, Oscar (OS), *tiers et le tout* (TT) and Borel's martingale (BM) with $\alpha = 1/G$ in bets in red and black with no house advantage. All strategies play until reaching G or a maximum of 200 rounds, whichever arrives first. The initial capital is chosen large enough to cover all possible outcomes of a given 200 round session. Note: the choice of L in Labouchere scales the bet sizes in order to be consistent with the desired goal of G in each case. The choice of α in Borel's martingale is chosen to be consistent with the strategy's projected gain of $1/\alpha$ when the number of rounds is large. Choice of $\epsilon = 10^{-10}$ because some strategies, such as BM, can only achieve the goal in the asymptotic limit.

We add the extra parameter $\epsilon > 0$ in $\mathbf{G}(\epsilon)$ because some strategies, such as Borel's martingale, can only achieve the goal asymptotically, but can reach a gain within a small tolerance ϵ of G in a finite number of steps. For our simulations, we have chosen $\epsilon = 10^{-10}$. (Note that we omited these columns from Table 1: $\mathbf{P}(G \mid \texttt{Gain}_{\mathcal{S}} > 0) = 100\%$ and $\mathbf{E}(\texttt{Gain}_{\mathcal{S}}/G \mid \texttt{Gain}_{\mathcal{S}} > 0) = 1$ for the systems in Table 1 for $G = 1$ because each of these has a minimum profit of 1 on the event $\texttt{Gain}_{\mathcal{S}} > 0$.)

Table 2 shows these values for 100,000 simulations of a 200 round sequence for each of the listed strategies. When simulating the Labouchere we assume the initial list of $L = (3, 4, 5, 6, 7)$ is scaled so that it sums to the goal G. This way, the key feature of the Labouchere is realized in that it reaches its desired profit exactly when the list becomes empty. We also assume that all strategies begin with sufficient capital to either achieve the goal of G or complete the session of all 200 rounds of play. Under these assumptions, the Labouchere will produce the same statistics regardless of the goal, as Table 2 shows.

Notice that even though all strategies stop after achieving a gain of G, some strategies, e.g., small parolis, d'Alembert and *le tiers et le tout*, are able to produce gains in excess of G because their bet sizes prior to stopping depend only on the previous outcomes and do not specifically depend on the goal. Although these strategies can net a larger than targeted profit, they incur more risk than is needed to attain the stated goal. This aspect of each strategy causes the martingaling ratio to be lower than it would be if bets were otherwise chosen to target the specific goal.

For $G = 2, 3, 4$, a single implementation of the d'Alembert has a 98-99% probability of netting a profit, which on average is about 115-120% the size of the stated goal. When an event has probability 98%, it will likely happen on each of the first few trials. The probability that it will happen on the first 4 trials, for example, is 92%. The probability that it will happen on the first 10 trials is more than 80%. Thus, a gambler who plays the d'Alembert strategy is more than 80% likely to net a profit on each of the first 10 times he implements it, each time approximately doubling the capital put into play.

By inspecting Table 2, we see that the Labouchere with an initial list scaled with the goal G achieves a consistently high martingaling ratio and comparable mean upside return to the d'Alembert. Because the list L is chosen to exactly achieve the goal G at the moment of stopping, the Labouchere does not overshoot the goal as the d'Alembert does. On these metrics, the Labouchere may be regarded as a superior system in principle, but inferior in practice because of the difficulty of remembering the running list L throughout a long sequence of play. In addition, we notice that the Labouchere generally requires a larger out-of-pocket risk $\mathbf{S}_{\mathcal{S}}$ and, thus, makes the risks of the strategy more apparent to the gambler. As a byproduct of this higher probability of gain and larger out-of-pocket risk, however, the Labouchere tends to achieve its goal in much fewer rounds of play, especially as the goal increases.

