A game-theoretic proof of the Erdos-Feller-Kolmogorov-Petrowsky law of the iterated logarithm for fair-coin tossing

Akimichi Takemura (U.Tokyo)
(joint with Takeyuki Sasai and Kenshi Miyabe)

Nov.13, 2014, CIMAT Mexico

Manuscript:

“A game-theoretic proof of Erdos-Feller-Kolmogorov-Petrowsky law of the iterated logarithm for fair-coin tossing”

T.Sasai, K.Miyabe and A.Takemura

arXiv:1408.1790 (version 2)

Outline

1. **LIL in the EFKP form**
2. **Fair-coin tossing game**
3. **Outline of our proof**
4. **Summary and topics for further research**

LIL in the EFKP form (EFKP-LIL)

Law of the iterated logarithm for fair-coin tossing (A.Khintchin (1924))

- $P(X_i = +1) = P(X_i = -1) = 1/2$, **independent**, $S_n = \sum_{i=1}^{n} X_i$.

$$\limsup_n \frac{S_n}{\sqrt{2n \ln \ln n}} = 1, \quad \liminf_n \frac{S_n}{\sqrt{2n \ln \ln n}} = -1, \ \ a.s.$$

- **We want to evaluate the behavior of S_n more closely.**
  $\rightarrow$ difference form rather than ratio form

- **Terminology (Lévy)**
  - $\psi(n)$ **belongs to the *upper class*:**

$$P(S_n > \sqrt{n}\psi(n) \quad i.o.) = 0.$$

  - $\psi(n)$ **belongs to the *lower class*:**

$$P(S_n > \sqrt{n}\psi(n) \quad i.o.) = 1.$$

- **Kolmogorov-Erdős's LIL (Erdős (1942))**

$$\psi(t) \in \begin{cases} \text{Upper} \\ \text{Lower} \end{cases} \quad \text{if} \quad \int^{\infty} \frac{\psi(t)}{t} e^{-\psi(t)^2/2} dt \begin{cases} < \infty \\ = \infty \end{cases}$$

- **For any** $k > 0$ **denote** $\ln_k t = \underbrace{\ln \ln \ldots \ln}_{k\textbf{times}} t$**.**

- **Consider** $\psi(t)$ **of the following form:**

$$\sqrt{2 \ln \ln t + 3 \ln \ln \ln t + 2 \ln_4 t + \cdots + (2 + \epsilon) \ln_k t}$$

- **By the condition above**

  $\epsilon > 0$**: upper class,** $\quad \epsilon \leq 0$**: lower class**

- **This follows from the convergence or divergence of the following integral:**

$$\int^{\infty} \frac{1}{t \ln t \ln_2 t \ldots \ln_{k-1}^{(1+\epsilon/2)}} \, dt \quad \begin{cases} < \infty, & \epsilon > 0 \\ = \infty, & \epsilon \leq 0 \end{cases}$$

- **We want to prove this theorem in game-theoretic framework.**

Fair-coin tossing game

Protocol (Fair-Coin Game)

$\mathcal{K}_0 := 1$.

FOR $n = 1, 2, \ldots$:

Skeptic announces $M_n \in \mathbb{R}$.

Reality announces $x_n \in \{-1, 1\}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Collateral Duty: Skeptic has to keep $\mathcal{K}_n \geq 0$.

Reality has to keep $\mathcal{K}_n$ **from tending to infinity.**

Let

$$I(\psi) = \int_1^\infty \frac{\psi(t)}{t} e^{-\psi(t)^2/2} dt$$

Theorem 1. *Let* $\psi(t) > 0$, $t \geq 1$, *be continuous and monotone non-decreasing. In Fair-Coin Game*

$$I(\psi) < \infty \Rightarrow \textit{Skeptic can force } S_n < \sqrt{n}\psi(n) \quad a.a. \qquad (1)$$

$$I(\psi) = \infty \Rightarrow \textit{Skeptic can force } S_n \geq \sqrt{n}\psi(n) \quad i.o. \qquad (2)$$

- (1) **is the *validity*,** (2) **is the *sharpness*.**
- **Game-theoretic result implies the measure-theoretic result (Chap.8 of S-V book).**

Motivations of our investigation:

- **When I saw EFKP-LIL, I wanted to know whether the line of the proof in Chap.5 of S-V book for LIL is strong enough to prove EFKP-LIL.**