Of the other strategies considered, only the Oscar competes with Labouchere and d'Alembert in terms of the martingaling ratio for small goals of 2, 3 or 4, and also offers a consistently higher mean return on risk. The Oscar is less

$\mathcal{S}$	$\mathbf{P}(\texttt{Gain}_{\mathcal{S}} > 0)$	$\mathbf{M}_{\mathcal{S}}$	$\mathbf{U}_{\mathcal{S}}$	$\mathbf{S}_{\mathcal{S}}$	$\mathbf{N}_{\mathcal{S}}$	$\mathbf{G}_{\mathcal{S}}(\epsilon)$	$\mathbf{R}_{\mathcal{S}}$
G = 1:							
DA	98.8%	80	0.63	5	3	100.0%	1.22
LA	98.8%	86	1.05	4.5	10	100.0%	1
TT	98.6%	72	0.31	6.0	6	100.0%	1.53
G = 5:							
DA	94.4%	16.5	1.36	13.5	14	100.0%	1.11
LA	94.4%	17.5	1.10	13	9	100.0%	1
TT	91.7%	11.0	1.20	14.5	26	100.0%	1.66
G = 20:							
DA	81.4%	4.4	2.45	26	48	100.0%	1.05
LA	80.8%	4.2	1.35	29	7	100.0%	1
TT	77.9%	3.5	2.52	26	52	100.0%	1.35
G = 50:							
DA	63.9%	1.8	3.9	36	108	99.6%	1.02
LA	59.2%	1.5	1.6	42	5	100.0%	1
TT	60.9%	1.6	4.3	34	72	99.8%	1.25

Table 3: Simulation results for d'Alembert (DA), Labouchere (LA) with $L = \frac{G}{25}(3, 4, 5, 6, 7)$ and *le tiers et le tout* (TT) for goals $G = 1, 5, 20, 50$. For all strategies we assume initial capital of $\mathcal{K}_0 = 100$. All strategies play until reaching the goal G, going broke, or a maximum of 200 rounds, whichever arrives first. Note: the choice of L in the Labouchere scales the bet sizes in order to be consistent with the desired goal of G in each case. For $\mathbf{G}_{\mathcal{S}}(\epsilon)$ we choose $\epsilon = 0$.

effective at achieving the larger goal of 20 units, in that it has a smaller success probability of about 77% and requires an average of more than 180 rounds of play to achieve the goal. As we might expect from its description, *le tiers et le tout* performs comparatively poorly for achieving small gains with high probability because it is primarily designed to achieve a large gain with small probability. We investigate its performance for achieving more ambitious gains in Table 3.

From these experiments, the gambler with unlimited wealth who faces no limitations on bet size may prefer the Labouchere to the other systems listed. In practice, the gambler will be limited in both initial capital and the maximum amount he can bet. Moreover, a gambler with a large amount of capital may be unsatisfied with a modest goal of just a few units, as the simulations in Table 2 are for achieving a goal of 2, 3, 4 or 20 units for a gambler whose bankroll is effectively unlimited in size. In Table 3, we compare the d'Alembert, Labouchere and *le tiers et le tout* with an initial capital of 100 units and a goal G of 1, 5, 20 and 50 units, representing appreciable fractions of their total bankrolls. In this case, we see that d'Alembert and Labouchere perform similarly in terms of martingaling ratio. When the goal is small ($G = 1$), Labouchere outperforms on mean upside return. But as the goal increases the mean upside return of d'Alembert overtakes Labouchere. As the goal increases, the average ratio

$\mathcal{S}$	$\mathbf{P}(\mathtt{Gain}_{\mathcal{S}} > 0)$	G	$\mathbf{U}_{\mathcal{S}}$	$\mathbf{S}_{\mathcal{S}}$	$\mathbf{N}_{\mathcal{S}}$	$\mathbf{G}_{\mathcal{S}}(\epsilon)$	$\mathbf{R}_{\mathcal{S}}$
$\mathbf{M}_{\mathcal{S}} = 1.5$:							
DA	60.1%	60	5.33	38	128	98.1%	1.01
LA	60.0%	47	2.58	39.5	6	100.0%	1
$\mathbf{M}_{\mathcal{S}} = 2.5$:							
DA	71.4%	36	4.27	32	80	100.0%	1.03
LA	72.1%	31	2.38	35	6	100.0%	1
$\mathbf{M}_{\mathcal{S}} = 5$:							
DA	83.0%	18	3.34	25	43	100.0%	1.05
LA	83.0%	17	2.23	26	7	100.0%	1
$\mathbf{M}_{\mathcal{S}} = 10$:							
DA	91.5%	8	2.65	17	22	100.0%	1.08
LA	92.4%	7	2.11	16	9	100.0%	1

Table 4: Simulation results for comparing d'Alembert (DA) and Labouchere (LA) with $L = \frac{G}{25}(3, 4, 5, 6, 7)$ for fixed values of martingaling coefficient $\mathbf{M}_{\mathcal{S}}$. For all strategies we assume initial capital of $\mathcal{K}_0 = 100$. All strategies play until reaching the goal G, going broke, or a maximum of 200 rounds, whichever arrives first. Note: to equalize the martingaling coefficients across different strategies, we must allow the goal G to vary. For $\mathbf{G}_{\mathcal{S}}(\epsilon)$ we choose $\epsilon = 0$.

$\mathtt{Gain}_{\mathcal{S}}/G$ (on the event $\mathtt{Gain}_{\mathcal{S}} > 0$) approaches 1, its value for Labouchere. For the large goal of $G = 50$, *le tiers et le tout* outperforms Labouchere in terms of martingaling ratio and outperforms both d'Alembert and Labouchere on mean upside return.

Table 3 thus reveals the magic of the d'Alembert. Though it lags behind in terms of mean upside return for small goals, it matches Labouchere in terms of martingaling ratio and requires fewer rounds of play on average to achieve the goal. But the d'Alembert improves consistently as the goal becomes more ambitious. For a desired gain of at least 5% of the initial capital, d'Alembert requires more time to reach the goal, but does so with higher frequency and by requiring the gambler to take less money out of his pocket. d'Alembert also competes with *le tiers et le tout* on mean upside return, even though TT is designed to produce large gains.

Our final comparison in Table 4 compares the d'Alembert and Labouchere for fixed values of the martingaling coefficient. For a given value of $\mathbf{M}_{\mathcal{S}}$, we compute the corresponding goal G, mean upside return, and other statistics based on an assumed initial capital of 100 for sessions that last until the gambler either plays 200 rounds or goes broke. For a given martingaling coefficient, we note that the mean upside return of DA exceeds that of LA in all cases, as does the total number of rounds required to reach the goal and the average ratio of the gain to the goal on the event that the goal is achieved. Interestingly, we find that the amount taken out of pocket is roughly the same for a given level of $\mathbf{M}_{\mathcal{S}}$, with the d'Alembert requiring a slightly smaller average spent $\mathbf{S}_{\mathcal{S}}$ in a few cases. Table 4 thus reveals the magic of the d'Alembert, which achieves a higher goal and higher mean upside return than Labouchere for a fixed probability of gain, while requiring about the same amount of risked capital on average.

5 Conclusion

As we have learned, there can be a vast difference between the capital a strategy requires and the amount a player risks. A strategy cannot be genuinely implemented unless it begins with enough capital to cover its bets in the worst case. A player is required only to cover his bets as he makes them, and even if he thinks he is following a strategy, the capital he needs will be random and usually much less than the capital required in the strategy's worst case. Because its denominator is random, the player's return can be surprisingly high on any given implementation of the strategy.

We have also learned something about how martingaling works. By modestly increasing your bet after losses and modestly decreasing it after wins, you can assure a very high return with very high probability. Similar but more aggressive strategies can achieve even better results when you have more capital to risk.

What do these lessons tell us about life outside the casino?

5.1 Lessons for finance and business

Finance theory often aspires to be a theory of investment, not a theory of speculation. To the extent that this encourages financiers and businessmen to behave accordingly, a citizen can only applaud. But the failure to teach about pure gambling can blind the public to what is happening when financiers and businessmen, wittingly or unwittingly, use strategies resembling the ones we have been studying.

Financial theorists acknowledge that the arithmetic average of successive returns on investment is a crude measure of financial success; it can paint a rosy picture quite different from the more meaningful geometric average. But the arithmetic average return and its theoretical counterpart, the mean return, remain the dominant metrics when money managers and corporate executives boast of their performance. It is difficult, moreover, to find any acknowledgement in the academic or popular press that this metric can be affected by the randomness of its denominator, even though examples of this affect abound. The randomness of capital risked is plainly in view whenever an enterprise seeks additional capital to overcome a setback.

Corporate executives are also surely not immune to the temptations of martingaling and parlaying when they make internal investment decisions. How often has a dashing corporate executive, celebrated for their exceptional success, simply played a paroli? How often, when a corporate executive succeeds by risking more when investments go bad, are we seeing the seduction of the d'Alembert? The public, including investors and politicians, might be better served if journalists reporting on these successes understood the mathematics of gambling.