- **My student, Takeyuki Sasai, worked hard and got it.**

- **We now have version 2 of the manuscript on arXiv.**

Outline of our proof

- **We construct Skeptic's strategies for validity and for sharpness.**
- **We employ (continuous) mixtures of strategies with constant betting ratios.**
- **We call them "Bayesian strategies", since the mixture weights correspond to the prior distribution in Bayesian inference.**
- **Our strategy depends on a given ψ.**
- **We have a very short validity proof (less than 2 pages).**

- **Our sharpness proof is about 9 pages in version 2.**
- **Although we give so many inequalities, the entire proof is explicit and elementary.**

Proof of Validity

- **Discretization of the integral**

$$\sum_{k=1}^{\infty} \frac{\psi(k)}{k} e^{-\psi(k)^2/2} < \infty$$

- **Strategy with constant betting proportion γ:**

$$M_n = \gamma \mathcal{K}_{n-1}$$

- **The capital process of this strategy:**

$$\mathcal{K}_n^{\gamma} = \prod_{i=1}^{n} (1 + \gamma x_i)$$

- **We bound this process from above and below**

$$e^{-\gamma^3 n} e^{\gamma S_n - \gamma^2 n/2} \leq \mathcal{K}_n^\gamma \leq e^{\gamma^3 n} e^{\gamma S_n - \gamma^2 n/2}.$$

(We use only the lower bound for validity)

- **Choose an infinite sequence $a_k \uparrow \infty$ such that**

$$\sum_{k=1}^{\infty} a_k \frac{\psi(k)}{k} e^{-\psi(k)^2/2} = Z < \infty.$$

- **Define p_k, γ_k by**

$$p_k = \frac{1}{Z} a_k \frac{\psi(k)}{k} e^{-\psi(k)^2/2}, \qquad \gamma_k = \frac{\psi(k)}{\sqrt{k}}$$

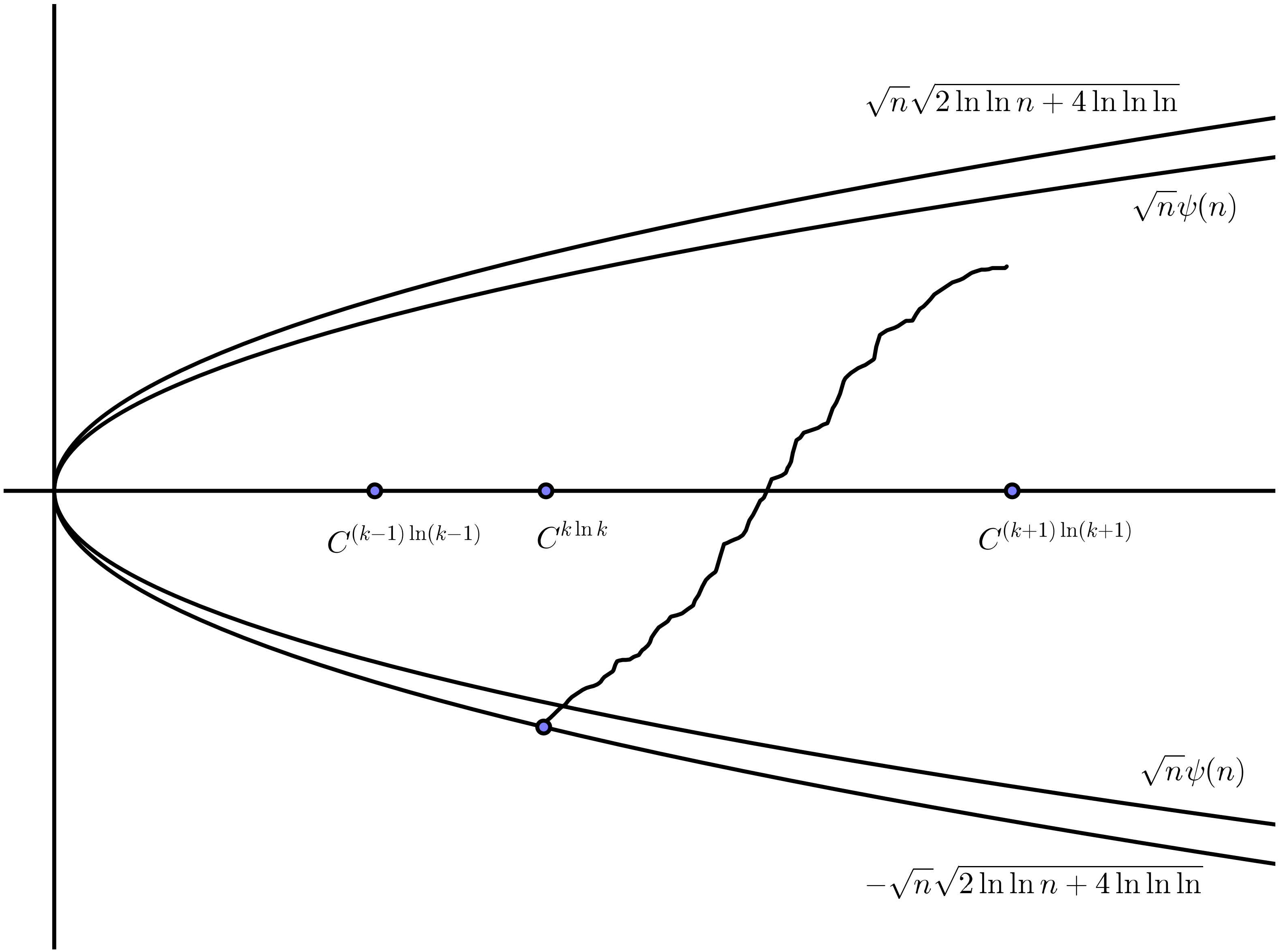
- **The following mixture strategy forces the validity.**