5.2 Lessons for statistical testing

The mathematical theory of probability began as a theory of betting. But proponents of its use for statistical inference, beginning with Jacob Bernoulli at the beginning of the 18th century [2, 3], have steadily sought to purge it of this heritage. Here, too, the failure to teach about betting has hampered understanding and contributed to today's replication crisis.

This is particularly evident in the case of statistical testing, where we still struggle to account for multiple testing in reasonable but principled ways. See [25] for a recent discussion, building on [26, 27], of how understanding and practice can be enhanced by generalizing statistical tests, which correspond to all-or-nothing bets against a probabilistic hypothesis, to bets that are not all-or-nothing. The amount by which such a bet multiplies its capital can then be used as a measure of the evidence against the hypothesis, much as p-values are now used. One obstacle to putting this idea into practice is the scientific public's limited understanding of the difference between amount of capital required by a strategy and the random risk of a bettor. When statisticians choose successive tests depending on the outcome of previous tests and ignore the resulting randomness, they are making an error analogous to that of a martingaler. We can avoid this pitfall, even when we cannot settle on a strategy for continued testing in advance of all testing, if we begin with fixed (notional) capital for testing and allow later bets only when they can be covered with what remains of that capital together with winnings from earlier bets.

Harry Crane goes further, arguing that because mathematical probability is just as much a theory of gambling now as it was at its origin, probabilistic claims attain a real-world meaning only when those who assert them incur the risks they imply [8]. Crane calls this the *Fundamental Principle of Probability*. Casinos attract investors by making claims about the probabilities against their customers. Aficionados of betting systems claim that their systems can beat these probabilities. The risking of real money on the outcomes makes both of these claims meaningful in a way that is lacking for almost all scientific claims based on p-values or other statistical methods. Crane attributes today's replication crisis to widespread neglect of this principle. Under the FPP, if either side is wrong, they will suffer financial loss in the long run. In the absence of the FPP, and the tangible, real-world risks associated with it, statisticians are incentivized to tout their analyses without consequence for faulty or misleading results, just as casino operators tout gambling systems on the false promise of riches.

6 Appendix

6.1 Concerning the word *martingale*

Many mathematicians know *martingale* as a technical term in probability theory. Any sequence of random variables $(\mathcal{K}_n)_{n\geq 0}$ satisfying (3) is called a martingale.[9] As we saw in §3.1, this condition is satisfied when $(\mathcal{K}_n)_{n\geq 0}$ is the capital process resulting from beginning with capital $\mathcal{K}_0$ and following a strategy for betting on successive outcomes $Y_1, Y_2, \ldots$, provided that the strategy uses only the outcomes of the preceding rounds and the bets are at prices given by expected values conditional on those preceding outcomes. But in this article we have been using *martingale* in an older sense, as the name not for a strategy's capital process but for the strategy itself. How are the two uses of the word related historically?

The mathematician's use of the word derives, of course, from the gambler's use of it. As Roger Mansuy explains in [19], the practice of doubling one's bet in order to cover one's loss was already called a *martingale* in the 18th century, probably because such daring play was associated with the inhabitants of Martigues, a French city on the Mediterranean. In the 19th century, the name was extended to other betting systems, and by the middle of the 19th century some people were calling any betting system in a game of many rounds a martingale. It was Jean Ville, in his 1939 book [31], who began to call the capital process a martingale. He made this change because the capital processes are in a one-to-one correspondence with the strategies, and the capital processes are often simpler to describe.

Although the point is sometimes not even mentioned when martingales are taught as part of probability theory, any sequence of random variables satisfying (3) can be interpreted as the capital process for a betting strategy in some fair game — i.e., some game in which the odds at which one bets on the nth round are given by the same conditional probability distribution that is used to calculate the expected value. The modern mathematical theory of martingales is, in this sense, still a theory about betting.

Why did 19th-century gamblers tend to call all betting systems martingales? Perhaps it was because any system can usually be turned into a strategy that is nearly certain to net a modest amount by the simple device of stopping when you are ahead.

As we mentioned at the beginning of §2, betting systems usually involved more than the *massage*, which tells how much to bet. They also had a *marche* or *attaque*, which told how to vary the color on which to bet and perhaps how to wait to bet. But even those touting the systems often knew that the *marche* or *attaque* does not matter. So in practice, *martingale* was often a synonym for *massage*.