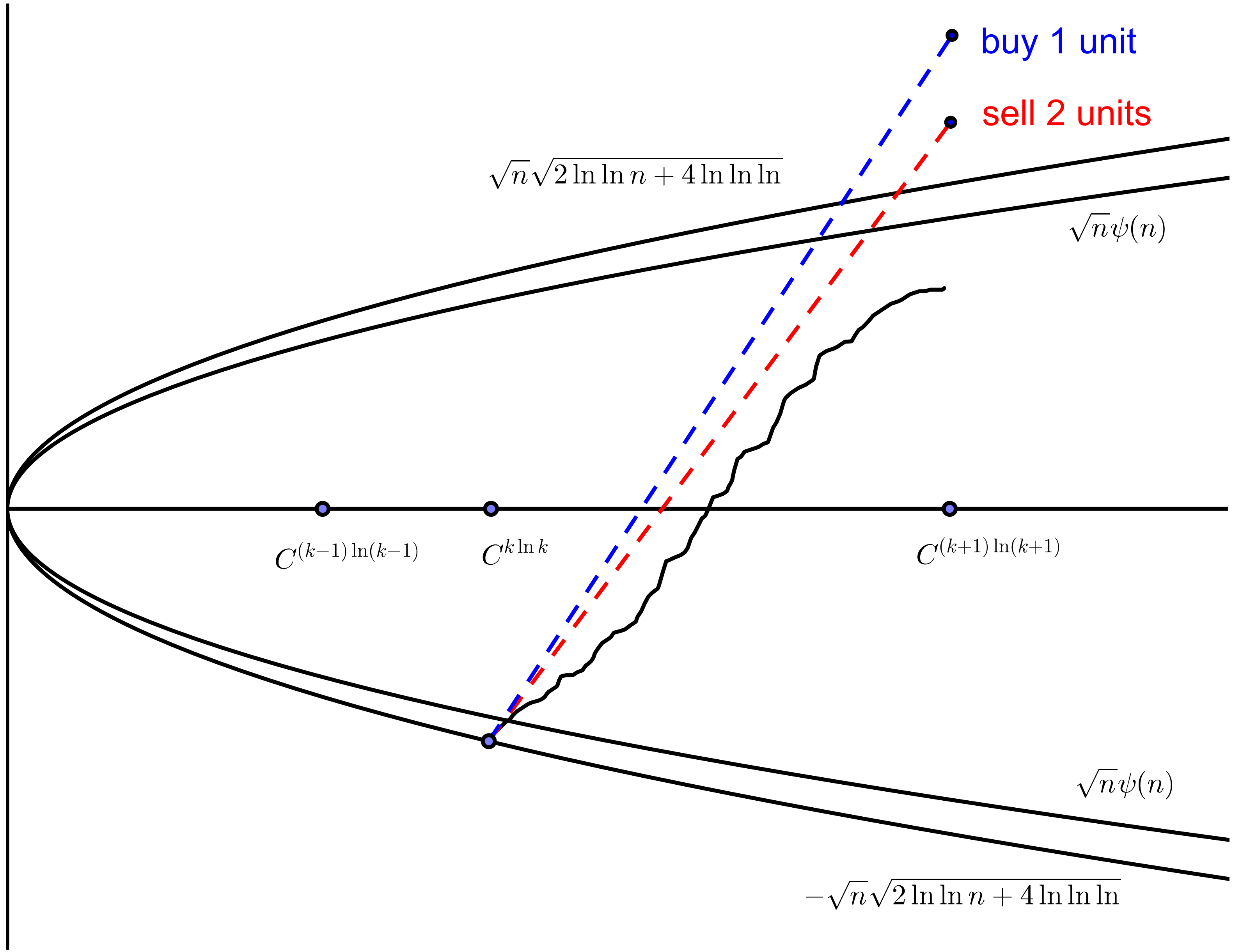
$$\mathcal{K}_n = \sum_{k=1}^{\infty} p_k \mathcal{K}_n^{\gamma_k},$$

Outline of the Sharpness proof

- We combine selling and buying of strategies as in Chapter 5 of S-V book and Miyabe and Takemura (2013).
- However, unlike them, in Version 2 of our manuscript, we only hedge from above. In Chapter 5 of S-V book and Miyabe and Takemura (2013), we need hedges both from above and from below.
- This is possible because $|x_n| = 1$.

- **Furthermore we divide the time axis $[0, \infty)$ into subintervals at time points $C^{k \ln k}$, $k = 1, 2, \ldots$, which is somewhat sparser than the exponential time points, used in proofs of usual LIL.**
- **This is also different from Erdős (1942).**
- **At the endpoint of each subinterval, Skeptic makes money if $S_n \le \sqrt{n}\psi(n)$, by the selling strategy.**


$\sqrt{n}\sqrt{2\ln\ln n + 4\ln\ln\ln}$
$\sqrt{n}\psi(n)$
$C^{(k-1)\ln(k-1)}$
$C^{k\ln k}$
$C^{(k+1)\ln(k+1)}$
$\sqrt{n}\psi(n)$
$-\sqrt{n}\sqrt{2\ln\ln n + 4\ln\ln\ln}$



buy 1 unit
sell 2 units
$\sqrt{n}\sqrt{2\ln\ln n + 4\ln\ln\ln}$
$\sqrt{n}\psi(n)$
$C^{(k-1)\ln(k-1)}$
$C^{k\ln k}$
$C^{(k+1)\ln(k+1)}$
$\sqrt{n}\psi(n)$
$-\sqrt{n}\sqrt{2\ln\ln n + 4\ln\ln\ln}$

- **The selling strategy is based on the following integral mixture of constant proportion strategies $\mathcal{K}_n^{\gamma}$**

$$\frac{1}{\ln k}\int_0^{\ln k}\int_{2/e}^{1}\mathcal{K}_n^{ue^{-w}\gamma}dudw$$

- **This smoothing seems to be essential for our proof.**

Summary and topics for further research

- **Usual LIL in the ratio form was already given in S-V's book.**
- **Also see Miyabe and Takemura (2013) ([3]).**
- **We gave EFKP-LIL in GTP for the first time.**
- **Although we only considered fair-coin tossing, our proof can be generalized to other cases (work in progress, in particular to the case of self-normalized sums).**

Topics

- **Generalization to self-normalized sums, where the population variance is replaced by the sample variances (like t-statistic).**
  - **We are hopeful to finish this generalization soon.**
  - **Some results for the case of self-normalized sums is given in measure-theoretic literature.**
  - **We seem to get stronger results.**

- **What happens if $\psi(n)$ is announced by Forecaster each round? Can Skeptic force**

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n} e^{-\psi(n)^2/2} = \infty \quad \Leftrightarrow \quad S_n \geq \sqrt{n}\psi(n) \quad i.o. \qquad \textbf{?} \tag{3}$$

  - **A related mathematical question: "is there a sequence of functions approaching the lower limit of the upper class?"**

- **Simplified question: does there exists a double array of positive reals a_{ij}, $i, j \geq 1$, such that**
  - **for each i, $\sum_j a_{ij} = \infty$.**
  - **a_{ij} is decreasing in i: $a_{1j} \geq a_{2j} \geq \dots, \ \forall j$.**
  - **for every divergent series of positive reals $b_j > 0$, $\sum_j b_j = \infty$, there exists some i_0 and j_0 such that**

$$a_{i_0 j} \leq b_j, \quad \forall j \geq j_0.$$

- **Probably the answer is NO. If it is YES, then by countable mixture of strategies we can show that (3) is true.**

References

[1] **P. Erdős. On the law of the iterated logarithm. *Annals of Mathematics, Second Series*, 43:419–436, 1942.**

[2] **A. Khinchine. Über einen Satz der Wahrscheinlichkeitsrechnung. *Fundamenta Mathematica*, 6:9–20, 1924.**

[3] **K. Miyabe and A. Takemura. The law of the iterated logarithm in game-theoretic probability with quadratic and stronger hedges. *Stochastic Process. Appl.*, 123(8):3132–3152, 2013.**