[9] When (3) holds possibly only with the equal sign replaced by $\leq$, $(\mathcal{K}_n)_{n\geq 0}$ is called a *supermartingale*. This neologism was introduced in the 1960s; see [27, p. 29].

6.2 How to prove Ville's inequality

We begin with Markov's inequality, which says that when $c > 0$ and X is a nonnegative random variable with $0 < \mathbf{E}(X) < \infty$,

$$\mathbf{P}(X \geq c\mathbf{E}(X)) \leq \frac{1}{c}.$$

Andrei Markov proved this inequality in 1900 [20] using a simple calculation:

$$\mathbf{P}(X \geq c\mathbf{E}(X)) = \mathbf{E}\left(\mathbf{1}_{X \geq c\mathbf{E}(X)}\right) \leq \mathbf{E}\left(\frac{X}{c\mathbf{E}(X)}\right) = \frac{1}{c},$$

where $\mathbf{1}_E$, whenever E is an event, designates the random variable that has the value 1 when E happens and 0 when E fails.

We also need the rule of iterated expectation, which tells us, for example, that for all $n \in \mathbb{N}$,

$$\mathbf{E}(X) = \mathbf{E}(\mathbf{E}_{n-1}(X)). \tag{16}$$

Combining this with (3) for the process $(\mathcal{K}_n)_{n\geq 0}$, we find that

$$\mathbf{E}(\mathcal{K}_n) = \mathcal{K}_0 \tag{17}$$

for all $n \in \mathbb{N}$. This is true whenever $(\mathcal{K}_n)_{n\geq 0}$ is the capital process for a betting strategy.

Given a positive real number $\mathcal{K}_0$, a strategy $\mathcal{S}$ that produces a nonnegative capital process $\mathcal{K}_0, \mathcal{K}_1, \ldots$ when it begins with $\mathcal{K}_0$, and a positive real number c, consider the alternative strategy that also begins with $\mathcal{K}_0$ and makes the same bets as $\mathcal{S}$ except that it makes no bets after the first round n for which $\mathcal{K}_n \geq c\mathcal{K}_0$. This strategy's capital process, say $\mathcal{K}_0, \mathcal{K}'_1, \mathcal{K}'_2, \ldots$, is nonnegative and satisfies $\mathbf{E}(\mathcal{K}'_n) = \mathcal{K}_0$ for all $n \in \mathbb{N}$. So Markov's inequality yields

$$\mathbf{P}(\mathcal{K}_k \geq c\mathcal{K}_0 \text{ for some } k \leq n) = \mathbf{P}(K'_n \geq c\mathcal{K}_0) \leq \frac{1}{c}$$

for all $n \in \mathbb{N}$. But the increasing sequence of events

$$\{\mathcal{K}_k \geq c\mathcal{K}_0 \text{ for some } k \leq n\}$$

has $\{\mathcal{K}_n \geq cK_0 \text{ for some } n\}$ as its union, and hence the axiom of continuity (a.k.a. countable additivity) yields Ville's inequality, (4).

This entire argument also works, with some equal signs replaced by inequalities, when $(\mathcal{K}_n)_{nge0}$ is merely a supermartingale and not a martingale. It is essentially the same argument that Ville gave in 1939 [31]. It is rigorous only if the conditional expectation operators $\mathbf{E}_n$ are well defined and satisfy the rule of iterated expectation. These conditions are certainly satisfied in the casino, on the assumption that $Y_1, Y_2, \ldots$ are independent. For the general case, Ville relied on Paul Lévy's theory of conditional probability, which assumed that any random variable can be represented as a real-valued function on the unit interval $[0, 1]$. Joseph L. Doob later provided a proof of Ville's inequality that relied instead on Kolmogorov's theory, which makes no topological assumptions but obtains conditional expectations that are defined only modulo a set of measure zero and may not have all the properties that they enjoy in elementary probability theory. With this formulation, Ville's inequality becomes a topic in advanced measure-theoretic probability, and this has made it less widely known than it deserves to be. It is not mentioned, for example, in Ethier's admirably comprehensive treatment of probability in games of chance [15].

6.3 Doob's conservation of fairness

In 1936, before Ville published his inequality, Doob published a different theorem concerning the futility of betting strategies [11]. This theorem says that you cannot change the probability distribution for the net gain of a betting strategy by omitting some rounds. If, for example, you have a strategy for betting on three rounds, it makes no difference if you play it on the second, eighth, or tenth rounds rather than the first three. It also makes no difference if you take account of the outcomes of previous rounds in deciding whether to play on a given round.

The conclusion of Doob's theorem was dubbed *the conservation of fairness* by Dubins and Savage in 1965 [13, p. 5], and the theorem has been generalized in various ways; see [15, §8.2, p. 315]. But Ville's inequality may be more incisive as a statement of the futility of betting strategies. Even after we have learned from Doob that the selection of rounds has not changed the probabilities, and that the expected net gain of a strategy remains zero (or less than zero if the house has an advantage), we still learn something from Ville's inequality: we learn how unlikely it is that the strategy will multiply the capital it requires substantially. Doob's theorem adds nothing to the scope of this conclusion, because any selection of rounds on which to bet can be considered part of the strategy to which Ville's inequality is applied.

Doob and Ville were both responding to the work of Richard von Mises, who contended that the futility of selecting rounds on which to bet was an essential aspect of mathematical probability. Doob saw his theorem as a way of reducing von Mises's insight to a minor result within mathematical probability as axiomatized by Andrei Kolmogorov. Ville, on the other hand, saw von Mises's insight as incomplete. Instead of saying that any way of selecting rounds on which to bet is futile, we should say that any strategy for betting, so long as what you can risk is bounded, is futile. As it happened, the view that Kolmogorov's axioms say all that needs to be said about the foundations of probability carried the day after World War II, and Ville's insights about the role betting can play in understanding probability theory were subsequently neglected.

6.4 Games of red and black

The flipping or tossing of a coin has long been the canonical example of a random experiment with binary outcomes, the two outcomes being "heads" and "tails". The frequenters of casinos are seldom willing, however, to bet on the outcomes of coin flips. It is too easy for a clever fellow with a quick hand to cheat one way or another.

Most of the betting systems discussed in this article were developed instead for casino games where the gambler could make even-money bets on "red" or "black". The French playing cards that we still use have equal numbers of red and black cards in a deck — 26 red and and 26 black. Perhaps the first bets on "red" or "black" were bets on a card drawn from such a deck. But cheating is easy here too.

In the late 18th century, when the earliest surviving books about betting systems were written, the most popular game of "red" and "black" was Trente et Quarante, where cheating is remarkably difficult. This game continued to be popular in the 19th century and is still played in some European casinos, but its place as the most popular game was taken, from the beginning of the 19th century, by Roulette. Here we provide some information about these games and the banker's advantage in each. For more historical details, see [24].

American Roulette The idea of spinning a wheel with alternately red and black zones and betting on whether the spin will stop on red or black goes back many centuries, but it was only around 1800 that such a game became popular in European casinos. Its popularity can be attributed to the precision with which the wheel was constructed and the variety of bets it allowed. Each time the wheel is spun, a ball falls in one of 38 pockets, which are supposed to be equally likely. Two of the pockets are labeled '0' and '00' and are the house's: the house collects all money bet on red and black when the ball falls into one of them. The other 36 pockets are alternately red and black and are numbered 1 through 36; the numbers permit other bets but are irrelevant to a bet on red and black. As 18 of the 36 are red and 18 are black, the probability of a bet on black or a bet on red winning is

$$\frac{18}{38} \approx 0.473684211.$$

The house's *advantage*, by definition the fraction of the money bet that it retains on average, is

$$\frac{2}{38} \approx 0.052631579.$$

Roulette with 38 pockets became and remained popular in the United States, and it is now known as the American version of Roulette.

European Roulette In the second half of the 19th century, some casinos reduced the house's advantage in Roulette in order to make the game more attractive. In this new Roulette, which became standard in Europe by the end of the century, there are 37 pockets, 18 red, 18 black, and only one for the house, labeled '0'. So the probability of winning a bet on red or a bet on black is

$$\frac{18}{37} \approx 0.0486486486,$$

and the house's advantage is only

$$\frac{1}{37} \approx 0.027027027.$$

Trente et Quarante Trente et Quarante (meaning "Thirty and Forty" in French) is also sometimes called Trente-un (French for "thirty-one") or Rouge et Noir (French for "Red and Black"). It is played with 6 decks of cards — $6 \times 52 = 312$ cards — all shuffled together as one deck. On each round, two hands of cards are dealt face-up. For no good reason, the two hands are called "red" and "black". Cards are dealt to each hand until the total of the numbers on their faces (an ace being counted as 1 and a face card as 10) is greater than 30. The hand with the greater total is the winner. In the case of a tie, the round is ignored (this is a "push"), and the players retrieve the money they have put on a color, except that when the tie is 31 to 31, the players lose half of this money to the house. Eventually, after 20 or so rounds, there will not be enough cards left in the deck for another round, and a new deck will be prepared.

As players can easily see, cheating by the house is practically impossible, and "red" and "black" are equally likely on each round. The probability of a tie, and of a 31 to 31 tie in particular, does change depending on what cards have already been dealt, but in ways far too complicated to allow any player to gain an advantage or to allow the house to enhance its advantage.

The game was popular not only because of its transparency but also because the house's advantage, which is limited to the share of the stakes it takes on 31 to 31 ties, is smaller than its advantage in any of the other 18th and 19th century casino games. According to Ethier [15], the probabilities on each round, to nine decimal places, are

- Red wins: 0.445200543
- Black wins: 0.445200543
- 31 to 31 tie: 0.021891370
- Push: 0.087707543

Ignoring pushes, the house takes on average

$$\frac{\frac{1}{2} \times 0.021891370}{1 - 0.087707543} = 0.011998000$$

of the money the player bets. The player would lose on average this same fraction of the money she bets if her bets won

$$\frac{1}{2} - \frac{1}{2} \times 0.011998000 = 0.494001000$$

of the time and lost the rest of the time. As commentators, beginning with the casino owner Pierre-Nicolas Huyn in 1788 [17], frequently noted, this advantage for the house is far less than the advantage in other casino games.

The complicated way that the house's advantage is implemented in Trente et Quarante creates some uncertainty about how some of the betting systems discussed in this article should be implemented. How should a 31 to 31 tie affect the next move if you are playing the d'Alembert? Should you increase your bet because you lost? We have not seen this question raised in the 19th century literature, and our computer simulations suggest that the answer does not make much difference to the issues discussed in this article.

6.5 Limits set by the casino

As we have mentioned, the limits on betting set by casinos need to be considered when we turn betting systems into betting strategies by specifying when they will stop betting. What were typical limits during the period when the best known betting systems were invented?

Boreux reports that the elegant casinos of the 1790s had two types of tables for Trente et Quarante. At the lower-class table the minimum bet was 1 écu and the maximum was 25 écu. At the upper-class table the minimum was 1 louis and the maximum was 100 louis, but you were allowed to violate this limit to bet double what you had lost [29].

According to Boreux, a taille in Trente et Quarante (a shuffled and cut deck of 312 cards) can produce between 18 to 32 rounds of play, taking about half an hour to play, and a séance consists of 2 to 4 tailles. So you might play about $25 \times 3 = 75$ rounds in a séance or about 150 a day. Some 19th-century authors wanted players to implement their systems over multiple days. So it might be reasonable to run a system for several hundred rounds.

Roulette is much faster. According to Marcel Boll [6, p. 180], a single Roulette wheel might be spun 600 times a day at Monte Carlo in the 1930s. The speed of Roulette made it difficult to impossible to implement a complicated system if you tried to play every round; authors who promoted systems for Roulette would sometimes explain that they were writing books about the systems rather then using them to enrich themselves because they had grown old and found the implementation too exhausting. But you if you played only every third round, you might be able to play a system for 200 rounds in a single day.

In 1902, G. d'Albigny [9, pp. 15, 22–23, 31] reported that the minimum and maximum bets at Monte Carlo were 5 francs and 6,000 francs in Roulette, 20 francs and 12,000 francs in Trente et Quarante. In 1986 [23, pp. 406–407], John Scarne reported that the maximum color bet in Roulette in casinos all over the world was usually 500 times the minimum.

These reports suggest that a system should not require more than a few hundred rounds of play, with bets more than a few hundred times the minimum. To catch on, a system would probably need to appear much quicker and less risky than this. The typical gambler would not want to play for hours or days on the promise of a small gain.

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